# SURFACES IN CONSTANT CURVATURE THREE-MANIFOLDS AND THE INFINITESIMAL TEICHMÜLLER THEORY

Andrea Seppi

Thesis defended on December, 1st 2015

Advisor: Prof. Francesco Bonsante

Jury:
Prof. Pietro Rigo (President)
Prof. Roberto Frigerio (Member)
Prof. Paolo Stellari (Member and secretary)

# Abstract

In this thesis are exploited several instances of the relationship between convex Cauchy surfaces $S$ in flat Lorentzian (2+1)-dimensional maximal globally hyperbolic manifolds $M$ and the tangent bundle of Teichmüller space $\mathcal{T}(S)$ of the topological surface $S$. This relationship was first pointed out by Geoffrey Mess in the case of closed surfaces.

The first case presented is the case of simply connected surfaces, and $M$ is a domain of dependence in $\mathbb{R}^{2,1}$. We prove a classification of entire surfaces of constant curvature in $\mathbb{R}^{2,1}$ in terms of Zygmund functions on the circle, which represent tangent vectors of universal Teichmüller space $\mathcal{T}(\mathbb{D})$ at the identity. An important ingredient is the solvability of Minkowski problem for Cauchy surfaces in any domain of dependence $M$ contained in the future cone over some point of $\mathbb{R}^{2,1}$, which is proved by analyzing the Dirichlet problem for the Monge-Ampère equation $\det D^2u(z) = (1/\psi(z))(1-|z|^2)^{-2}$ on the disc, where $\psi$ is a smooth positive function. Moreover, when $S$ is a surface of constant curvature, the principal curvatures are bounded if and only if $\varphi$ is in the Zygmund class.

The situation of $S$ a closed surface, and $M$ is a maximal globally hyperbolic flat spacetime diffeomorphic to $S \times \mathbb{R}$, is next discussed. We provide an explicit relation between the embedding data of any strictly convex Cauchy surface in $M$ and the holonomy of $M$, which was used by Mess to parametrize the moduli space of manifolds $M$ as above by means of the tangent bundle of $\mathcal{T}(S)$. The techniques used in this thesis are amenable to be extended to the case of globally hyperbolic flat spacetimes with $n > 0$ particles, namely cone singularities along timelike lines, where the cone angle is assumed in $(0, 2\pi)$. The analogue of Mess' parametrization is then proved, showing that the corresponding moduli space is parametrized by the tangent bundle of Teichmüller space of the closed surface $S$ with $n$ punctures.

The above connections can be regarded as an infinitesimal version of the relation of Teichmüller space $\mathcal{T}(S)$ and universal Teichmüller space $\mathcal{T}(\mathbb{D})$ with surfaces in maximal globally hyperbolic Anti-de Sitter manifolds (either with the topological type of a closed surface, or with trivial topology) and in quasi-Fuchsian hyperbolic manifolds (or in $\mathbb{H}^3$ itself). In the last part of the thesis this perspective is discussed, and the behavior of zero mean curvature surfaces in $\mathbb{H}^3$ and $\mathbb{A}\mathrm{d}\mathbb{S}^3$ close to the Fuchsian locus is discussed. The main result in hyperbolic space is a sublinear estimate of the supremum of principal curvatures of a minimal embedded disc in $\mathbb{H}^3$ spanning a quasicircle $\Gamma$ in the boundary at infinity in terms of the norm of $\Gamma$ in the sense of universal Teichmüller space, provided $\Gamma$ is sufficiently close to being the boundary of a totally geodesic plane. As a by-product, there is a universal constant C

such that if the Teichmüller distance between the ends of a quasi-Fuchsian manifold $M$ is at most C, then $M$ is almost-Fuchsian, independently of the genus.

In Anti-de Sitter space, an estimate is proved for the principal curvatures of any maximal surface with boundary at infinity the graph of a quasisymmetric homeomorphism $\phi$ of the circle. The supremum of the principal curvatures is estimated again in a sublinear way, in terms of the cross-ratio norm of $\phi$. This also provides a bound on the maximal distortion of the quasiconformal minimal Lagrangian extension to the disc of a given quasisymmetric homeomorphism.

# Acknowledgements

During the course of my PhD, several people contributed not only to make this work possible, but also to make these years a delightful time of my life. I owe a debt of gratitude to them, which can only partially be fulfilled by being mentioned here.

I would primarily like to thank my advisor Francesco Bonsante, for his constant willingness and patience in the guidance of my work, and for his support which in many instances has gone far beyond the mathematical content of my thesis.

I am very grateful to Jean-Marc Schlenker for giving me the opportunity to spend a fruitful period at University of Luxembourg, for his advices and encouragement. I am thankful to Thierry Barbot, François Fillastre, Zheng Huang and Micheal Wolf for their interest in my work, their repeated encouragement and for several useful conversations.

A special thank goes to those who have been colleagues - and friends - as PhD students in Pavia. Although it would be unfair to try to list all their names, they certainly contributed to make our environment an enjoyable place to do mathematics, and much more.

I would like to warmly thank my family, which has constantly supported me, like in all the previous steps of my education. Finally, and most importantly, there are no words to thank my girlfriend Laura, whose love has surrounded every moment of my life.

# Contents

# List of Figures

# Introduction

The cornerstone of this thesis is the idea that the theory of convex surfaces in Minkowski (2+1)-space is strongly interrelated with the infinitesimal theory of Teichmüller spaces. A first instance of this fact was described by Geoffrey Mess in his groundbreaking work [Mes07], in which the class of compact globally hyperbolic manifolds locally modelled on Minkowski space was studied. These objects are flat Lorentzian three-manifolds $M$, diffeomorphic to $S \times \mathbb{R}$ where $S$ is a closed surface (assumed here to be of genus $g \geq 2$), such that the horizontal slices are Cauchy surfaces for the causal structure. Mess proved that every $M$ as above is obtained as the quotient of a convex domain in Minkowski space under the free and proper discontinuous action of an affine deformation of a Fuchsian group. Using this construction, the space of maximal globally hyperbolic flat Lorentzian manifolds (up to isometries isotopic to the identity) was parametrized by the tangent bundle of the Teichmüller space of the surface $S$. Further developments in this direction were then exploited in [Bar05, Bon05].

The relation between Teichmüller spaces and hyperbolic three-manifolds containing a closed surface of genus $g \geq 2$ has been widely studied. Anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is the model Lorentzian manifold of constant curvature $-1$, analogous of hyperbolic space $\mathbb{H}^3$ in the Lorentzian setting, and its relations with Teichmüller theory were largely investigated and appreciated after the pioneering work again by Geoffrey Mess. Indeed, the parameter space of maximal globally hyperbolic Anti-de Sitter spacetimes is diffeomorphic to $\mathcal{T}(S) \times \mathcal{T}(S)$, i.e. the cartesian product of two copies of Teichmüller space of $S$. To some extent the latter result - first proved in [Mes07] - is the analogue of Bers' Simultaneous Uniformization Theorem ([Ber60]), which enabled to parametrize the space of quasi-Fuchsian hyperbolic manifolds by two copies of $\mathcal{T}(S)$.

One of the main themes of this work is the idea that the assumption that $S$ is a closed surface can be relaxed, in (at least) two different directions.

Employing the perspective of quasiconformal mappings for the theory of Teichmüller spaces, it turns out that a very interesting object to consider in relation with the geometry of convex surfaces is universal Teichmüller space $\mathcal{T}(\mathbb{D})$, which can be thought of as the space of quasiconformal deformations of the disc $\mathbb{D}$, as a Riemann surface. This is an infinite-dimensional space which generalizes the notion of Teichmüller space of a surface, in fact $\mathcal{T}(\mathbb{D})$ contains an embedded copy of the Teichmüller space of any hyperbolic Riemann surface. Universal Teichmüller space has been studied in relation with surfaces in hyperbolic space, for instance in [Eps86, Eps87], and in Anti-de Sitter space, in [BS10, Sca12, SK13].

On the other hand, another generalization of the above correspondence between geometric structures on three-manifolds of constant curvature and the Teichmüller theory of surfaces comes from allowing certain types of singularities. As already exploited in [KS07, BS09, BBS11, BBS14, Tou15] for the Anti-de Sitter case, and in [KS07, MS09, LS14] for the hyperbolic case, Teichmüller space of a punctured surface is a relevant object in relation with geometric structures on three-manifolds with cone singularities. When dealing with Lorentzian manifold, the usual assumption is to consider cone singularities along timelike lines, corresponding to the physical notion of "particle". The parametrizations of maximal globally hyperbolic/quasi-Fuchsian manifolds discussed above, and other remarkable features of the theory for closed surfaces, thus extend to the case of manifolds with cone singularities and lead to interesting new constructions.

Part II of this thesis is mostly concerned with the study of convex surfaces in Minkowski space $\mathbb{R}^{2,1}$ and in flat Lorentzian manifolds (possibly with particles), in relation with the tangent space of, respectively, universal Teichmüller space and the Teichmüller spaces of (punctured) surfaces.

The developments pursued in this thesis are often permeated with another notion: the idea of geometric transition. Minkowski space is identified to the tangent space at a fixed point of both de Sitter space - namely, the model space $\mathrm{d}\mathbb{S}^3$ of positive constant curvature Lorentzian manifolds - and Anti-de Sitter space. Hence, Minkowski space can be thought of as a rescaled limit of Anti-de Sitter and de Sitter geometries, by "zooming in" from a fixed point. When the scale of zooming gets infinitely larger, any information around the fixed point blows up and is recorded - at an infinitesimal level - in a copy of Minkowski space. Let us briefly remark that $\mathrm{d}\mathbb{S}^3$ is a dual space to $\mathbb{H}^3$: indeed, both spaces have a projective model, and the duality arises from the projective duality of $\mathbb{R}\mathrm{P}^3$. The analogous construction provides a duality of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ to itself.

In his PhD thesis ([Dan11]), Jeffrey Danciger described a different transition procedure which can be regarded as the blow-up of a totally geodesic plane. This transition involves both Anti-de Sitter space and hyperbolic space, while the limit object is a three-manifold endowed with a degenerate metric, called half-pipe geometry. See also [Dan13, Dan14, DGK13].

The underlying space of half-pipe geometry $\mathbb{HP}^3$ can be identified to $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane, while its degenerate metric only reads the first component, where it coincides with the hyperbolic metric. It turns out that this space is in a natural way the space which parametrizes spacelike planes in Minkowski space. This produces a "duality" between $\mathbb{R}^{2,1}$ and $\mathbb{HP}^3$ which is naturally the rescaled limit - in a projective sense - of the duality between $\mathrm{d}\mathbb{S}^3$ and $\mathbb{H}^3$, and of the self-duality of $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

This transitional geometry gives an account for the fact that Minkowski space is strictly interrelated with the infinitesimal theory of Teichmüller spaces. For instance, when $S$ is a closed surface, it is natural to regard the parameter space $T\mathcal{T}(S)$ for flat Lorentzian maximal globally hyperbolic manifolds as a blow-up of the Anti-de Sitter analogous parameter space, given by $\mathcal{T}(S)\times\mathcal{T}(S)$, close to the Fuchsian locus which is represented by the diagonal. However, the assumption that $S$ is a closed surface is not essential here, and very often $S$ will be replaced by the disc $\mathbb{D}$, as a Riemann surface. Several geometric quantities associated to surfaces in $\mathbb{R}^{2,1}$, describing for instance the curvature, or the asymptotic behaviour at infinity, are therefore the mirror at first order of the analogous quantities in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ or $\mathrm{d}\mathbb{S}^3$. Such amount of information is very often translated in simple terms in half-pipe geometry, by means of the already mentioned duality.

In Part III of this thesis, we will consider the behaviour of surfaces in hyperbolic space and Anti-de Sitter space which are topologically discs and are "close" to the Fuchsian locus, i.e. close (in the sense of universal Teichmüller space $\mathcal{T}(\mathbb{D})$) to being a totally geodesic plane, with special interest towards the properties of curvature of the surface.

## Convex surfaces in Minkowski space

The first objects studied, in the presentation of this thesis, are convex surfaces in Minkowski space - especially surfaces of constant curvature - and the tangent space of universal Teichmüller space.

As first observed by Hano and Nomizu ([HN83]), the standard embedding of $\mathbb{H}^2$ into the hyperboloid of $\mathbb{R}^{2,1}$ is not the unique isometric embedding of $\mathbb{H}^2$. This is a striking difference with the case of Euclidean geometry, where by a classical theorem, any isometric embedding of the sphere of constant curvature into $\mathbb{R}^3$ is equivalent to the standard embedding of the round sphere (i.e. up to post-composition with an ambient isometry).

A strictly convex spacelike surface $S$ in $\mathbb{R}^{2,1}$ gives rise, by means of the duality between Minkowski space and half-pipe geometry, to a spacelike surface $S^*$ in $\mathbb{HP}^3$, namely a surface in $\mathbb{H}^2 \times \mathbb{R}$ which is a graph over an open connected domain of $\mathbb{H}^2$. If $S^*$ is the graph of the function $\bar{u}$, it turns out that the inverse of the shape operator of $S$ can be expressed as $\mathrm{Hess}\,\bar{u} - \bar{u}\,E$, where Hess denotes the covariant hyperbolic Hessian and $E$ is the identity operator. The function $\bar{u}$ is the analogous in the Lorentzian setting of the classical support function for Euclidean convex bodies.

This point of view enables to give an explanation for the difference with the rigidity of the sphere in the Euclidean case. Indeed, in the Euclidean case, the equation for a constant curvature surface in terms of the support function $\bar{u} : \mathbb{S}^2 \to \mathbb{R}$ is

$$\det(\mathrm{Hess}\,\bar{u} + \bar{u}\,E) = 1\,,$$

where Hess now is the covariant Hessian on the sphere. By using the comparison principle, it turns out that the difference of any two solutions must be the restriction on $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$. This allows to conclude that every solution is of the form $\bar{u}(x) = 1 + \langle x, \xi \rangle$ for some $\xi \in \mathbb{R}^3$. But this is exactly the support function of the round sphere of radius 1 centered at $\xi$.

In the Minkowski case, the support function $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$ of a constant curvature spacelike surface (with surjective Gauss map) satisfies the equation

$$\det(\mathrm{Hess}\,\bar{u} - \bar{u}\,E) = 1\,.$$

Here the maximum principle cannot be directly used by the non-compactness of $\mathbb{H}^2$. This is a general indication that some boundary condition must be taken into account to determine the solution and the isometric immersion of $\mathbb{H}^2$.

More generally, we will deal with the Lorentizian version of the classical Minkowski problem in Euclidean space. Given a smooth spacelike strictly convex surface $S$ in $\mathbb{R}^{2,1}$, the curvature function is defined as $\psi : G(S) \to \mathbb{R}$, $\psi(x) = -K_S(G^{-1}(x))$, where $G : S \to \mathbb{H}^2$ is the Gauss map and $K_S$ is the scalar intrinsic curvature on $S$. Minkowski problem consists in finding a convex surface in Minkowski space whose curvature function is a prescribed positive function $\psi$. Using the support function technology, the problem turns out to be equivalent to solving the equation

$$\det(\mathrm{Hess}\bar{u} - \bar{u}\, E) = \frac{1}{\psi}\,. \tag{1}$$

Using the Klein model of $\mathbb{H}^2$, Equation (1) can be reduced to a standard Monge-Ampère equation over the unit disc $\mathbb{D}$. In particular solutions of (1) explicitly correspond to solutions $u : \mathbb{D} \to \mathbb{R}$ of the equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2}\,. \tag{MA}$$

The problem is going to be well-posed once a boundary condition is imposed, of the form

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

The boundary value of the solution $u$ has a direct geometric interpretation. Indeed, as $\partial\mathbb{D}$ (regarded as the set of lightlike directions) parameterizes lightlike linear planes, the restriction of the support function on $\partial\mathbb{D}$ gives the height function of lightlike support planes of $S$, where $u(\eta) = +\infty$ means that there is no lightlike support plane orthogonal to $\eta$. Equivalently, when $S$ is the graph of a convex function $f : \mathbb{R}^2 \to \mathbb{R}$, the condition $u|_{\partial\mathbb{D}} = \varphi$ is also equivalent to requiring that

$$\lim_{r\to+\infty} (r - f(rz)) = \varphi(z)$$

for every $z \in \partial\mathbb{D}$. This is the type of asymptotic condition considered for instance in [Tre82] and [CT90], where the existence problem for constant mean curvature surfaces is treated.

The first result of Chapter 4 concerns the solvability of Minkowski problem in Minkowski space.

**Theorem 4.A.** *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a lower semicontinuous and bounded function and $\psi : \mathbb{D} \to [a, b]$ for some $0 < a < b < +\infty$. Then there exists a unique spacelike entire graph $S$ in $\mathbb{R}^{2,1}$ whose support function $u$ extends $\varphi$ and whose curvature function is $\psi$.*

We say that a convex surface is a spacelike entire graph if $S = \{(p, f(p)) \,|\, p \in \mathbb{R}^2\}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a $C^1$ function on the horizontal plane such that $||Df(p)|| < 1$ for all $p \in \mathbb{R}^2$.

In [Li95], Li studied the Minkowski problem in Minkowski space in any dimension showing the existence and uniqueness of the solution of (MA) imposing $u|_{\partial\mathbb{D}} = \varphi$, for a given smooth $\varphi$. The result was improved in dimension 2+1 by Guan, Jian and

Schoen in [GJS06], where the existence of the solution is proved assuming that the boundary data is only Lipschitz. The solutions obtained in both cases correspond to spacelike entire graphs.

A remarkable result in [Li95] is that under the assumption that the boundary data is smooth, the corresponding convex surface $S$ has principal curvatures bounded from below by a positive constant. As a partial converse statement, if $S$ has principal curvatures bounded from below by a positive constant, then the corresponding function $u : \mathbb{D} \to \mathbb{R}$ extends to a continuous function of the boundary of $\mathbb{D}$.

A natural question to ask is whether the condition that the principal curvatures are bounded from below by a positive constant can be characterized in terms of the boundary value of the support function. An indication that the results of [Li95, GJS06] are not fully satisfactory comes from the solution of Minkowski problem in any maximal globally hyperbolic flat spacetime, due to Barbot, Béguin and Zeghib ([BBZ11]). Their result can be expressed in the following way: given any cocompact Fuchsian group $G$ and any affine deformation $\Gamma$ of $G$, for every positive $G$-invariant function $\psi$, there exists a unique $\Gamma$-invariant convex surface $S$ with curvature $K_S(x) = -\psi(x)$ for $x \in \mathbb{H}^2$.

Indeed, if $u : \mathbb{D} \to \mathbb{R}$ is the support function corresponding to some $\Gamma$-invariant surface $S$, combining the result by Li and the cocompactness of $\Gamma$, it turns out that $u$ extends to the boundary of $\mathbb{D}$. It is not difficult to see that the extension on the boundary only depends on $\Gamma$ and in particular it is independent of the curvature function; on the other hand, the extension $u|_{\partial\mathbb{D}}$ is in general not Lipschitz although the principal curvatures are bounded from below, by cocompactness.

The second result we obtain is the determination of the exact regularity class of the extension on $\partial\mathbb{D}$ of functions $u : \mathbb{D} \to \mathbb{R}$ corresponding to surfaces with principal curvatures bounded from below.

**Theorem 4.B.** *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. There exists a spacelike entire graph in $\mathbb{R}^{2,1}$ whose principal curvatures are bounded from below by a positive constant and whose support function at infinity is $\varphi$ if and only if $\varphi$ is in the Zygmund class.*

A function $\varphi : S^1 \to \mathbb{R}$ is in the Zygmund class if there is a constant $C$ such that, for every $\theta, h \in \mathbb{R}$,

$$|\varphi(e^{i(\theta+h)}) + \varphi(e^{i(\theta-h)}) - 2\varphi(e^{i\theta})| < C|h| .$$

Functions in the Zygmund class are $\alpha$-Hölder for every $\alpha \in (0, 1)$, but in general they are not Lipschitz.

Universal Teichmüller space $\mathcal{T}(\mathbb{D})$ can be defined as the space of quasisymmetric homeomorphisms of the circle up to Möbius transformations. It turns out that the boundary value $u|_{\partial\mathbb{D}} = \varphi$ of the support function of a convex surface $S$ in Minkowski space can be identified in a natural way to a vector field on $S^1$, and $\varphi$ is in the Zygmund class - which is equivalent to the boundedness from below of the principal curvatures of $S$, by Theorem 4.B - if and only if such vector field is an element of the tangent space of $\mathcal{T}(\mathbb{D})$ at the identity.

Hence Theorem 4.B can be thought of as the infinitesimal version of a statement in Anti-de Sitter geometry, asserting that if a convex surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ has principal curvatures bounded from below, then the boundary at infinity is the graph of a quasisymmetric homeomorphism $\phi : S^1 \to S^1$. Vice versa, given $\phi$ quasisymmetric, it is possible to construct convex surfaces with bounded principal curvatures and with $gr(\phi)$ as asymptotic boundary. It is possible to give a precise meaning to the above heuristic argument which compares the situation in Minkowski space to the infinitesimal picture of the Anti-de Sitter case, by means of the notion of geometric transition, and this will be the content of the last chapter of the thesis.

It is also possible, in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, to give a third condition equivalent to the above. Reinterpreting - in light of the work of Mess - a result anticipated by Thurston ([Thu86]) and later proved independently by Gardiner ([GHL02]) and Šarić ([Šar06]), it turns out that $\phi : S^1 \to S^1$ is quasisymmetric if and only if the bending lamination of the upper boundary of the convex hull of $gr(\phi)$ is a measured geodesic lamination with finite Thurston norm.

In the Minkowski setting, there is a measured geodesic lamination associated to any function $\varphi : S^1 \to \mathbb{R}$, by considering the bending lamination of the convex envelope of $\varphi$ in $\mathbb{D} \times \mathbb{R}$. Heuristically, the convex envelope of $\varphi$ is a pleated surface and is the graph of a piecewise affine function $u$. The bending lines provide a geodesic lamination over $\mathbb{H}^2$, whereas a transverse measure encodes the amount of bending. Moreover, it turns out that $u$ is the support function of the so-called domain of dependence of $S$, namely the largest convex domain in $\mathbb{R}^{2,1}$ for which $S$ is a Cauchy surface.

Given a support function at infinity $\varphi$, the domain of dependence $D$ associated to any surface $S$ as above is uniquely determined by $\varphi$. Using again a theorem proved in [GHL02] or [MŠ12, Appendix], we show that the condition that $\varphi$ is in the Zygmund class is also equivalent to the fact that the measured geodesic lamination associated to $D$ has finite Thurston norm. This interpretation will actually be very useful in the proof of Theorem 4.B.

The solutions of the Minkowski problem in Theorem 4.B are obtained by approximation from solutions which are invariant for affine deformations of cocompact Fuchsian groups, using the theorem of Barbot, Béguin and Zeghib.

We mention that another important step to the proof of Theorem 4.B is the use of barriers, which enable to show that the surface we construct does not develop singularities and is therefore a spacelike entire graph. To construct such barriers, we consider constant curvature surfaces invariant under a one-parameter family of isometries, thus reducing the partial differential equation (MA) to an ODE. Hano and Nomizu in [HN83] studied the constant curvature surfaces invariant for a one-parameter hyperbolic group fixing the origin, thus exhibiting for the first time non-standard immersions of the hyperbolic plane in $\mathbb{R}^{2,1}$. Here are considered surfaces invariant under a one-parameter parabolic group, which are suited to be used as barriers in a more general context.

Theorem 4.B implies that spacelike entire graphs of constant curvature $-1$ and with a uniform bound on the principal curvatures correspond to functions $u$ whose extension to $\partial\mathbb{D}$ is Zygmund. We prove that also the converse holds. This gives a complete classification of such surfaces in terms of Zygmund functions.

**Theorem 4.C.** *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a function in the Zygmund class. For every*

*$K < 0$ there is a unique spacelike entire graph $S$ in $\mathbb{R}^{2,1}$ of constant curvature $K$ and with bounded principal curvatures whose corresponding function $u$ extends $\varphi$.*

Finally, we show - using again the fact that the same statement holds in maximal globally hyperbolic flat spacetimes, as proved by Barbot, Béguin and Zeghib - that all the surfaces of constant curvature in a domain of dependence $D$ contained in the future cone over a point provide a foliation of $D$.

**Theorem 4.D.** *If $D$ is a domain of dependence contained in the future cone of a point, then $D$ is foliated by surfaces of constant curvature $K \in (-\infty, 0)$.*

The condition that $D$ is contained in the cone over a point is easily seen to be equivalent to the fact that the support function at infinity is bounded (while it might be in general not continuous, but only lower semicontinuous).

## Maximal globally hyperbolic flat spacetimes

The next aim of the thesis is a deeper understanding of the aforementioned relation between maximal globally hyperbolic flat spacetimes and Teichmüller spaces, initiated by Mess. A different motivation for the study of these objects comes from the observation that in dimension 2+1, Lorentzian metrics which are solutions of Einstein equation are precisely metrics of constant curvature. If the cosmological constant is 0, then the solutions reduce to flat metrics.

From the physical point of view ([Wit89]), a reasonable requirement to put on flat metrics on a manifold $M$ is global hyperbolicity. Choquet-Bruhat in [CB68] proved that the embedding data (i.e. the first and the second fundamental form) of any Cauchy surface determine a maximal globally hyperbolic flat spacetime. A globally hyperbolic spacetime $M$ is maximal if there is no isometric embedding of $M$ in a larger spacetime $M'$, sending a Cauchy surface of $M$ to a Cauchy surface of $M'$.

In dimension 2+1, the constraint equation reduce to the Lorentzian version of the Gauss-Codazzi equations for a metric $I$ (the first fundamental form) and a symmetric 2-tensor $B$ (the shape operator). When the cosmological constant (and thus also the ambient curvature) is 0, the Gauss-Codazzi equations can be written as:

$$\begin{cases} \det B = -K_I \\ d^{\nabla^I} B = 0\,. \end{cases} \qquad \text{(GC-}\mathbb{R}^{2,1}\text{)}$$

Here $d^{\nabla^I} B$ denotes the exterior derivative of $B$ with respect to the Levi-Civita connection of $I$.

As already remarked, when $S$ is a closed surface of genus $g \geq 2$, Mess provided a parametrization of the space of maximal globally hyperbolic flat structures on $S \times \mathbb{R}$, by means of the holonomy representation. In fact he showed that the linear part of the holonomy is a discrete and faithful representation $\rho : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^2)$, where $\mathrm{Isom}(\mathbb{H}^2)$ is the group of orientation-preserving isometries of hyperbolic plane, and is isometric to $\mathrm{SO}_0(2,1)$, the connected component of the identity of $SO(2,1)$. This provides an element $X_\rho = [\mathbb{H}^2/\rho]$ of Teichmüller space $\mathcal{T}(S)$.

On the other hand, the translation part is a cocycle $t \in H^1_\rho(\pi_1(S), \mathbb{R}^{2,1})$. Using the $\mathrm{SO}_0(2,1)$-equivariant identification between $\mathbb{R}^{2,1}$ and $\mathfrak{so}(2,1)$ given by the Lorentzian cross product, $t$ can be directly regarded as an element of the cohomology group $H^1_{Ad\circ\rho}(\pi_1(S), \mathfrak{so}(2,1))$ which is canonically identified to the tangent space $T_\rho(\mathcal{R}(\pi_1(S), \mathrm{SO}_0(2,1))/\!\!/\mathrm{SO}_0(2,1))$ of the character variety, [Gol84].

By a celebrated result of Goldman (see for instance [Gol80]) the map which gives the holonomy of the uniformized surface

$$\mathbf{hol} : \mathcal{T}(S) \to \mathcal{R}(\pi_1(S), \mathrm{SO}_0(2,1))/\!\!/\mathrm{SO}_0(2,1)\,,$$

is a diffeomorphism of $\mathcal{T}(S)$ over a connected component of the character variety $\mathcal{R}(\pi_1(S), \mathrm{SO}_0(2,1))/\!\!/\mathrm{SO}_0(2,1)$. Through this map we identify $H^1_{Ad\circ\rho}(\pi_1(S), \mathfrak{so}(2,1))$ and $T_{X_\rho}\mathcal{T}(S)$, and consider $t$ as a tangent vector of Teichmüller space.

One of the results we present is an explicit relation between the embedding data $(I, B)$ of any Cauchy surface $S$ in a maximal globally hyperbolic flat spacetime $M$ and the holonomy of $M$, which provides the correspondence with the parametrization of Mess. We will work under the assumption that $S$ inherits from $M$ a spacelike metric $I$ of negative curvature, which, by the Gauss equation for spacelike surfaces in Minkowski space, is equivalent to the fact that the shape operator $B$ has positive determinant and corresponds to a local convexity of $S$. This assumption will permit the following convenient change of variables. Instead of the pair $(I, B)$, one can in fact consider the pair $(h, b)$, where $h$ is the third fundamental form $h = I(B\cdot, B\cdot)$ and $b = B^{-1}$. The fact that $(I, B)$ solves Gauss-Codazzi equations corresponds to the conditions that $h$ is a hyperbolic metric and $b$ is a self-adjoint solution of Codazzi equation for $h$.

It is simple to check that the holonomy of the hyperbolic surface $(S, h)$ is the linear part of the holonomy of $M$, so the isotopy class of $h$ does not depend on the choice of a Cauchy surface in $M$ and corresponds to the element $X_\rho$ of Mess parameterization. Recovering the translation part of the holonomy of $M$ in terms of $(h, b)$ is subtler, and is based on the fact that $b$ solves the Codazzi equation for the hyperbolic metric $h$. Oliker and Simon in [OS83] proved that any $h$-self-adjoint operator on the hyperbolic surface $(S, h)$ which solves the Codazzi equation can be locally expressed as $\operatorname{Hess} u - u\,E$ for some smooth function $u$. Using this result we construct a short sequence of sheaves

$$0 \to \mathcal{F} \to \mathrm{C}^\infty \to \mathcal{C} \to 0\,, \tag{2}$$

where $\mathcal{C}$ is the sheaf of self-adjoint Codazzi operators on $S$ and $\mathcal{F}$ is the sheaf of flat sections of the $\mathbb{R}^{2,1}$-valued flat bundle associated to the holonomy representation of $h$. Passing to cohomology, this gives a connecting homomorphism

$$\delta : \mathcal{C}(S, h) \to H^1(S, \mathcal{F})\,.$$

It is a standard fact that $H^1(S, \mathcal{F})$ is canonically identified with $H^1_{\mathrm{hol}}(\pi_1(S), \mathbb{R}^{2,1})$. Under this identification we prove the following result.

**Theorem 5.A.** *Let $M$ be a globally hyperbolic spacetime and $S$ be a uniformly convex Cauchy surface with embedding data $(I, B)$. Let $h$ be the third fundamental form of $S$ and $b = B^{-1}$. Then*

- *the linear holonomy of $M$ coincides with the holonomy of $h$;*
- *the translation part of the holonomy of $M$ coincides with $\delta b$.*

It should be remarked that the construction of the short exact sequence (5.1) and the proof of Theorem 5.A are carried out just by local computations, so they hold for any uniformly convex spacelike surface in any flat globally hyperbolic spacetime without any assumption on the compactness or completeness of the surface.

In the case where $S$ is closed, we also provide a 2-dimensional geometric interpretation of $\delta b$. This is based on the simple remark that $b$ can also be regarded as a first variation of the metric $h$. As any Riemannian metric determines a complex structure over $S$, $b$ determines an infinitesimal variation of the complex structure $X$ underlying the metric $h$, giving in this way an element $\Psi(b) \in T_{[X]}\mathcal{T}(S)$.

**Theorem 5.B.** *Let $h$ be a hyperbolic metric on a closed surface $S$, $X$ denote the complex structure underlying $h$ and $\mathcal{C}(S,h)$ be the space of self-adjoint $h$-Codazzi tensors. Then the following diagram is commutative*

$$\begin{tikzcd} \mathcal{C}(S,h) \arrow[r, "\Lambda\circ\delta"] \arrow[d, "\Psi"'] & H^1_{Ad\circ hol}(\pi_1(S), \mathfrak{so}(2,1)) \\ T_{[X]}\mathcal{T}(S) \arrow[r, "\mathcal{J}"'] & T_{[X]}\mathcal{T}(S) \arrow[u, "d\mathbf{hol}"] \end{tikzcd} \tag{3}$$

*where $\Lambda : H^1_{hol}(\pi_1(S), \mathbb{R}^{2,1}) \to H^1_{Ad\circ hol}(\pi_1(S), \mathfrak{so}(2,1))$ is the natural isomorphism, and $\mathcal{J}$ is the complex structure on $\mathcal{T}(S)$.*

As a consequence we get the following corollary.

**Corollary 5.C.** *Two embedding data $(I,B)$ and $(I',B')$ correspond to Cauchy surfaces contained in the same spacetime if and only if*

- *the third fundamental forms $h$ and $h'$ are isotopic;*
- *the infinitesimal variation of $h$ induced by $b = B^{-1}$ is Teichmüller equivalent to the infinitesimal variation of $h'$ induced by $b' = (B')^{-1}$.*

The key point to prove Theorem 5.B is to relate $\delta b$ to the first-order variation of the holonomy of the family of hyperbolic metrics $h_t$ obtained by uniformizing the metrics $h((E+tb)\cdot,(E+tb)\cdot)$. This computation can be made explicit in the case $b = b_q$ is a harmonic Codazzi tensor, in which case $b_q$ is the first variation of a family of hyperbolic metrics. It turns out that the variation of the holonomy for $\Psi(b_q)$ is $\Lambda\delta(b_{iq})$.

However in general $b$ is not tangent to a deformation of $h$ through hyperbolic metrics, thus the proof uses the decomposition $b = b_q + \operatorname{Hess} u - u\,E$. Heuristically $u\,E$ is a conformal variation, whereas $\operatorname{Hess} u$ is trivial in the sense that correspond to the Lie derivative of the metric through the gradient field $\operatorname{grad} u$. So one has $\Psi(b) = \Psi(b_q)$ and the commutativity of the diagram (5.2) follows by the computation on harmonic differentials.

In the last part of Chapter 5, we apply the commutativity of the diagram (5.2) to hyperbolic geometry to obtain a new proof of the fact, proven by Goldman in [Gol84], that the Weil-Petersson symplectic form on $\mathcal{T}(S)$ coincides up to a factor with the Goldman pairing on the character variety through the map **hol**. In this proof, by explicit computations we show that the pull-back of those forms through the maps $\Lambda \circ \delta$ and $\Psi$ coincide (up to a factor) on $\mathcal{C}(S, h)$. The computation of the Weil-Petersson metric is quite similar to that obtained by Fischer and Tromba [FT84b] and the result is completely analogous. The computation of the Goldman pairing follows in a simple way using a different characterization of self-adjoint Codazzi tensors. The inclusion of $\mathbb{H}^2 \to \mathbb{R}^{2,1}$ projects to a section $\iota$ of the flat $\mathbb{R}^{2,1}$-bundle $F$ associated with $\mathrm{hol} : \pi_1(S) \to \mathrm{SO}_0(2, 1)$. The differential of this map provides an inclusion $\iota_*$ of $TS$ into $F$ corresponding to the standard inclusion of $T\mathbb{H}^2$ into $\mathbb{R}^{2,1}$. Thus any operator $b$ on $TS$ corresponds to an $F$-valued one-form $\iota_* b$; moreover $b$ is Codazzi and self-adjoint for $h$ if and only if the form $\iota_* b$ is closed. From this point of view the connecting homomorphism $\delta : \mathcal{C}(S, h) \to H^1(S, \mathcal{F})$ associated to the short exact sequence in (5.1) can be expressed as $\delta(b) = [\iota_* b]$, where we are implicitly using the canonical identification between $H^1(S, \mathcal{F})$ and the de Rham cohomology group $H^1_{dR}(S, F)$. The fact that the Goldman pairing coincides with the cup product in the de Rham cohomology proves immediately the coincidence of the two forms.

## Flat spacetimes with massive particles

We then apply this machinery to study globally hyperbolic spacetimes containing particles. Particles in a Lorentzian manifold of constant curvature are cone singularities along timelike lines with angle in $(0, 2\pi)$; we will focus here on the case of flat metrics. In order to develop a reasonable study of Cauchy surfaces in a spacetime with particles, some assumption are needed about the behavior of the surface around a particle. Here the assumption we consider is very weak: we only assume that the shape operator of the surface is bounded and uniformly positive (meaning that the principal curvatures are uniformly far from 0 and $+\infty$). We will briefly say that the Cauchy surface is bounded and uniformly convex.

Under this assumption we prove that the surface is necessarily orthogonal to the singular locus and intrinsically carries a Riemannian metric with cone angles equal to the cone singularities of the particle - using the definition given by Troyanov [Tro91] of metrics with cone angles on a surface with variable curvature. It turns out that the third fundamental form of such a surface is a hyperbolic metric with the same cone angles and $b = B^{-1}$ is a bounded and uniformly positive Codazzi operator for $(S, h)$. More precisely we prove the following statement.

**Theorem 6.A.** *Let us fix a* divisor $\boldsymbol{\beta} = \sum \beta_i p_i$ *on a surface with* $\beta_i \in (-1, 0)$ *and consider the following sets:*

- $\mathbb{E}_{\boldsymbol{\beta}}$ *is the set of embedding data* $(I, B)$ *of bounded and uniformly convex Cauchy surfaces on flat spacetimes with particles so that for each* $i = 1, \ldots, k$ *a particle of angle* $2\pi(1 + \beta_i)$ *passes through* $p_i$*.*
- $\mathbb{D}_{\boldsymbol{\beta}}$ *is the set of pairs* $(h, b)$*, where* $h$ *is a hyperbolic metric on* $S$ *with a cone*

*singularity of angle* $2\pi(1+\beta_i)$ *at each* $p_i$ *and* $b$ *is a self-adjoint solution of Codazzi equation for* $h$*, bounded and uniformly positive.*

*Then the correspondence* $(I, B) \to (h, b)$ *induces a bijection between* $\mathbb{E}_{\boldsymbol{\beta}}$ *and* $\mathbb{D}_{\boldsymbol{\beta}}$*.*

We remark that, by Gauss-Bonnet formula, in order to have $\mathbb{D}_{\boldsymbol{\beta}}$ (and consequently $\mathbb{E}_{\boldsymbol{\beta}}$) nonempty one has to require that $\chi(S, \boldsymbol{\beta}) := \chi(S) + \sum \beta_i$ is negative.

The difficult part of the proof is to show that for $(h, b) \in \mathbb{E}_{\boldsymbol{\beta}}$, the corresponding pair $(I, B)$ is the embedding data of some Cauchy surface in a flat spacetime with particles. Clearly the regular part of $S$ can be realized as a Cauchy surface in some flat spacetime $M$; it remains to prove that $M$ can be embedded in a spacetime with particles. Reducing to a local analysis, we first prove that a neighborhood of a puncture $p_i$ can be realized as a surface in a small flat cylinder with a particle, and then use some standard cut-and-paste procedure to construct the thickening of $M$.

As a by-product, the above construction allows to prove the following result concerning Riemannian metrics with cone points, which might have an interest on its own:

**Theorem 6.B.** *Let* $h$ *be a hyperbolic metric with cone singularities and let* $b$ *be a Codazzi, self-adjoint operator for* $h$*, bounded and uniformly positive. Then* $I = h(b\cdot, b\cdot)$ *defines a singular metric with the same cone angles as* $h$*.*

Another goal of this part is to show the analogue of Theorem 5.B in the context of cone singularities, proving that the relevant moduli space is the tangent bundle of the Teichmüller space of the punctured surface, and in particular it is independent of the cone angles.

To give a precise statement we use the Troyanov uniformization result [Tro91] which ensures that, given a conformal structure on $S$, there is a unique conformal hyperbolic metric with prescribed cone angles at the points $p_i$ (notice we are assuming $\chi(S, \boldsymbol{\beta}) < 0$). So once the divisor $\boldsymbol{\beta}$ is chosen we have a holonomy map

$$\mathbf{hol} : \mathcal{T}(S, \mathfrak{p}) \to \mathcal{R}(\pi_1(S \setminus \mathfrak{p}), \mathrm{SO}_0(2,1)) /\!\!/ \mathrm{SO}_0(2,1)\,,$$

where $\mathfrak{p} = \{p_1, \dots, p_k\}$ is the support of $\boldsymbol{\beta}$, and $\mathcal{T}(S, \mathfrak{p})$ is the Teichmüller space of the punctured surface.

As in the closed case fix a hyperbolic metric $h$ on $S$ with cone angles $2\pi(1+\beta_i)$ at $p_i$. Let $X$ denote the complex structure underlying $h$. Any Codazzi operator $b$ on $(S, h)$ can be regarded as an infinitesimal deformation of the metric on the regular part of $S$. If $b$ is bounded this deformation is quasiconformal so it extends to an infinitesimal deformation of the underlying conformal structure at the punctures, providing an element $\Psi(b)$ in $T_{[X]}\mathcal{T}(S, \mathfrak{p})$.

**Theorem 6.C.** *Let* $\mathcal{C}_\infty(S, h)$ *be the space of bounded Codazzi tensors on* $(S, h)$*. The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{C}_\infty(S,h) & \xrightarrow{\Lambda\circ\delta} & H^1_{Ad\circ hol}(\pi_1(S\setminus\mathfrak{p}), \mathfrak{so}(2,1)) \\ {\scriptstyle\Psi}\big\downarrow & & d\mathbf{hol}\big\uparrow \\ T_{[X]}\mathcal{T}(S,\mathfrak{p}) & \xrightarrow[\mathcal{J}]{} & T_{[X]}\mathcal{T}(S,\mathfrak{p}) \end{array}\,, \tag{4}$$

*where* $\Lambda : H^1_{hol}(\pi_1(S \setminus \mathfrak{p}), \mathbb{R}^{2,1}) \to H^1_{Ad \circ hol}(\pi_1(S \setminus \mathfrak{p}), \mathfrak{so}(2,1))$ *is the natural isomorphism, and* $\mathcal{J}$ *is the complex structure on* $\mathcal{T}(S, \mathfrak{p})$.

In order to repeat the argument used in the closed case, we show that also in this context bounded Codazzi tensors can be split as the sum of a trivial part and a harmonic part. More precisely we prove that any square-integrable Codazzi tensor on a surfaces with cone angles in $(0, 2\pi)$ can be expressed as the sum of a trivial Codazzi tensor and a Codazzi tensor corresponding to a holomorphic quadratic differential with at worst simple poles at the punctures. As a consequence we have the following corollary.

**Corollary 6.D.** *Two embedding data* $(I, B)$ *and* $(I', B')$ *in* $\mathbb{E}_{\boldsymbol{\beta}}$ *correspond to Cauchy surfaces contained in the same spacetime with particles if and only if*

- *the third fundamental forms* $h$ *and* $h'$ *are isotopic;*
- *the infinitesimal variation of* $h$ *induced by* $b = B^{-1}$ *is Teichmüller equivalent to the infinitesimal variation of* $h'$ *induced by* $b' = (B')^{-1}$.

It should be remarked that in this context, at least if the cone angles are in $[\pi, 2\pi)$, the holonomy does not distinguish the structures, so this corollary is not a direct consequence of Theorem 6.C, but an additional argument is required.

It is a natural question to ask whether the condition of containing a uniformly convex surface is restrictive. To point out some counterexamples, it is sufficient to double a cylinder in Minkowski space based on some polygon on $\mathbb{R}^2$. However the spacetimes obtained in this way have the property that Euler characteristic $\chi(S, \boldsymbol{\beta})$ of its Cauchy surfaces is 0. In the last section we provide some more elaborate counterexamples, with negative Euler characteristic, producing spacetimes with particles which do not contain strictly convex Cauchy surfaces. The construction is based on some simple surgery idea. In all those exotic examples at least one particle must have cone angle in $[\pi, 2\pi)$, and there exists a convex Cauchy surface, although not strictly convex. Similar problems regarding the existence of spacetimes with certain properties on the Cauchy surfaces have been tackled in [BG00].

To conclude Part II, we finally address the question of the coincidence of the Weil-Petersson metric and the Goldman pairing in this context of structures with cone singularities. Once a divisor $\boldsymbol{\beta}$ is fixed, the hyperbolic metrics with prescribed cone angles allow to determine a Weil-Petersson product on $\mathcal{T}(S, \mathfrak{p})$, as it has been studied in [ST11]. In [Mon10], Mondello showed that also in this singular case the Weil-Petersson product corresponds to an intersection form on the subspace of $H^1_{\mathrm{Ad} \circ \mathrm{hol}}(\pi_1(S \setminus \mathfrak{p}), \mathfrak{so}(2,1))$ corresponding to cocycles trivial around the punctures. Actually Mondello's proof is based on a careful generalization of Goldman argument in the case with singularity. Like in the closed case, we give a substantially different proof of this coincidence by using the commutativity of (6.1).

## Minimal surfaces in hyperbolic space

In Part III of the thesis, the main object of study are surfaces in hyperbolic space and Anti-de Sitter space which are "close" to being a totally geodesic plane. An important focus will be on surfaces of zero mean curvature. The meaning of how far a surface is from being totally geodesic is provided by the asymptotic curve in the boundary at infinity, in relation with the theory of universal Teichmüller space. We start by considering the case of minimal surfaces in hyperbolic space. Recall that a surface embedded in a Riemannian manifold is minimal if its mean curvature vanishes, i.e. the principal curvatures at every point $x$ have opposite values $\lambda = \lambda(x)$ and $-\lambda$.

It was proved by Anderson ([And83, Theorem 4.1]) that for every Jordan curve $\Gamma$ in $\partial_\infty \mathbb{H}^3$ there exists a minimal embedded disc $S$ such that its boundary at infinity coincides with $\Gamma$. It can be proved that if the supremum $||\lambda||_\infty$ of the principal curvatures of $S$ is in $(-1, 1)$, then $\Gamma = \partial_\infty S$ is a quasicircle, i.e. is the image of a round circle under a quasiconformal homeomorphism of $\partial_\infty \mathbb{H}^3$.

However, uniqueness does not hold in general. Anderson proved the existence of a curve at infinity $\Gamma$ invariant under the action of a quasi-Fuchsian group (hence a quasicircle) spanning several distinct minimal embedded discs, see [And83, Theorem 5.3]. More recently in [HW13a] invariant curves spanning an arbitrarily large number of minimal discs were constructed. On the other hand, if the supremum of the principal curvatures of a minimal embedded disc $S$ satisfies $||\lambda||_\infty \in (-1, 1)$ then, by an application of the maximum principle, $S$ is the unique minimal disc asymptotic to the quasicircle $\Gamma = \partial_\infty S$.

By the classical work of Ahlfors and Bers, the space of quasicircles up to Möbius transformations can be identified to universal Teichmüller space $\mathcal{T}(\mathbb{D})$. The main purpose here is to study the supremum $||\lambda||_\infty$ of the principal curvatures of a minimal embedded disc, in relation with the norm of the quasicircle at infinity, in the sense of universal Teichmüller space. The relations we obtain are interesting for "small" quasicircles, that are close in $\mathcal{T}(\mathbb{D})$ to a round circle. The "size" of a quasicircle $\Gamma$ can be measured in different ways: the classical Bers norm is one possibility, which uses the identification of $\mathcal{T}(\mathbb{D})$ with an open subset in the space of bounded holomorphic quadratic differentials; another possibility is to take the optimal constant $K$ such that the $\Gamma$ is the image of a $K$-quasiconformal mapping. In the latter case, $\Gamma$ is called a $K$-quasicircle.

**Theorem 7.A.** *There exist universal constants $K_0 > 1$ and $C$ such that every minimal embedded disc in $\mathbb{H}^3$ with boundary at infinity a $K$-quasicircle $\Gamma \subset \partial_\infty \mathbb{H}^3$, with $1 \leq K \leq K_0$, has principal curvatures bounded by*

$$||\lambda||_\infty \leq C \log K\,.$$

There are two direct consequences of Theorem 7.A. The first is the following corollary:

**Corollary 7.B.** *There exists a universal constant $K_0'$ such that every $K$-quasicircle $\Gamma \subset \partial_\infty \mathbb{H}^3$ with $K \leq K_0'$ is the boundary at infinity of a unique minimal embedded disc.*

Corollary 7.B is obtained by choosing $K_0' < \min\{K_0, e^{1/C}\}$ and recalling that the minimal disc with prescribed quasicircle at infinity is unique if $||\lambda||_\infty < 1$.

A second application concerns almost-Fuchsian manifolds, namely quasi-Fuchsian manifold contaning a closed minimal surface with principal curvatures in $(-1,1)$ (according to the definition given in [KS07]). The minimal surface in an almost-Fuchsian manifold is unique, by the above discussion, as first observed by Uhlenbeck ([Uhl83]). As an application of Theorem 7.A, the following corollary is proved.

**Corollary 7.C.** *If the Teichmüller distance between the conformal metrics at infinity of a quasi-Fuchsian manifold $M$ is smaller than a universal constant $d_0$, then $M$ is almost-Fuchsian.*

We remark that Theorem 7.A, when restricted to the case of quasi-Fuchsian manifolds, is a partial converse of results presented in [GHW10], giving a bound on the Teichmüller distance between the hyperbolic ends of an almost-Fuchsian manifold in terms of the maximum of the principal curvatures.

The proof of Theorem 7.A is composed of several steps. By means of the technique of "description from infinity" (see [Eps84] and [KS08]), we construct a foliation of $\mathbb{H}^3$ by equidistant surfaces, such that all the leaves of the foliation have the same boundary at infinity, a quasicircle $\Gamma$. Using a theorem proved in [ZT87] and [KS08, Appendix], which relates the curvatures of the leaves of the foliation with the Schwarzian derivative of the map which uniformizes the conformal structure of one component of $\partial_\infty\mathbb{H}^3 \setminus \Gamma$, we obtain an explicit bound for the distance between two surfaces of this foliation, one concave and one convex, in terms of the Bers norm of $\Gamma$. This distance goes to 0 when $\Gamma$ approaches a circle in $\partial_\infty\mathbb{H}^3$.

A fundamental property of a minimal surface $S$ with boundary at infinity a curve $\Gamma$ is that $S$ is contained in the convex hull of $\Gamma$. Hence, by the previous step, every point $x$ of $S$ lies on a geodesic segment orthogonal to two planes $P_-$ and $P_+$ such that $S$ is contained in the region bounded by $P_-$ and $P_+$. The length of such geodesic segment is bounded by the Bers norm of the quasicircle at infinity, in a way which does not depend on the chosen point $x \in S$.

The next step in the proof is then a Schauder-type estimate. Considering the function $u$, defined on $S$, which is the hyperbolic sine of the distance from the plane $P_-$, it turns out that $u$ solves the equation

$$\Delta_S u - 2u = 0\,. \tag{L}$$

where $\Delta_S$ is the Laplace-Beltrami operator of $S$. We then apply classical theory of linear PDEs, in particular Schauder estimates, to prove that

$$||u||_{C^2(\Omega')} \leq C||u||_{C^0(\Omega)}\,,$$

where $\Omega' \subset\subset \Omega$ and $u$ is expressed in normal coordinates centered at $x$.

The final step is then estimating the principal curvatures at $x \in S$, by observing that the shape operator can be expressed in terms of $u$ and the first and second derivatives of $u$. The Schauder estimate above then gives a bound on the principal curvatures just in terms of the supremum of $u$ in a geodesic ball of fixed radius centered at $x$. By using the first step, since $S$ is contained between $P_-$ and the nearby plane $P_+$, we finally get an estimate of the principal curvatures of a minimal embedded disc in terms of the Bers norm of the quasicircle at infinity.

All the previous estimates do not depend on the choice of $x \in S$. Hence the following theorem is actually proved.

**Theorem 7.D.** *There exist constants $K_0 > 1$ and $C > 4$ such that the principal curvatures $\pm\lambda$ of every minimal surface $S$ in $\mathbb{H}^3$ with $\partial_\infty S = \Gamma$ a $K$-quasicircle, with $K \leq K_0$, are bounded by:*

$$||\lambda||_\infty \leq \frac{C||\Psi||_{\mathcal{B}}}{\sqrt{1 - C||\Psi||^2_{\mathcal{B}}}}\,, \tag{5}$$

*where $\Gamma = \Psi(S^1)$, $\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal map, conformal on $\widehat{\mathbb{C}} \setminus \mathbb{D}$, and $||\Psi||_{\mathcal{B}}$ denotes its Bers norm.*

Observe that the estimate holds in a neighborhood of the identity (which represents circles in $\partial_\infty \mathbb{H}^3$), in the sense of universal Teichmüller space. Theorem 7.A is then a consequence of Theorem 7.D, using the well-known fact that the Bers embedding is locally bi-Lipschitz.

## Maximal surfaces in Anti-de Sitter space

We then move to an application of similar techniques to maximal surfaces in Anti-de Sitter space, which are the analogue of minimal surfaces in Riemannian manifolds. The object of interest in the parametrization of maximal surfaces is here again universal Teichmüller space $\mathcal{T}(\mathbb{D})$, which can be defined also as the space of quasisymmetric homeomorphisms of the circle up to Möbius transformations.

The strong relation between these two objects was pointed out in [BS10], where the authors tackled the classical problem of the existence of quasiconformal extensions to the disc of quasisymmetric homeomorphisms of the circle. Based on a construction of Krasnov and Schlenker, the proof is translated in terms of existence and uniqueness of a maximal disc $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with prescribed boundary at infinity. For a spacelike surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, the asymptotic boundary is regarded as the graph of an orientation-preserving homeomorphism $\phi : S^1 \to S^1$, using the structure of doubly ruled quadric of the boundary at infinity. The quasiconformality of the extension of $\phi$ is then directly related to the fact that the principal curvatures of $S$ are (in absolute value) uniformly smaller than 1.

The main result in the Anti-de Sitter context is an estimate of the maximal dilatation of the minimal Lagrangian extension of a quasisymmetric homemomrphism $\phi$ of the circle (whose existence was proved by Bonsante and Schlenker), in terms of the cross-ratio norm $||\phi||_{cr}$ of $\phi$. The latter measures the distortion of quadruple of points, and vanishes on Möbius transformations, this providing a norm on universal Teichmüller space $\mathcal{T}(\mathbb{D})$.

**Theorem 8.A.** *There exist universal constants $\delta$ and $C$ such that, for any quasisymmetric homeomorphism $\phi$ of $S^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian quasiconformal extension $\Phi_{ML} : \mathbb{D} \to \mathbb{D}$ has maximal dilatation $K(\Phi_{ML})$ bounded by the relation*

$$\log K(\Phi_{ML}) \leq C||\phi||_{cr}\,.$$

The maximal dilatation of classical quasiconformal extensions of a quasisymmetric $\phi : S^1 \to S^1$ has been widely studied, and Theorem 8.A is similar to estimates obtained for instance for the Beurling-Ahlfors extension in [BA56] (then improved in [Leh83]) and for the Douady-Earle extension in [DE86] (see also [HM12] for further developments).

The proof widely uses the geometry of Anti-de Sitter space and is composed again of several steps. As pointed out already in [BS10], a very relevant quantity in this problem is the width of the convex hull of $S$, which is defined as the supremum of the length of timelike paths contained in the convex hull. Indeed, if $S$ is a maximal surface with boundary at infinity the graph of $\phi : S^1 \to S^1$, the convex hull of $S$ has width $\leq \pi/2$, and it turns out that $\phi$ is quasisymmetric if and only if the width of the convex hull of $S$ is strictly smaller than $\pi/2$.

The first step of the proof is a more quantitative statement in this direction.

**Proposition 8.B.** *Given any quasisymmetric homeomorphism $\phi$, let $w$ be the width of the convex hull of the graph of $\phi$ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$. Then*

$$\tanh\left(\frac{||\phi||_{cr}}{4}\right) \leq \tan(w) \leq \sinh\left(\frac{||\phi||_{cr}}{2}\right).$$

In particular, the second inequality is used in the proof of Theorem 8.A. On the other hand, the first inequality (that will be used to prove an inequality in the opposite direction of Theorem 8.A, see Theorem 8.D below) is not interesting when $w$ is larger than $\pi/4$.

The second part of the proof of Theorem 8.A involves - as in the hyperbolic case - Schauder estimates. Indeed the function $u$, now defined as the sine of the distance from a support plane $P_-$ of the lower boundary of the convex hull of $gr(\phi)$, satisfies again the equation

$$\Delta_S u - 2u = 0\,. \tag{L}$$

We will use again an explicit expression for the shape operator of the maximal surface $S$ in terms of the value of $u$, the first derivatives of $u$, and the second derivatives of $u$. Hence, using the Schauder-type estimate, the principal curvatures are bounded in terms of the supremum of $u$ on a geodesic ball $B_S(x, R)$. The latter is finally bounded in terms of the width. The sketched construction will not depend on the choice of the point $x \in S$, and thus will prove:

**Theorem 8.C.** *There exists a constant $C$ such that, for every maximal surface $S$ with bounded principal curvatures $\pm\lambda$ and width $w = w(\mathcal{CH}(\partial_\infty S))$,*

$$||\lambda||_\infty \leq C \tan w\,.$$

The differential of the minimal Lagrangian extension of $\phi$ can be expressed (as noted in [BS10] and [KS07]) in terms of the shape operator of $S$. Using this relation, together with Proposition 8.B and Theorem 8.C, the maximal dilatation of $\Phi$ is finally estimated. We actually obtain - as in the hyperbolic case - a more precise estimate, from which Theorem 8.A follows. This is stated in Theorem 8.E.

We also obtain an estimate in the other direction, namely, a bound from below of the quasiconformal distortion of the minimal Lagrangian extension of a quasisymmetric homeomorphism, in terms of the cross-ratio norm of the latter. This is stated in full generality in Theorem 8.F at the end of Chapter 8. A consequence is:

**Theorem 8.D.** *There exist universal constants $\delta$ and $C_0$ such that, for any quasisymmetric homeomorphism $\phi$ of $S^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian quasiconformal extension $\Phi : \mathbb{D} \to \mathbb{D}$ has maximal dilatation $K(\Phi_{ML})$ bounded by the relation*

$$C_0||\phi||_{cr} \leq \log K(\Phi_{ML}) \, .$$

*The constant $C_0$ can be taken arbitrarily close to* $1/2$.

Although investigation of the best value of the constant $C$ in Theorem 8.A was not pursued in this work, this shows that $C$ cannot be taken smaller than $1/2$.

## Geometric transition of surfaces

Finally, we will discuss in more detail how the above results are related to one another through the idea of geometric transition. In particular, Theorem 4.B provided a characterization of the support function at infinity of convex surfaces in Minkowski space with principal curvatures bounded from below by a positive constant. The regularity of those support functions is the Zygmund regularity, namely, the infinitesimal version of quasisymmetric homeomorphism.

Heuristically, this statement is the infinitesimal version of the following statement in Anti-de Sitter space: an orientation-preserving homeomorphism of the circle $\phi : S^1 \to S^1$ is quasisymmetric if and only if its graph is the asymptotic boundary in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ of a convex surfaces with bounded principal curvatures.

Indeed, Minkowski space is the zoom-in limit of Anti-de Sitter space, and Zygmund fields are tangent vectors to smooth paths (for the smooth structure of universal Teichmüller space) of quasisymmetric homeomorphisms.

In the last part of the thesis a precise meaning will be given to this heuristic idea. The support function at infinity $u|_{\partial\mathbb{D}} = \varphi$ of a convex entire graph $S$ in $\mathbb{R}^{2,1}$ is identified, by means of the duality with half-pipe geometry $\mathbb{HP}^3$, to the boundary at infinity of the convex surface in $\mathbb{HP}^3$ dual to $S$. We prove that, if $\phi_t$ is a smooth path of quasisymmetric homeomorphisms, such that $\phi_0 = \mathrm{id}$ and the vector field $\dot\phi$ is identified to the function $\varphi : S^1 \to \mathbb{R}$, then the graph of $\phi_t$ in $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$ converges (up to a factor) to the graph of $\varphi$ in $\partial_\infty\mathbb{HP}^3$, under the transition to half-pipe geometry obtained by a blow-up close to a totally geodesic plane.

On the other hand, we show that it is possible to have control also on the degeneration of curvature. We introduce a natural connection on $\mathbb{HP}^3$ which enables to define the second fundamental form of a surface in $\mathbb{HP}^3$. Now let $S_t$ be a smooth family of surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, with $S_0$ a totally geodesic plane. The aforementioned blow-up transition provides a rescaled limit of the surfaces $S_t$ for $t \to 0$, which is a surface $S$ in $\mathbb{HP}^3$. By these definitions, the shape operator of the rescaled limit $S$, is precisely the first-order variation at $t = 0$ of the shape operator of $S_t$.

Finally, the self-duality of Anti-de Sitter space is well-behaved for the transitions to $\mathbb{HP}^3$ (by blowing-up a plane) and to $\mathbb{R}^{2,1}$ (by blowing-up the dual point). Putting these ingredients together, we obtain the following, which is the last result proved in the thesis.

**Proposition 9.A.** *Let $\sigma_t : \mathbb{H}^2 \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ be a $C^2$ family of smooth embeddings with image surface $S_t = \sigma_t(\mathbb{H}^2)$. Suppose the boundary at infinity of $S_t$ is the graph of the quasisymmetric homeomorphism $\phi_t : S^1 \to S^1$, satisfying the following:*

- *For $t = 0$, $\sigma_t$ is an isometric embedding of the totally geodesic plane $\{x^3 = 0\}$;*
- *The principal curvatures of $S_t$ are $\lambda_i(x) = O(t)$, for $i = 1, 2$, i.e. are uniformly bounded by $Ct$, for small $t$, independently of the point $x$;*
- *The path $\phi_t$ is tangent at $\phi_0 = \mathrm{id}$ to a Zygmund field $\dot{\phi}$ on $S^1$.*

*Then the rescaled limit in $\mathbb{R}^{2,1}$ of the surfaces $S_t^*$ dual to $S_t$ is a spacelike entire graph in $\mathbb{R}^{2,1}$, with principal curvatures bounded from below by a positive constant and with support function at infinity the function $\varphi$ (in the Zygmund class) which corresponds to $\dot{\phi}/2$ under the standard trivialization of $TS^1$.*

This statement should make precise the idea that the geometry of surfaces in Minkowski space is intimately related to the tangent space of (universal) Teichmüller space, in a way which reflects at first order the connections of Anti-de Sitter space (or hyperbolic space) with the theory of Teichmüller spaces.

# Outline of the thesis

The thesis is organized as follows. Part I contains the preliminary notions which will be used in the original contents. In particular, Chapter 1 introduces the geometric three-dimensional models which will be treated, namely Minkowski, hyperbolic, de Sitter and Anti de Sitter geometries.

Chapter 2 is a review of the theory of Teichmüller spaces of Riemann surfaces. The cases of closed surfaces, punctured surfaces and the disc are mentioned.

Chapter 3 is meant to be a brief collection of known results on some partial differential equations of interest in the thesis, in particular some type of linear elliptic equations and Monge-Ampère equations.

In Part II are collected the results concerning Minkowski geometry. Chapter 4 deals with convex surfaces in Minkowski space, with special interest towards surfaces of constant curvature, the Minkowski problem, and the boundedness of curvature in relation with universal Teichmüller space. The material of this chapter can be found in:

[BS15b] Francesco Bonsante and Andrea Seppi. Spacelike convex surfaces with prescribed curvature in (2+1)-Minkowski space. *ArXiv: 1505.06748v1*, 2015.

Chapter 5 is mostly concerned about maximal globally hyperbolic flat spacetimes contaning a closed Cauchy surface, while Chapter 6 concerns the case of spacetimes with particles. The material of these two chapters is contained in:

[BS15a] Francesco Bonsante and Andrea Seppi. On Codazzi Tensors on a Hyperbolic Surface and Flat Lorentzian Geometry. *To appear on International Mathematics Research Notices*, 2015. ArXiv: 1501.04922.

Part III deals with three-manifolds of negative constant curvature. Chapter 7 is focused on minimal surfaces in hyperbolic space, while the purpose of Chapter 8 is to treat maximal surfaces in Anti-de Sitter space and quasiconformal extensions of quasisymmetric homeomorphisms. The content of Chapter 7 and 8 is essentially in:

[Sep14] Andrea Seppi. Minimal surfaces in Hyperbolic space and maximal surfaces in Anti-de Sitter space. *ArXiv: 1411.3412v1*, 2014.

Chapter 9 is concerned with geometric transitions in relation with the results of the previous chapters, and is unpublished at the time of writing this thesis.

# Part I

# Preliminaries

# Chapter 1

# The geometric models

The main setting of this thesis are three-manifolds endowed with Riemannian or Lorentzian metrics of constant curvature. In this chapter we give a description of the models we are most interested in, by studying several properties which will be of use. These will include the theory of embedding of surfaces, which will be discussed case-by-case, trying to highlight the common features and the remarkable differences. In each case, we will also consider some special classes of three-manifolds locally modelled on the constant curvature models, which contain a closed surface (spacelike, when the metric is Lorentzian).

## 1.1 Minkowski space and hyperbolic plane

The $(2+1)$-dimensional Minkowski space is the vector space $\mathbb{R}^3$ endowed with the bilinear quadratic form

$$\langle x, y\rangle_{2,1} = x^1y^1 + x^2y^2 - x^3y^3 \,. \tag{1.1}$$

It will be denoted by $\mathbb{R}^{2,1}$ in this thesis. The group $\mathrm{Isom}_0(\mathbb{R}^{2,1})$ of orientation-preserving and time-preserving isometries is isomorphic to

$$\mathrm{SO}_0(2,1) \rtimes \mathbb{R}^{2,1}$$

where $\mathrm{SO}(2,1)$ is the group of linear isometries of Minkowski product, $\mathrm{SO}_0(2,1)$ is the connected component of the identity, and $\mathbb{R}^{2,1}$ acts on itself by translations.

Vectors in $T_x\mathbb{R}^{2,1} \cong \mathbb{R}^{2,1}$ are classified according to their causal properties. In particular:

$$v \in T_x\mathbb{R}^{2,1} \text{ is } \begin{cases} \textit{timelike} & \text{if } \langle v, v\rangle_{2,1} < 0 \\ \textit{lightlike} & \text{if } \langle v, v\rangle_{2,1} = 0 \\ \textit{spacelike} & \text{if } \langle v, v\rangle_{2,1} > 0 \end{cases}$$

A vector $v$ is *causal* if it is either timelike or lightlike. By convention, such a $v$ is *future-directed* if the $x^3$-component is positive. The set of future-directed causal vectors at $x_0 \in \mathbb{R}^{2,1}$ is the *future cone* at $x_0$ and is denoted by $\mathrm{I}^+(x_0)$. Clearly $\mathrm{I}^+(x_0)$ is the $x_0$-translate of

$$\mathrm{I}^+(0) = \{x \in \mathbb{R}^{2,1} : (x^1)^2 + (x^2)^2 < (x^3)^2, x^3 > 0\} \,.$$

A plane $P$ is *spacelike* if its orthogonal vectors are timelike; an embedded differentiable surface $S$ in $\mathbb{R}^{2,1}$ is a *spacelike surface* if its tangent plane $T_xS$ is spacelike for every point $x \in S$. In this case, the symmetric 2-tensor induced on $S$ by the Minkowski product is a Riemannian metric.

*Example* 1.1.1. An example of spacelike embedded surface is the *hyperboloid*

$$\mathbb{H}^2 = \{x \in \mathbb{R}^{2,1} : \langle x, x\rangle_{2,1} = -1, x^3 > 0\},$$

which is the analogue of the sphere of radius 1 in $\mathbb{R}^3$. It is easy to see that the following parametrization in *polar coordinates* $r \in (0, \infty)$, $\theta \in [0, 2\pi]$,

$$(r, \theta) \mapsto (\sinh r \cos\theta, \sinh r \sin\theta, \cosh r)$$

gives the induced first fundamental form

$$dr^2 + (\sinh r)^2 d\theta^2$$

which is a complete Riemannian metric. The Riemannian manifold $\mathbb{H}^2$ endowed with this metric is called *hyperbolic plane*. It can be showed that in the hyperboloid model the geodesics are the intersections of $\mathbb{H}^2$ with planes of $\mathbb{R}^{2,1}$ through the origin (when this intersection is nonempty and contains more than one point).

Isometries of $\mathbb{H}^2$ are obtained as restrictions of linear isometries of the ambient $\mathbb{R}^{2,1}$. Hence there is a natural description of $\mathrm{Isom}(\mathbb{H}^2)$, the group of orientation-preserving isometries of $\mathbb{H}^2$, as

$$\mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{SO}_0(2, 1).$$

It is also useful to consider the *projective model* of hyperbolic plane, namely

$$\{x \in \mathbb{R}^{2,1} : \langle x, x\rangle_{2,1} < 0\}/\sim,$$

where $x \sim x'$ if there exists $\lambda$ such that $x = \lambda x'$. This is an open domain in the projective space $\mathbb{RP}^2$. Geodesics in the projective model are lines of $\mathbb{RP}^2$ which intersect this domain. Considering the affine chart $\{x^3 \neq 0\}$ and the affine coordinates $(x, y) = (x^1/x^3, x^2/x^3)$, one obtains the *Klein model*

$$\{(x, y) : x^2 + y^2 < 1\}.$$

We will always identify the Klein model to the open disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

In this model, geodesics of hyperbolic space coincide with straight lines in $\mathbb{C}$. Several other models of hyperbolic space (and its higher dimensional analogue) will be discussed later.

Analogously to the definition for spacelike planes, we say that a plane $P$ is *lightlike* if its orthogonal complement is composed of lightlike vectors. An example of *lightlike surface*, for which the tangent plane is lightlike at every point, is the following.

*Example* 1.1.2. The *null cone*

$$N = \partial \mathrm{I}^+(0) \setminus \{0\} = \{x \in \mathbb{R}^{2,1} : \langle x, x \rangle_{2,1} = 0, x^3 > 0\}$$

is a lightlike surface in $\mathbb{R}^{2,1}$. Indeed, by means of the parametrization

$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta, r)\,.$$

one can check that the induced bilinear form has the degenerate form $r^2 d\theta^2$.

The projectivization $\mathbb{P}N = N/\sim$, where again $x \sim x'$ if and only if $x = \lambda x'$ for some $\lambda > 0$, is the *boundary at infinity* $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$, meaning that every complete geodesic of $\mathbb{H}^2$ is asymptotic to two points in $\partial_\infty \mathbb{H}^2$. Conversely, every pair of distinct points in $\partial_\infty \mathbb{H}^2$ uniquely determines a geodesic. In the Klein model, $\partial_\infty \mathbb{H}^2$ is identified to $\partial \mathbb{D}$.

The boundary at infinity $\partial_\infty \mathbb{H}^2$ is naturally endowed with a structure of real projective line. Under this identification $\partial_\infty \mathbb{H}^2 \cong \mathbb{R}\mathrm{P}^1$, every isometry acts as a projective transformation, and conversely every projective transformation of $\mathbb{R}\mathrm{P}^1$ uniquely extends to an isometry of $\mathbb{H}^2$. This provides a natural isomorphism

$$\mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{PSL}_2\mathbb{R}\,.$$

Finally, the reader can guess that a timelike plane is such that the orthogonal complement is spacelike.

*Example* 1.1.3. An example of an embedded timelike surface is the double cover of de Sitter space, namely

$$\widehat{\mathrm{d}\mathbb{S}^2} = \{x \in \mathbb{R}^{2,1} : \langle x, x \rangle_{2,1} = +1\}\,.$$

Indeed, by the parametrization

$$(t, \theta) \mapsto (\cosh t \cos\theta, \cosh t \sin\theta, \sinh t)\,,$$

the induced metric on $\widehat{\mathrm{d}\mathbb{S}^2}$ is of the form

$$-dt^2 + (\cosh t)^2 d\theta^2\,.$$

It is now easy to see that $\widehat{\mathrm{d}\mathbb{S}^2}$ parametrizes oriented geodesics of $\mathbb{H}^2$. Indeed, for every point $x$ in $\widehat{\mathrm{d}\mathbb{S}^2}$, the orthogonal complement $x^\perp$ is a timelike plane in $\mathbb{R}^{2,1}$, which intersects $\mathbb{H}^2$ in a complete geodesic. The same geodesic is obtained as the intersection of $\mathbb{H}^2$ with $(-x)^\perp$. Using the orientation of $\mathbb{R}^{2,1}$, points $x \in \widehat{\mathrm{d}\mathbb{S}^2}$ determine the orientation of the geodesic by ordering its endpoints at infinity. Hence the quotient

$$\mathrm{d}\mathbb{S}^2 = \widehat{\mathrm{d}\mathbb{S}^2}/\pm 1$$

is naturally identified to the space of geodesics of $\mathbb{H}^2$.

### 1.1.1 The geometry of immersed surfaces in Minkowski space

In this section we discuss the theory of immersions of surfaces in Minkowski space, which is a straightforward adaptation of the classical theory for Euclidean space. First, let us remark that the Levi-Civita connection $\nabla^{\mathbb{R}^{2,1}}$ of the Lorentzian flat metric (1.1), in the $(x^1, x^2, x^3)$-coordinates, is simply

$$\nabla^{\mathbb{R}^{2,1}}_v w = Dw(v) ,$$

for any pair of smooth vector fields $v, w$. Now, given a smooth immersion $\sigma : S \to \mathbb{R}^{2,1}$ with image a spacelike surface $\sigma(S)$ in $\mathbb{R}^{2,1}$, recall that the *first fundamental form* is the pull-back of the induced metric, namely

$$I(v, w) = \langle \sigma_*(v), \sigma_*(w)\rangle_{2,1} .$$

The Levi-Civita connection $\nabla^S$ of the first fundamental form $I$ of $S$ is obtained from the Levi-Civita connection of Minkowski space: given vector fields $v, w$ on $S$, $\nabla^S_v w$ is the orthogonal projection to the tangent space of $S$ of $\nabla^{\mathbb{R}^{2,1}}_{\sigma_* v}(\sigma_* w)$.

Let us denote by $N$ the future unit normal vector field on $S$, namely for every point $x \in S$, $N_x$ is the future-directed timelike vector orthogonal to $T_{\sigma(x)}\sigma(S)$ with $\langle N_x, N_x\rangle_{2,1} = -1$. By means of the flat metric on $\mathbb{R}^{2,1}$, we can identify all the tangent spaces $T_x\mathbb{R}^{2,1} \cong \mathbb{R}^{2,1}$ in a natural way, and this enables to define the *Gauss map* $G : S \to \mathbb{H}^2$, with values in the hyperboloid:

$$G(x) = N_x .$$

The *second fundamental form* $I\!I$ is a bilinear form on $S$ defined by

$$\nabla^{\mathbb{R}^{2,1}}_{\sigma_* v}(\sigma_* \hat{w}) = \nabla^S_v \hat{w} + I\!I(v, w)N$$

where $\hat{w}$ denotes any smooth vector fields on $S$ extending the vector $w$. It turns out to be symmetric, thus showing that it only depends on the vectors $v$ and $w$, not on the extension of any of them. The *shape operator* of $S$ is the $(1, 1)$-tensor $B \in \mathrm{End}(TS)$ defined as

$$B(v) = \nabla^{\mathbb{R}^{2,1}}_v N . \tag{1.2}$$

The above equation is not quite correct, as by an abuse of notation we are identifying $T_xS$ to $T_{\sigma(x)}\sigma(S)$, by means of $\sigma_*$, which is an injective linear map. The more precise expression should be

$$B(v) = (\sigma_*)^{-1}\nabla^{\mathbb{R}^{2,1}}_{\sigma_* v} N .$$

Indeed, by applying the condition of compatibility of the metric of the Levi-Civita connection to the condition $\langle N, N\rangle_{2,1} = -1$, it is easily checked that $\nabla^{\mathbb{R}^{2,1}}_{\sigma_* v} N$ is orthogonal to $N$, hence is in $T_{\sigma(x)}\sigma(S)$. Therefore for every $v \in T_xS$, $\nabla^{\mathbb{R}^{2,1}}_{\sigma_* v} N$ can be identified to the vector $B(v)$ in $T_xS$ by means of $\sigma_*$. We will very often implicitly adopt the above abuse of notation.

It turns out that $B$ is self-adjoint with respect to $I$, namely

$$I(B(v), w) = I(v, B(w)) \tag{1.3}$$

Moreover the expression in (1.3) coincides with the second fundamental form $I\!I(v, w)$. Since $B$ is self-adjoint with respect to $I$, it is diagonalizable at every point. The eigenvalues of $B$ are called *principal curvatures*.

*Example* 1.1.4. The simplest example is, as usual, the hyperboloid $\mathbb{H}^2$. In this case, the Gauss map is the identity, in the sense that

$$G_{\mathbb{H}^2}(x) = x \in \mathbb{H}^2 \,.$$

Hence it turns out that the second fundamental form coincides with the first fundamental form, and the shape operator is the identity $B = E : TS \to TS$.

The first fundamental form and the shape operator of an immersed spacelike surface obey two very important relations. The first relation is provided by the Minkowski version of Gauss Theorem, and is called *Gauss equation*:

$$K_I = -\det B \tag{G-$\mathbb{R}^{2,1}$}$$

where $K_I$ is the curvature of the first fundamental form. The second relation is *Codazzi equation*:

$$(\nabla^I_v B)(w) - (\nabla^I_w B)(v) = 0 \tag{Cod}$$

The expression in (Cod) is usually called *exterior derivative* of $B$ and can be expressed in the following way:

$$d^{\nabla^I} B(v,w) := (\nabla^I_v B)(w) - (\nabla^I_w B)(v) = \nabla^I_v(B(\hat{w})) - \nabla^I_w(B(\hat{v})) - B([\hat{v},\hat{w}])\,,$$

where $\hat{v}$ and $\hat{w}$ are arbitrary extensions of $v$ and $w$ to $S$.

*Remark* 1.1.5. For a spacelike immersion $\sigma$ of a surface $S$ in $\mathbb{R}^{2,1}$, it is easy to compute the pull-back by the Gauss map of the hyperbolic metric of $\mathbb{H}^2$. Indeed, for $v \in T_xS$,

$$d(G \circ \sigma)(v) = \sigma_*(B(v))$$

and therefore the 2-tensor obtained by pull-back is

$$\langle d(G \circ \sigma)(v), d(G \circ \sigma)(w) \rangle_{2,1} = I(B(v), B(w))\,.$$

The latter term is called *third fundamental form*:

$$I\!I\!I(v,w) = I(B(v), B(w))\,.$$

If $S$ is strictly convex, $\det B$ never vanishes, hence the third fundamental form is a Riemannian metric on $S$. In this case the Gauss map is a diffeomorphism, and $I\!I\!I$ is clearly a hyperbolic metric. This can be seen also by applying the formulae for the Levi-Civita connection and the curvature of the third fundamental form, see [Lab92] or [KS07].

$$\nabla^{I\!I\!I} = B^{-1} \nabla^I B \tag{1.4}$$

$$K_{I\!I\!I} = \frac{K_I}{\det B} \tag{1.5}$$

Indeed, by the Gauss Equation (G-$\mathbb{R}^{2,1}$), we have $K_{I\!I\!I} = -1$.

The fundamental theorem of the theory of immersed surfaces states that every immersion of a simply connected surface is determined, up to isometries of the ambient space, by its *embedding data* $I$ and $B$. See for instance [Pet06].

**Theorem 1.1.6** (Fundamental theorem of immersed surfaces in Minkowski space)**.** *Let $\tilde{S}$ be a simply connected surface. Given any pair $(I, B)$, where $I$ is a Riemannian metric on $\tilde{S}$ and $B$ is a $(1,1)$-tensor self-adjoint for $I$, such that the Gauss-Codazzi equations*

$$\begin{cases} \det B = -K_I \\ d^{\nabla^I} B = 0 \end{cases} \tag{GC-$\mathbb{R}^{2,1}$}$$

*are satisfied, there exists a smooth immersion $\sigma : \tilde{S} \to \mathbb{R}^{2,1}$ such that the first fundamental form is $I$ and the shape operator is $B$. Moreover, given any two such immersions $\sigma$ and $\sigma'$, there exists $\mathcal{R} \in \mathrm{Isom}(\mathbb{R}^{2,1})$ such that $\sigma' = \mathcal{R} \circ \sigma$.*

### 1.1.2 The "dual" space of Minkowski space: half-pipe geometry

From the definition, $\mathbb{H}^2$ parametrizes spacelike planes in $\mathbb{R}^{2,1}$ through the origin. Indeed, a point $x \in \mathbb{H}^2$ corresponds uniquely to the spacelike plane $x^{\perp}$. We now want to introduce the space of spacelike planes of $\mathbb{R}^{2,1}$, not necessarily containing the origin. It should be clear to the reader that the natural parameter space is $\mathbb{H}^2 \times \mathbb{R}$.

Let us make more precise this correspondence. We will define a map

$$\{\text{spacelike planes of } \mathbb{R}^{2,1}\} \to \mathbb{H}^2 \times \mathbb{R}\,.$$

Let $P = p + x^{\perp}$ be a spacelike plane in $\mathbb{R}^{2,1}$, where $p \in \mathbb{R}^{2,1}$ and $x \in \mathbb{H}^2$. Clearly $x$ is the point of $\mathbb{H}^2$ which represents the image of $P$ under its Gauss map. We define

$$P \mapsto (x, \langle p, x \rangle_{2,1})\,, \tag{1.6}$$

where $p$ can be chosen arbitrarily in $P$. This map is bijective and explicitly parametrizes spacelike planes of Minkowski space. The parameter space $\mathbb{H}^2 \times \mathbb{R}$ can be obtained in the following *half-pipe model*, first introduced by Danciger in [Dan11]. Let us consider the following degenerate bilinear form on $\mathbb{R}^4 = \{x = (x^1, x^2, x^3, x^4)\}$.

$$\langle x, y \rangle_{2,0,1} = (x^1)^2 + (x^2)^2 - (x^4)^2\,.$$

The half-pipe geometry is defined as

$$\mathbb{HP}^3 = \{x \in \mathbb{R}^{2,0,1} : \langle x, x \rangle_{2,0,1} < 0, x^4 > 0\}$$

endowed with the degenerate metric induced by the bilinear form of $\mathbb{R}^{2,0,1}$. It is clear that the half-pipe metric can be expressed on $\mathbb{H}^2 \times \mathbb{R}$ as

$$g_{\mathbb{H}^2} + 0 \cdot dt^2 \tag{1.7}$$

where $g_{\mathbb{H}^2}$ is the metric of the hyperbolic plane and $t$ is the coordinate of the second component.

Also for half-pipe geometry, we can define a *projective model*, namely we consider

$$\{x \in \mathbb{R}^{2,0,1} : \langle x, x \rangle_{2,0,1} < 0\} / \sim\,,$$

where $x \sim x'$ if there exists $\lambda$ such that $x = \lambda x'$. Thanks to this definition, we have the notions of geodesics and planes in half-pipe geometry, which are just the intersections of the above domain with projective lines and planes of $\mathbb{R}\mathrm{P}^3$. We say that a line or a plane in $\mathbb{HP}^3$ is spacelike if its induced metric is a Riemannian metric. By the form (1.7) of the metric, the induced metric on a spacelike plane $P$ is always the hyperbolic metric.

We now *define* the group $\mathrm{Isom}(\mathbb{HP}^3)$ of *isometries of half-pipe geometry* (which preserve the orientation and the degenerate direction) as the group of projective transformations mapping $\mathbb{HP}^3$ to itself, with unit determinant, which preserve the degenerate metric (1.7).

The projective model of half-pipe geometry enables to define in a natural way the *boundary at infinity* of $\mathbb{HP}^3$:

$$\partial_\infty \mathbb{HP}^3 = \{x \in \mathbb{R}^{2,0,1} : \langle x, x\rangle_{2,0,1} = 0\}/\sim \ .$$

Every spacelike line in $\mathbb{HP}^3$ is asymptotic to two points in $\partial_\infty \mathbb{HP}^3$.

Using the affine chart $\{x^4 \neq 0\}$, we obtain a *Klein model* for $\mathbb{HP}^3$, which is thus identified to $\mathbb{D} \times \mathbb{R}$. By the above discussion, it is clear that is this model spacelike planes are graphs of affine functions over $\mathbb{D}$, in the sense that

$$P = \{(z, t) \in \mathbb{D} \times \mathbb{R} : t = u(z)\} \, ,$$

where $u : \mathbb{D} \to \mathbb{R}$ is an affine function. See Figure 1.1. In this model, the boundary at infinity of $\mathbb{HP}^3$ is identified to $\partial\mathbb{D} \times \mathbb{R}$.

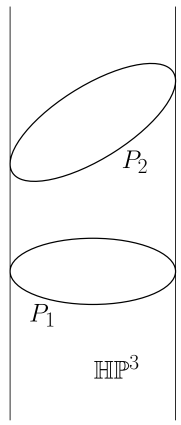



**Figure 1.1:** Two planes in $\mathbb{HP}^3$, in the affine model $\mathbb{D} \times \mathbb{R}$.

We conclude this section by showing that the correspondence between $\mathbb{R}^{2,1}$ and $\mathbb{HP}^3$ is a honest duality. So far we have associated to every spacelike plane of $\mathbb{R}^{2,1}$ a point in $\mathbb{HP}^3$. We want to show:

- The vice versa holds, namely that points of $\mathbb{R}^{2,1}$ correspond in a natural way to planes in $\mathbb{HP}^3$;
- Given a point $p$ and a plane $P$ in $\mathbb{R}^{2,1}$, and their dual plane $p^*$ and point $P^*$ in $\mathbb{HP}^3$, $p \in P$ if and only if $P^* \in p^*$;
- The correspondence is natural, in the sense that it preserves the actions of the isometry groups of $\mathbb{R}^{2,1}$ and $\mathbb{HP}^3$.

For the first point, let $p \in \mathbb{R}^{2,1}$. Recalling the definition (1.6), a point $(x,t)$ in $\mathbb{H}^2 \times \mathbb{R}$ corresponds to a plane through $p$ if and only if $\langle x, p\rangle_{2,1} = t$. It is clear that this condition extends to a homogeneous equation in $\mathbb{R}^{2,0,1}$, hence it defines a plane for the projective structure we defined on $\mathbb{HP}^3$. The second point follows directly.

The third point is also straightforward. Indeed, let $p \mapsto \gamma(p) = Ap + v$ an isometry of $\mathbb{R}^{2,1}$, for $A \in \mathrm{SO}_0(2,1)$ and $v \in \mathbb{R}^{2,1}$. Given a plane $P$ in $\mathbb{R}^{2,1}$ with normal vector $x \in \mathbb{H}^2$, $\gamma(P)$ has normal vector $Ax$, while for every point $p \in P$,

$$\langle Ax, \gamma(p)\rangle_{2,1} = \langle Ax, Ap + v\rangle_{2,1} = \langle x, p\rangle_{2,1} + \langle Ax, v\rangle_{2,1} \,.$$

Therefore, by the correspondence defined in (1.6), $\gamma$ induces the following action on $\mathbb{HP}^3$:

$$(x,t) \in \mathbb{H}^2 \times \mathbb{R} \mapsto (Ax, t + \langle x, A^{-1}v\rangle_{2,1}) \,.$$

This defines a projective transformation of $\mathbb{HP}^3$, which also preserves the degenerate metric (1.7), hence is in $\mathrm{Isom}(\mathbb{HP}^3)$. Indeed the metric of $\mathbb{H}^2 \times \mathbb{R}$ is simply the hyperbolic metric on the first component, regardless of the second component, and is preserved by $A$. The determinant of such transformation is 1, since the associated matrix has the form

$$\begin{pmatrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ \star & \star & \star & 1 \end{pmatrix}$$

and $\det A = 1$. Moreover, it can be easily checked that this defines a group isomorphism $\mathrm{Isom}(\mathbb{R}^{2,1}) \cong \mathrm{Isom}(\mathbb{HP}^3)$.

### 1.1.3 The support function and duality for convex surfaces

The aim of this section is to discuss how the notion of duality between Minkowski space and half-pipe geometry extends to a duality of convex surfaces, and how this is related to the Lorentzian analogue of the support function of Euclidean convex bodies. Roughly speaking, to a smooth spacelike strictly convex surface in $\mathbb{R}^{2,1}$ one can associate the *dual surface* $S^*$ in $\mathbb{HP}^3$, which is defined as

$$S^* = \{X \in \mathbb{HP}^3 : X \text{ is dual to a plane in } \mathbb{R}^{2,1} \text{ tangent to } S\} \,.$$

It turns out that $S^*$ is a strictly convex surface in $\mathbb{HP}^3$ (where the notion of convexity is well-defined, thanks to the projective structure) and is smooth if $S$ is smooth.

Under the smoothness assumption, this construction also provides an immersion of the dual surface $S^*$ in $\mathbb{HP}^3$. Let us define the map $d : S \to S^*$ which associates to $x \in S$ the point in $\mathbb{HP}^3$ which represents the plane $T_xS$ of $\mathbb{R}^{2,1}$. Hence, given an immersion $\sigma : S \to \mathbb{R}^{2,1}$, we obtain the immersion $d \circ \sigma$ of the dual surface in $\mathbb{HP}^3$. If we consider $\mathbb{HP}^3$ as $\mathbb{H}^2 \times \mathbb{R}$, by construction the first component of $d \circ \sigma(x)$ is precisely the Gauss map $G(x) \in \mathbb{H}^2$. Therefore, by Remark 1.1.5 and the degenerate form (1.7) of the metric of $\mathbb{HP}^3$, the first fundamental form of the immersion $d \circ \sigma$ is precisely the third fundamental form

$$I^*(v,w) = I\!I\!I(v,w) = I(B(v), B(w)) \,.$$

Before proceeding further, we prefer however to extend the theory of surfaces in Minkowski space to the case of non-smooth convex surfaces. This will enable to define a dual surface for this more general class of objects.

*Remark* 1.1.7. Mess proved [Mes07] that if the first fundamental form of a spacelike immersion is complete, then the image of the immersion is a *spacelike entire graph.* This means that it is of the form $\{(x^1, x^2, x^3) \mid x^3 = f(x^1, x^2)\}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a convex function satisfying the spacelike condition $||Df|| < 1$, where $Df$ is the Euclidean gradient of $f$. Notice however that if $S$ is an entire spacelike graph in general it might not be complete.

We will consider the case of convex entire graphs which are not smooth and possibly contain lightlike rays. Those correspond to convex functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $||Df|| \leq 1$ almost everywhere. We will extend the notion of Gauss map to this more general class.

A *future-convex domain* in $\mathbb{R}^{2,1}$ is a closed convex set which is obtained as the intersection of future half-spaces bounded by spacelike planes. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a convex function satisfying the condition $||Df|| \leq 1$, then the epigraph of $f$ is a future-convex domain, and conversely the boundary of any future-convex domain is the graph of a convex function as above. Basic examples of future-convex domains are the future of any smooth convex surface, or, somehow on the opposite side of the range of examples, $\mathrm{I}^+(0)$.

A *support plane* for a future-convex domain $D$ is a plane $P = y + x^\perp$ such that $P \cap \mathrm{int}(D) = \emptyset$ and every translate $P' = P + v$, for $v$ in the future of $x^\perp$, intersects $\mathrm{int}(D)$. A future-convex domain can admit spacelike and lightlike support planes. We define the *spacelike boundary* of $D$ as the subset

$$\partial_s D = \{p \in \partial D : p \text{ belongs to a spacelike support plane of } D\}\,.$$

It can be easily seen that $\partial D \setminus \partial_s D$ is a union of lightlike geodesic rays. So $\partial D$ is an entire spacelike graph if and only if it does not contain lightlike rays.

We can now define the Gauss map for the spacelike boundary of a future-convex set. We allow the Gauss map to be set-valued, namely

$$G(p) = \{x \in \mathbb{H}^2 : p + x^\perp \text{ is a support plane of } D\}\,.$$

By an abuse of notation, we will treat the Gauss map as a usual map with values in $\mathbb{H}^2$. The following fact, which is well-known, has to be interpreted in this sense.

*Fact* 1.1.8. Given a future-convex domain $D$ in $\mathbb{R}^{2,1}$, the Gauss map of $\partial_s D$ has image a convex subset of $\mathbb{H}^2$. If $S$ is a strictly convex embedded spacelike surface, then its Gauss map is a homeomorphism onto its image.

We can finally introduce the Lorentzian analogue of the support function of Euclidean convex bodies. Roughly speaking, the support function encodes the information about the support planes of a future-convex domain.

Given a future-convex domain $D$ in $\mathbb{R}^{2,1}$, the *support function* of $D$ is the function $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R} \cup \{\infty\}$ defined by

$$U(x) = \sup_{p \in D} \langle p, x \rangle_{2,1}\,.$$

Several properties of support functions can be straightforwardly deduced from the definition and the above discussion.

- Given two future-convex domains $D_1$ and $D_2$ with support functions $U_1$ and $U_2$, $U_1 \geq U_2$ if and only if $D_1 \subseteq D_2$;
- Given the isometry $p \mapsto \gamma(p) = Ap + v$, with $A \in \mathrm{SO}_0(2,1)$, the support function of $D' = \gamma(D)$ is
$$U'(x) = U(A^{-1}x) + \langle x, t \rangle_{2,1}\,; \tag{1.8}$$
- $U$ is lower semicontinuous, since it is defined as the supremum of continuous functions;
- $U$ is convex, hence it is continuous on $\mathrm{I}^+(0)$;
- $U$ is 1-homogeneous, namely $U(\lambda x) = \lambda U(x)$ for every $\lambda > 0$.

The last property is quite important, as it ensures that the graph of $U$ is a well-defined projective subset of $\mathbb{HP}^3$. Indeed, it turns out that in $\mathbb{R}\mathrm{P}^3$
$$[\lambda x, U(\lambda x)] = [\lambda x, \lambda U(x)] = [x, U(x)]\,,$$
for every $\lambda > 0$.

*Remark* 1.1.9. It should now be clear that the *dual surface* associated to $S = \partial D$, where $D$ is a future-convex domain, can be defined as
$$S^* = \{X \in \mathbb{HP}^3 : X \text{ is dual to a support plane of } D \text{ in } \mathbb{R}^{2,1}\}\,.$$
Moreover, if $U : \mathrm{I}^+(0) \to \mathbb{R} \cup \{\infty\}$ is the support function on the interior of the future cone of 0, then
$$S^* = \{[x,t] : x \in \mathrm{I}^+(0), t \in \mathbb{R}, t = U(x)\}\,,$$
namely $S^*$ is the graph of $U$.

Conversely, a convex spacelike surface $\Sigma$ in $\mathbb{HP}^3$ can be described locally as the graph of a 1-homogeneous convex function $U$ defined on a subset of $\mathrm{I}^+(0)$. We will say that $\Sigma$ is *entire* if it is the graph of a function $U$ defined over the whole $\mathrm{I}^+(0)$. (It will be in general useful to extend $U$ to $\overline{\mathrm{I}^+(0)}$ by lower semicontinuity.) See Figure 1.2. It was proved in [FV13, Lemma 2.21] that every such function defines a future-convex domain $D$ in $\mathbb{R}^{2,1}$ by means of
$$D = \{p \in \mathbb{R}^{2,1} : \langle p, x \rangle \leq U(x) \text{ for every } x \in \mathrm{I}^+(0)\}\,.$$
By construction, the support function of $D$ is precisely $U$. Moreover, the surface $\partial D$ is exactly the dual surface of $\Sigma$, namely
$$\Sigma^* = \{x \in \mathbb{R}^{2,1} : x \text{ is dual to a support plane of } \Sigma \text{ in } \mathbb{HP}^3\}\,,$$
where the definition of support planes in $\mathbb{HP}^3$ is completely analogous. Hence we have a duality between convex entire spacelike surfaces in half-pipe geometry and future-convex domains in Minkowski space with surjective Gauss map, satisfying the duality property that $S^{**} = S$ and $\Sigma^{**} = \Sigma$.

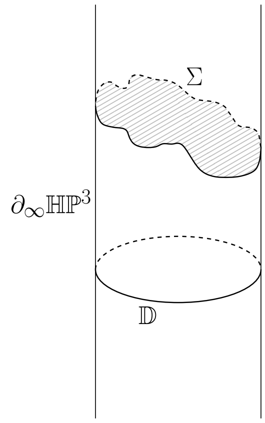



**Figure 1.2:** A spacelike surface $\Sigma$ in $\mathbb{HP}^3$, in the affine model $\mathbb{D} \times \mathbb{R}$.

*Example* 1.1.10. The support function of

$$D_1 = \mathrm{I}^+(\mathbb{H}^2)\,,$$

(which is the future of the hyperboloid $\mathbb{H}^2$) is

$$U_1(x) = -\sqrt{|\langle x, x\rangle_{2,1}|}\,.$$

On the other hand, given a point $p \in \mathbb{R}^{2,1}$, consider the domain

$$D_2 = \mathrm{I}^+(p)\,.$$

namely the future cone over $p$. Its support function, as in part we have already discussed, is

$$U_2(x) = \langle x, p\rangle_{2,1}\,.$$

Hence the surface in $\mathbb{HP}^3$ dual to $\partial D_2$, if regarded in the $\mathbb{D} \times \mathbb{R}$ model, is the graph of an affine function.

We now give a slightly more complicated example. Consider two points $p, q \in \mathbb{R}^{2,1}$ such that the segment $[p, q]$ is spacelike. Consider the domain

$$D_3 = \mathrm{I}^+([p, q])\,,$$

i.e. the set of points in the future of such segment. Then its support function is

$$U_3(x) = \max\{\langle x, p\rangle_{2,1}, \langle x, q\rangle_{2,1}\}\,.$$

In other words, the surface dual to $D_3$ is the graph of a piecewise affine function on $\mathbb{D}$. The geodesic $[p, q]^\perp$ of $\mathbb{H}^2$ is such that $U_3$ is affine on each half-plane bounded by such geodesic.

### 1.1.4 A special class of Minkowski manifolds containing a closed spacelike surface: maximal globally hyperbolic manifolds

In this section we want to introduce a wide class of examples of three-manifolds containing a closed (i.e. compact and without boundary) embedded spacelike surface. We will consider a special case of three-manifolds, locally modelled on $\mathbb{R}^{2,1}$, whose topology is $S \times \mathbb{R}$, containing a closed spacelike surface homotopic to $S \times \{0\}$, called *maximal globally hyperbolic manifolds*. First we need to introduce some definitions from the causal geometry of Lorentzian manifolds.

The future of a point and of a spacelike segment in $\mathbb{R}^{2,1}$, discussed in Example 1.1.10, are examples of future-convex domains with a special property, called *domains of dependence*. These were introduced in [Bon05]. A *future domain of dependence* is an open domain in $\mathbb{R}^{2,1}$ which is obtained as the intersection of at least two future half-spaces bounded by lightlike planes.

A classical notion in Lorentzian geometry is the *Cauchy development* of a surface. Let $S$ be a spacelike surface in $\mathbb{R}^{2,1}$ (or more generally in a Lorentzian manifold $M$). Its Cauchy development is

$$D(S) = \{p \in \mathbb{R}^{2,1} : \text{every inextensible causal path from } p \text{ intersects } S \text{ in one point}\}\,.$$

The Cauchy development of $S$ in $\mathbb{R}^{2,1}$ can be obtained as the intersection of the future of all lightlike planes which do not intersect $S$. This shows that $D(S)$ is a domain of dependence.

Given a surface $S$ in a Lorentzian three-manifold $M$, $S$ is a *Cauchy surface* for $M$ if every inextensible causal path intersects $S$ exactly once. Hence a surface $S$ in a domain of dependence $D$ is a Cauchy surface if and only if the Cauchy development of $S$ coincides with $D$. A Lorentzian manifold $M$ containing a Cauchy surface is called *globally hyperbolic*.

**Definition 1.1.11.** A Lorentzian three-manifolds $M$ is *maximal globally hyperbolic* if $M$ contains a Cauchy surface $S$ and, for every isometric embedding of $\iota : M \to \widehat{M}$ in a Lorentzian manifold $\widehat{M}$ such that $\iota(S)$ is a Cauchy surface, $\iota$ is surjective.

An isometric embedding of a globally hyperbolic manifold to another globally hyperbolic manifold, sending a Cauchy surface to a Cauchy surface, is called a *Cauchy embedding*.

Maximal globally hyperbolic flat spacetimes (i.e. endowed with a flat Lorentzian metric) were studied by Geoffrey Mess in this pioneering work [Mes07]. See also [ABB$^+$07]. We quickly review some important facts here. Mess described the classifying space for maximal globally hyperbolic spacetimes in terms of Teichmüller spaces. We defer the description of these objects to Section 2.2.

By a classical theorem of Choquet-Bruhat, see [CB68], every flat globally hyperbolic spacetime admits an extension to a maximal globally hyperbolic spacetime, unique up to global isometry. The meaning of this theorem in our context is that the embedding data $(I, B)$ (recall Subsection 1.1.1) of a Cauchy surface uniquely determine the extension to *the* maximal globally hyperbolic spacetime. Let us make this assertion more precise.

Let $(I, B)$ be the embedding data of the closed Cauchy surface $S$ in $M$. Take the universal cover $\pi : \tilde{S} \to S$ and consider the lift $(\tilde{I}, \tilde{B})$ of $(I, B)$ to $\tilde{S}$. Since the Gauss-Codazzi equations (GC-$\mathbb{R}^{2,1}$) have a local nature, and $\tilde{S}$ is simply connected, $(\tilde{I}, \tilde{B})$ satisfy the hypothesis of Theorem 1.1.6. Therefore there exists an immersion $\sigma : \tilde{S} \to \mathbb{R}^{2,1}$ with embedding data $(\tilde{I}, \tilde{B})$.

However, for every deck transformation $\gamma \in \pi_1(S)$ of $\tilde{S}$, $\tilde{I}$ and $\tilde{B}$ are clearly invariant under the action of $\gamma$. Hence $\sigma' = \sigma \circ \gamma$ is an isometric immersion of $\tilde{S}$ with the same embedding data as $\sigma$. By the uniqueness part of Theorem 1.1.6, there exists an isometry

$$\mathcal{R}(\gamma) \in \mathrm{Isom}(\mathbb{R}^{2,1})$$

such that

$$\sigma \circ \gamma = \mathcal{R}(\gamma) \circ \sigma \,.$$

It can be easily checked that this defines a group representation

$$\mathcal{R} : \pi_1(S) \cong \pi_1(M) \to \mathrm{Isom}(\mathbb{R}^{2,1}) \,,$$

called the *holonomy* of $S$ (or of $M$). A different choice of $\sigma$ (which again differs by post-composition with an isometry $\mathcal{R}_0$) corresponds to a different representation $\mathcal{R}$ in the same conjugacy class.

Mess proved that the subgroup $\mathcal{R}(\pi_1(S))$ acts freely and properly discontinuously on the Cauchy development $D(\tilde{S})$ of $\tilde{S}$. The quotient $D(\tilde{S})/\mathcal{R}(\pi_1(S))$ is a maximal globally hyperbolic spacetime and is the maximal extension provided by the Croquet-Bruhat Theorem. Moreover, $\sigma : \tilde{S} \to \mathbb{R}^{2,1}$ is an embedding.

Let us write $\mathcal{R}(\gamma)(x) = \rho(\gamma)x + t_\gamma$. It is easy to check that

$$\rho : \pi_1(S) \to \mathrm{SO}_0(2,1)$$

is a group representation, called *linear part* of the holonomy, while the *translation part*

$$\gamma \mapsto t_\gamma \in \mathbb{R}^{2,1}$$

satisfies the relation

$$t_{\alpha\beta} = \rho(\alpha)t_\beta + t_\alpha \tag{1.9}$$

A map $t : \pi_1(S) \to \mathbb{R}^{2,1}$ satisfying the condition (1.9) is called a *cocycle* with respect to the representation $\rho$.

*Example* 1.1.12. Simple examples of maximal globally hyperbolic flat spacetimes are provided by the *Fuchsian* spacetimes. Given a discrete and faithful representation $\rho : \pi_1(S) \to \mathrm{SO}_0(2,1)$, where $S$ is a closed surface, $\rho$ can be thought as a representation in $\mathrm{Isom}(\mathbb{R}^{2,1})$ with trivial translation part. Hence $\rho(\pi_1(S))$ preserves the origin in $\mathbb{R}^{2,1}$ and preserves all the hyperboloids

$$\mathbb{H}^2(-t) := \{\langle x, x\rangle_{2,1} = -1/t^2\} \,.$$

Moreover, $\mathbb{H}^2/\rho(\pi_1(S))$ is a hyperbolic surface, i.e. a surface endowed with a constant curvature $-1$ Riemannian metric $g_\rho$. This point will be discussed further in Chapter 2.

Clearly the domain of dependence of the hyperboloid is the cone $\mathrm{I}^+(0)$. The quotient $M = \mathrm{I}^+(0)/\rho(\pi_1(S))$ is a flat maximal globally hyperbolic spacetime. The foliation by surfaces homeomorphic to $S$ is provided by the quotient of the hyperboloids $\mathbb{H}^2(-t)$ as $t \in (0, \infty)$. By a direct computation, the metric on $M \cong S \times \mathbb{R}$ has the form

$$-dt^2 + t^2 g_\rho \,.$$

We highlight that the embedding of $\tilde{S}$ is unique only up to an isometry of $\mathbb{R}^{2,1}$. Post-composing the embedding $\sigma$ (and thus also the domain of dependence) with an isometry $\mathcal{R}_0$ of $\mathbb{R}^{2,1}$ changes the holonomy by conjugation by $\mathcal{R}_0$. In particular, if we fix the linear part of the holonomy to be $\rho$, and we compose with a translation by a vector $t_0 \in \mathbb{R}^{2,1}$, we see that the new cocycle $t'$ differs from $t$ by the *coboundary*

$$t_\alpha = \rho(\alpha)t_0 - t_0 \,.$$

Given a representation $\rho : \pi_1(S) \to \mathrm{SO}_0(2,1)$, the first cohomology group $H^1_\rho(\pi_1(S), \mathbb{R}^{2,1})$ is the vector space obtained as a quotient of cocycles over coboundaries. Mess proved that for every discrete and faithful representation $\rho$ and every cohomology class in $H^1_\rho(\pi_1(S), \mathbb{R}^{2,1})$, there exists a unique (up to translation) future domain of dependence with holonomy $\mathcal{R}(\gamma)(x) = \rho(\gamma)(x) + t_\gamma$.

## 1.2 Hyperbolic space and de Sitter space

Let us consider (3+1)-dimensional Minkowski space, namely $\mathbb{R}^{3,1} = (\mathbb{R}^4, \langle \cdot, \cdot \rangle_{3,1})$, where the bilinear form of interest is

$$\langle x, y \rangle_{3,1} = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4.$$

As in the 2-dimensional case (see Example 1.1.1), the hyperboloid model of hyperbolic 3-space is

$$\mathbb{H}^3 = \left\{ x \in \mathbb{R}^{3,1} : \langle x, x \rangle_{3,1} = -1, x^4 > 0 \right\}.$$

The group of orientation-preserving isometries of $\mathbb{H}^3$ is $\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{SO}_0(3,1)$, namely the group of linear isometries of $\mathbb{R}^{3,1}$ which preserve orientation and do not switch the two connected components of the quadric $\{\langle x, x \rangle_{3,1} = -1\}$. It turns out that $\mathbb{H}^3$ is homogeneous and isotropic. More precisely, $\mathrm{Isom}(\mathbb{H}^3)$ induces a transitive action on oriented orthonormal triples in tangent spaces of $\mathbb{H}^3$.

Analogously to the lower-dimensional case, the induced metric is a complete Riemannian metric of constant curvature $-1$. Actually, $\mathbb{H}^3$ is *the* simply connected complete Riemannian manifold of constant curvature $-1$, in the sense that any other simply connected complete Riemannian manifold of constant curvature $-1$ is isometric to $\mathbb{H}^3$ (see for instance [Rat48, Theorem 8.6.2]).

Again, complete geodesics of $\mathbb{H}^3$ are intersections of $\mathbb{H}^3$ with 2-dimensional planes of $\mathbb{R}^{3,1}$ through the origin (when nontrivial). By using this fact and the structure of the isometry group $\mathrm{Isom}(\mathbb{H}^3)$, it is easy to compute formulae involving the distance on $\mathbb{H}^3$ induced by the Riemannian metric, which will be denoted by $d_{\mathbb{H}^3}(\cdot, \cdot)$. For instance, given a point $p \in \mathbb{H}^3$ and a tangent vector $v \in T_p\mathbb{H}^3 = p^\perp$, the unit speed geodesic starting at $p$ with initial vector $v$ is parametrized by:

$$r \mapsto \gamma(r) = \cosh(r)p + \sinh(r)v\,.$$

It follows that, for $p, q \in \mathbb{H}^3$,

$$\cosh(d_{\mathbb{H}^3}(p,q)) = |\langle p, q\rangle_{3,1}|\,. \tag{1.10}$$

It is natural to define a *projective model* of hyperbolic space:

$$\{x \in \mathbb{R}^{3,1} : \langle x, x\rangle_{3,1} < 0\}/\sim\,,$$

The usual affine coordinates $(x, y, z) = (x^1/x^4, x^2/x^4, x^3/x^4)$, for the affine chart $\{x^4 \neq 0\}$, provide the *Klein model*, namely the unit ball

$$\mathbb{B} = \left\{(x,y,z) : x^2 + y^2 + z^2 < 1\right\}\,.$$

In the projective model it is evident that $\mathbb{H}^3$ has a boundary at infinity $\partial_\infty\mathbb{H}^3$, which is the projectivization of $\{x \in \mathbb{R}^{3,1} : \langle x, x\rangle_{3,1} = 0\}$. Every geodesic in $\mathbb{H}^3$ is asymptotic to points in this boundary at infinity. It is well-known that $\partial_\infty\mathbb{H}^3$ is a 2-sphere endowed with a natural conformal structure.

In the Klein model, the boundary at infinity is just the usual round sphere

$$\partial\mathbb{B} = \left\{(x,y,z) : x^2 + y^2 + z^2 = 1\right\}$$

and the conformal structure is the one induced by the round metric. Geodesics are straight lines and totally geodesic planes (see next section) are the intersection of $\mathbb{B}$ with an affine plane. We quickly mention that other useful models of $\mathbb{H}^2$ and $\mathbb{H}^3$ - which will be occasionally used also in this work - are the *Poincaré model* and the *half-plane/half-space model*. More details can be found in various references, for instance [Rat48, CFKP97, BP92].

As in the two-dimensional case, let us denote by $\widehat{\mathrm{d}\mathbb{S}^3}$ the region

$$\widehat{\mathrm{d}\mathbb{S}^3} = \left\{x \in \mathbb{R}^{3,1} : \langle x, x\rangle_{3,1} = 1\right\}$$

and we call de Sitter space the projectivization of $\widehat{\mathrm{d}\mathbb{S}^3}$,

$$\mathrm{d}\mathbb{S}^3 = \{\langle x, x\rangle_{3,1} > 0\}\,/\sim\,.$$

De Sitter space is a constant curvature +1 Lorentzian manifold. Its geodesics, as usual, are intersections of $\mathrm{d}\mathbb{S}^3$ with projective lines of $\mathbb{RP}^3$; totally geodesic planes are intersections with projective planes of $\mathbb{RP}^3$. Hence totally geodesic planes $P$ of $\mathrm{d}\mathbb{S}^3$ are parametrized by the point $P^\perp$ of $\mathbb{H}^3$ (here it is convenient to use the projective model of $\mathbb{H}^3$). Vice versa, totally geodesic planes $Q$ in hyperbolic space are parametrized by the dual points $Q^\perp$ in $\mathrm{d}\mathbb{S}^3 \subset \mathbb{RP}^3$.

### 1.2.1 The geometry of immersed surfaces in hyperbolic space

We briefly recall the theory of immersions of surfaces in hyperbolic space. The first and second fundamental form are defined exactly as in Subsection 1.1.1, namely for an immersion $\sigma : S \to \mathbb{H}^2 \subset \mathbb{R}^{3,1}$,

$$I(v, w) = \langle \sigma_*(v), \sigma_*(w) \rangle_{3,1} .$$

and the second fundamental form is defined by

$$\nabla^{\mathbb{H}^3}_{\sigma_* v}(\sigma_* \hat{w}) = \nabla^S_v \hat{w} + I\!I(v, w) N$$

for $v, w \in T_x S$, where $N \in T_{\sigma(x)}\sigma(S)$ is the unit normal vector to $S$. Here $\nabla^S$ is the Levi-Civita connection of the first fundamental form $I$ of $S$ which can be obtained as the induced connection of the ambient Levi-Civita connection $\nabla^{\mathbb{H}^3}$ (which is in turn induced by $\nabla^{\mathbb{R}^{3,1}}$). It turns out that

$$I\!I(v, w) = I(B(v), w) = I(v, B(w))$$

where the shape operator $B$, which again satisfies Codazzi equation (Cod), is now defined as (compare Equation (1.2) for Minkowski space)

$$B(v) = -\nabla^{\mathbb{H}^2} N .$$

Another remarkable difference with the Minkowski case is that *Gauss equation* now is:

$$K_I = -1 + \det B \tag{G-$\mathbb{H}^3$}$$

where $K_I$ is the curvature of the first fundamental form. This is indeed a very general phenomenon: the constant $-1$ represents the curvature of the ambient manifold, whereas when the ambient metric is Riemannian, the sign in front of the determinant of $B$ is positive.

*Example* 1.2.1. A totally geodesic plane in $\mathbb{H}^3$ (for which $B \equiv 0$) is nothing but an isometric copy of $\mathbb{H}^2$, which has curvature $-1$ at every point.

In the setting of hyperbolic space, the role of the Gauss map defined for surfaces in Euclidean or Minkowski space is replaced by two *hyperbolic Gauss maps.* This notion and several properties which we will use are widely discussed in [Eps84], [Eps86]. Given a smooth surface $S$ in $\mathbb{H}^3$, and a choice $N$ of a unit normal vector field to $S$, we can define the maps

$$G_\pm : S \to \partial_\infty \mathbb{H}^3$$

which associate to $x \in S$ the asymptotic limits of the geodesic orthogonal to $S$ at $x$. In other words,

$$G_\pm(x) = \lim_{t \to \pm\infty} \gamma_x(t) ,$$

where $\gamma_x$ is the unit speed parametrization of the geodesic such that $\gamma(0) = x$ and $\gamma'(0) = N_x$.

We conclude the section by stating the fundamental theorem of the theory of immersed surfaces for hyperbolic space.

**Theorem 1.2.2** (Fundamental theorem of immersed surfaces in hyperbolic space)**.** *Let $\tilde{S}$ be a simply connected surface. Given any pair $(I, B)$, where $I$ is a Riemannian metric on $\tilde{S}$ and $B$ is a $(1,1)$-tensor self-adjoint for $I$, such that the Gauss-Codazzi equations*

$$\begin{cases} -1 + \det B = K_I \\ d^{\nabla^I} B = 0 \end{cases} \tag{GC-$\mathbb{H}^3$}$$

*are satisfied, there exists a smooth immersion $\sigma : \tilde{S} \to \mathbb{H}^3$ such that the first fundamental form is $I$ and the shape operator is $B$. Moreover, given any two such immersions $\sigma$ and $\sigma'$, there exists $\mathcal{R} \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\sigma' = \mathcal{R} \circ \sigma$.*

We will be interested (especially in Section 7) in *minimal surfaces* in $\mathbb{H}^3$, which are defined as surfaces for which the trace of the shape operator is zero. This amounts to saying that the principal curvatures of $S$ at every point are opposite, let's say $\lambda$ and $-\lambda$. The definition is well-posed, since the trace of a matrix is invariant by conjugacy and thus the condition $\mathrm{tr}B = 0$ does not depend on the choice of a coordinate system. Moreover, it does not depend on the choice of the normal unit vector field. Indeed, switching the direction of the normal vector field $N$ replaces $B$ with $-B$. The condition $\mathrm{tr}B = 0$ is equivalent to the fact that the surface is a critical point of the area functional (for compactly supported deformations of the surface).

### 1.2.2 Some properties of convex surfaces and their duality

As in the already discussed case of Minkowski space and half-pipe geometry, the duality between planes in $\mathbb{H}^3$ and points in $\mathrm{d}\mathbb{S}^3$ (and vice versa, planes in $\mathrm{d}\mathbb{S}^3$ and points in $\mathbb{H}^3$) can be extended to convex surfaces. First, we remark again that the notion of convex surface is well-defined in $\mathbb{H}^3$ and $\mathrm{d}\mathbb{S}^3$, by means of the projective structure they are endowed with. For instance, a surface in $\mathbb{H}^3$ is convex if it is in the Klein model, in the Euclidean sense.

Given a smooth convex surface $S$ in $\mathbb{H}^3$, we can define its *dual surface* as

$$S^* = \{x \in \mathrm{d}\mathbb{S}^3 : x^\perp \text{ is a plane in } \mathbb{H}^3 \text{ tangent to } S\}\,.$$

Of course, the dual surface of a smooth convex surface in $\mathrm{d}\mathbb{S}^3$ is defined completely analogously. We can also define a map $d : S \to S^*$ by means of

$$d(x) = (T_x S)^\perp$$

namely, to a point of $S$ we associate the point in $\mathrm{d}\mathbb{S}^3$ which corresponds to the tangent plane to $S$ at $x$. If $S$ is strictly convex, this map is injective. Then $S^*$ is a strictly convex smooth spacelike surface and $S^{**} = S$ (see [Sch98, HR93]).

Moreover, we can also study the geometry of $S^*$ as an immersed surface. Given an immersion $\sigma : S \to \mathbb{H}^3$, with first fundamental form $I$ and shape operator $B$, we can describe the dual surface by means of the immersion

$$d \circ \sigma : S \to \mathrm{d}\mathbb{S}^3\,.$$

It can be proved ([Sch98]) that the first fundamental form of the dual surface coincides with the third fundamental form of $S$, namely

$$I^*(v,w) = I\!I\!I(v,w) = I(B(v), B(w))$$

where $B = \nabla N$ is the shape operator of $S^*$ in $\mathrm{d}\mathbb{S}^3$. The shape operator of $S^*$, by means of the parametrization $d \circ \sigma$, is easily expressed as $B^* = B^{-1}$. The same formulae hold when the roles of $\mathbb{H}^3$ and $\mathrm{d}\mathbb{S}^3$ are exchanged.

It will be important in the following to define a notion of *asymptotic boundary* of a surface. We say that the *boundary at infinity* of a surface $S$ in $\mathbb{H}^3$ is the intersection with $\partial_\infty \mathbb{H}^3$ of the closure of the surface in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$. In the Klein model, this is the intersection of $\bar{S}$ with the sphere at infinity, in the usual topology. Clearly the same is defined also for de Sitter space (in the same affine chart as for the Klein model, de Sitter space lies in the complement of the unit ball $B$). It turns out that, given a smooth strictly convex surface $S$, which is an embedded disc with boundary at infinity $\Gamma$, $S^*$ has $\Gamma$ as boundary at infinity.

Such duality can be extended to non-smooth surfaces. Indeed, given a convex surface in $\mathbb{H}^3$, we can again define the notion of *support plane*, as a plane such that every translate in its normal direction - in the direction in which the surface is convex - disconnects the surface. On the other hand, translates of the plane on the other side are disjoint to the surface. Hence it is again possible to define the *dual surface* of a convex surface $S$ in $\mathbb{H}^3$ as

$$S^* = \{x \in \mathrm{d}\mathbb{S}^3 : x^\perp \text{ is a support plane of } S \text{ in } \mathbb{H}^3\}$$

and vice versa (assuming that the convex surface in $\mathrm{d}\mathbb{S}^3$ is spacelike).

*Example* 1.2.3. Consider a convex surface $S$ which is obtained as the intersection of two totally geodesic planes $P_1, P_2$ in $\mathbb{H}^3$. The support planes of $S$ are all the planes containing the geodesic $P_1 \cap P_2$ which do not disconnect $S$, including $P_1$ and $P_2$. By composing with isometries in $\mathrm{SO}_0(3,1)$, we can assume $P_1 = p_1^\perp$ and $P_2 = p_2^\perp$, where

$$p_1 = [(1,0,0,0)]$$
$$p_2 = [(\cos\theta, \sin\theta, 0, 0)] .$$

By a direct observation, $\theta$ is the angle between the planes $P_1$ and $P_2$ in $\mathbb{H}^3$, and coincides also with the length of the spacelike geodesic in $\mathrm{d}\mathbb{S}^3$ connecting $p_1$ and $p_2$. The other support planes of $S$ are exactly the planes $p^\perp$ where $p$ is a point of the form $[(\cos\theta', \sin\theta', 0, 0)]$, with $\theta' \in (0,\theta)$. Hence the dual of $S$ is very degenerate: it is the geodesic spacelike segment $l$ connecting $p_1$ and $p_2$. However, considering the future of $l$, we obtain a convex surface (which is lightlike in some parts), sharing the same boundary at infinity as $S$.

Given a Jordan curve $\Gamma$ in the sphere at infinity $\partial_\infty \mathbb{H}^3$, the *convex hull* of $\Gamma$ is the smallest convex subset of $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ containing $\Gamma$. It can be defined, for instance, as the intersection of all half-spaces bounded by totally geodesic planes $P$ of $\mathbb{H}^3$ whose boundary at infinity $\partial_\infty P$ do not intersect $\Gamma$. We will denote the convex hull of a curve $\Gamma$ by $\mathcal{CH}(\Gamma)$.

Example 1.2.3 can be generalized to *pleated surfaces* in $\mathbb{H}^3$, namely hyperbolic surfaces which are bent along a measured geodesic lamination, as described in [EM06]. See Definition 2.3.6 for the definition of measured geodesic laminations. For instance, such surfaces arise as the boundaries of the convex hull of a Jordan curve in $\partial_\infty\mathbb{H}^3$. More precisely, the boundary of $\mathcal{CH}(\Gamma)$ in $\mathbb{H}^3$ has two connected components, one of which is a convex surface, and the other is concave (provided we choose the direction of the normal vector field coherently). The induced distance on pleated surfaces is well-defined and is a hyperbolic metric. If $S$ is a pleated surface, the above construction provides a spacelike connected set $S^*$ in $\mathrm{d}\mathbb{S}^3$. The induced metric endows $S^*$ with the structure of a real tree. This is exactly the real tree dual to the bending lamination of $S$. See [Sca96] for more details. We will introduce measured geodesic laminations in more detail in Chapter 2.

### 1.2.3 A special class of hyperbolic manifolds containing a closed surface: quasi-Fuchsian manifolds

In this subsection we want to consider - similarly to Subsection 1.1.4 - a class of hyperbolic three-manifolds containing a closed embedded surface. Observe that, if a hyperbolic three-manifold $M$ contains a closed embedded surface $S$, then by taking the covering $\widehat{M}$ of $M$ corresponding to the subgroup $\pi_1(S) < \pi_1(M)$, we obtain a hyperbolic three-manifold $\widehat{M}$ homeomorphic to $S \times \mathbb{R}$. We will consider only manifolds whose topology is $S \times \mathbb{R}$ in the present work.

Every complete hyperbolic three-manifold $M$ is isometric to a quotient of $\mathbb{H}^3$. In the language of $(G, X)$-structures (see for instance [Rat48, Thu97a, Thu97b]), a complete hyperbolic manifold is a smooth manifold endowed with a complete $(\mathrm{Isom}(\mathbb{H}^3), \mathbb{H}^3)$-structure. Hence the *developing map*

$$\mathrm{dev} : \tilde{M} \to \mathbb{H}^3 ,$$

which in general is only a local diffeomorphism, is a global diffeomorphism under the completeness assumption. Then $M$ is identified to $\mathbb{H}^3/\mathcal{R}(\pi_1(S))$, where

$$\mathcal{R} : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$$

is the *holonomy representation.* Compare also with Subsection 1.1.4.

The *limit set* of a discrete subgroup $G < \mathrm{Isom}(\mathbb{H}^3)$ is the set of accumulation points in $\partial_\infty\mathbb{H}^3$ of orbits of the action of $G$ on $\mathbb{H}^3$. We denote the limit set of a discrete subgroup $G < \mathrm{Isom}(\mathbb{H}^3)$ by $\Lambda(G)$.

**Definition 1.2.4.** A complete hyperbolic manifold $M = \mathbb{H}^3/G$ is a *quasi-Fuchsian manifold* if the limit set $\Lambda(G)$ is a Jordan curve.

Given a quasi-Fuchsian manifold $M = \mathbb{H}^3/G$, the limit set $\Lambda(G)$ can also be thought as the asymptotic boundary of the lift to the universal cover $\mathbb{H}^3$ of any embedded surface in $M \cong S \times \mathbb{R}$ homotopic to the standard inclusion $S \hookrightarrow S \times \{0\}$.

The convex hull of $\Lambda(G)$ is invariant for the action of $G$ on $\mathbb{H}^3$. The quotient $\mathcal{CH}(\Lambda(G))/G$ is called *convex core* of $M$.

*Example* 1.2.5. The most trivial example of quasi-Fuchsian manifolds is - of course - provided by the *Fuchsian* ones. If a discrete and faithful representation $\mathcal{R} : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$ preserves a totally geodesic plane $P$, then it also preserves all the surfaces equidistant from $P$, which have the same boundary at infinity as $P$. Hence the limit set is obviously $\partial_\infty P$. The convex hull is simply $P$. The image of $\mathcal{R}$ lies in a subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ which is an isomorphic copy of $\mathrm{Isom}(\mathbb{H}^2)$; hence $\mathcal{R}$ produces a discrete and faithful representation

$$\rho : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^2)\,.$$

By a direct computation, the metric is of the form

$$dr^2 + (\cosh r)^2 g_\rho\,,$$

where $g_\rho$ is the hyperbolic metric of $\mathbb{H}^2/\rho(\pi_1(S))$.

Quasi-Fuchsian representations can indeed be thought as *quasiconformal deformations* of Fuchsian representations. We will employ this point of view in the following chapters. It turns out, for instance, that the Hausdorff dimension of the limit set is equal to 1 precisely in the case of Fuchsian manifolds.

The moduli spaces of quasi-Fuchsian manifolds will be discussed in Subsection 2.2. We conclude this section by giving a definition, which will be very important in Section 7.

**Definition 1.2.6.** A quasi-Fuchsian manifold is *almost-Fuchsian* if it contains a closed surface with principal curvatures in $(-1, 1)$.

It was proved by Ben Andrews that an almost-Fuchsian manifold necessarily contains a minimal surface with principal curvatures in $(-1, 1)$. By an application of the maximum principle, the minimal surface is unique in its homotopy class.

## 1.3 Anti-de Sitter space

Consider $\mathbb{R}^{2,2}$, the vector space $\mathbb{R}^4$ endowed with the bilinear form of signature (2,2):

$$\langle x, y\rangle_{2,2} = x^1y^1 + x^2y^2 - x^3y^3 - x^4y^4$$

and define

$$\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3} = \left\{x \in \mathbb{R}^{2,2} : \langle x, x\rangle_{2,2} = -1\right\}.$$

The topology of $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ is that of a solid torus. We define Anti-de Sitter space as

$$\mathbb{A}\mathrm{d}\mathbb{S}^3 = \widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}/\pm\mathrm{I}\,.$$

Then $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ is a double cover of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. The pseudo-Riemannian metric induced on $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ descends to a metric on $\mathbb{A}\mathrm{d}\mathbb{S}^3$ of constant curvature -1. The definition of timelike/lightlike/spacelike/causal vectors, paths, planes, surfaces goes exactly like in Section 1.1.

As in the previous cases, the group of isometries of $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ which preserve orientation and time-orientation is $\mathrm{Isom}(\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}) \cong \mathrm{SO}_0(2,2)$, namely the connected component of the identity in the group of linear isometries of $\mathbb{R}^{2,2}$. Therefore the group of orientation-preserving and time-preserving isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ has a natural isomorphism

$$\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \cong \mathrm{SO}_0(2,2)/\left\{\pm \mathrm{I}\right\} .$$

Given a point $p$ and a timelike vector $v \in T_p\mathbb{A}\mathrm{d}\mathbb{S}^3$, the geodesic leaving $p$ with initial tangent vector $v$ is parametrized by

$$r \mapsto \gamma(r) = \cos(r)p + \sin(r)v \,.$$

Hence timelike geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ are closed ($\gamma$ is a representative of the generator of $\pi_1(\mathbb{A}\mathrm{d}\mathbb{S}^3, p) \cong \mathbb{Z}$) and have lenght $\pi$. We will denote by $d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(\cdot,\cdot)$ the *timelike distance* in $\mathbb{A}\mathrm{d}\mathbb{S}^3 \setminus Q$, where $Q$ is a totally geodesic spacelike plane. We underline that this is actually not a distance, hence it does not endow $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with a metric structure. It is defined as follows: given points $p$ and $q \in \mathrm{I}^+(p)$, the distance between $p$ and $q$ is the maximum lenght of timelike paths from $p$ to $q$:

$$d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p,q) = \sup_\gamma \int ||\dot\gamma||_{\mathbb{A}\mathrm{d}\mathbb{S}^3} \,.$$

The distance between two such points $p, q$ is achieved along the timelike geodesic connecting $p$ and $q$. The timelike distance satisfies the reverse triangle inequality, meaning that, if $q \in \mathrm{I}^+(p)$ and $r \in \mathrm{I}^+(q)$,

$$d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p,r) \geq d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p,q) + d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(q,r) \,.$$

Again, there are easy formulae (see also [BS10]) relating the distance between points and the bilinear form of $\mathbb{R}^{2,2}$: for instance, if $q \in \mathrm{I}^+(p)$,

$$\cos(d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p,q)) = |\langle p,q\rangle_{2,2}| \,. \tag{1.11}$$

If $v \in T_p\mathbb{A}\mathrm{d}\mathbb{S}^3$ is a lightlike vector, a parametrization of the lightlike geodesic whose tangent initial vector is $v$ is provided by

$$r \mapsto \gamma(r) = p + rv \,.$$

In this case, it does not make sense to talk about unit-speed parametrization, since the tangent vector to $\gamma$ is always a null vector. Finally, if $v$ is spacelike, then the geodesic from $p$ with initial velocity vector $v$ is

$$r \mapsto \gamma(r) = \cosh(r)p + \sinh(r)v \,.$$

If $p$ and $q$ are connected by a spacelike line, the lenght $l([p,q])$ of the geodesic segment connecting $p$ and $q$ is given by

$$\cosh(l([p,q])) = |\langle p,q\rangle_{2,2}|. \tag{1.12}$$

However, in this setting, it should be remarked that it is possible to find spacelike paths connecting $p$ and $q$ with length arbitrarily small (by considering paths which are closer and closer to piecewise lightlike geodesics) or arbitrarily large (for instance, by staying in a totally geodesic plane containing $p$ and $q$).

We define the *projective model* of Anti-de Sitter space to be the projective domain

$$\left\{x \in \mathbb{R}^{2,2} : \langle x, x\rangle_{2,2} < 0\right\} / \sim ,$$

which can be considered an open domain (not convex, though) of $\mathbb{R}\mathrm{P}^3$.

In the affine chart $\{x_4 \neq 0\}$, $\mathbb{A}\mathrm{d}\mathbb{S}^3$ fills the domain $\left\{x^2 + y^2 < 1 + z^2\right\}$, interior of a one-sheeted hyperboloid; however $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is not contained in a single affine chart. Indeed in this description we are missing a totally geodesic plane at infinity. Since geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ are intersections of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with linear planes in $\mathbb{R}^{2,2}$, in the affine chart geodesics are represented again by straight lines. Totally geodesic planes in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ arise as intersections with linear hyperplanes of $\mathbb{R}^{2,2}$. Every spacelike plane is an isometric copy of $\mathbb{H}^2$. See Figure 1.3 for a picture of the light cone of a point in the affine chart $\{x_4 \neq 0\}$.

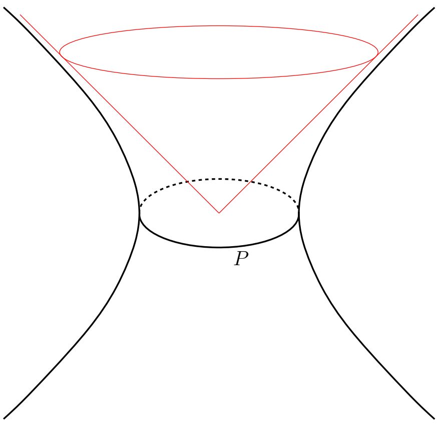



**Figure 1.3:** The lightcone of future null geodesic rays from a point and a totally geodesic plane $P$.

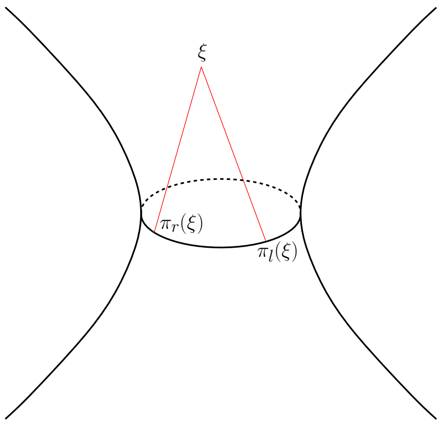



**Figure 1.4:** Left and right projection from a point $\xi \in \partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ to the plane $P = \{x_3 = 0\}$

The *boundary at infinity* of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is defined as the topological frontier of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ in $\mathbb{R}\mathrm{P}^3$, namely the doubly ruled quadric

$$\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3 = \left\{x \in \mathbb{R}^{2,2} : \langle x, x\rangle_{2,2} = 0\right\} / \sim \ .$$

It is naturally endowed with a conformal Lorentzian structure, for which the lightlike lines are precisely the left and right rulings. Given a spacelike plane $P$, which we recall is obtained as intersection of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with a linear hyperplane of $\mathbb{R}\mathrm{P}^3$ and is a copy of $\mathbb{H}^2$, $P$ has a natural boundary at infinity $\partial_\infty P$ which coincides with the usual boundary at infinity of $\mathbb{H}^2$. Moreover, $\partial_\infty P$ intersects each line in the left or right ruling in exactly one point. If a spacelike plane $P$ is chosen, $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ can be identified with $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2$ by means of the following description: $\xi \in \partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ corresponds to $(\pi_l(\xi), \pi_r(\xi))$, where $\pi_l$ and $\pi_r$ are the projections to $\partial_\infty P$ following the left and right ruling respectively (compare Figure 1.4).

Hence, given a map $\phi : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$, the graph of $\phi$ can be thought of as a curve $\Gamma = gr(\phi)$ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$. The notion of spacelike curve is also well-defined, since

$\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ is endowed with a conformal Lorentzian structure. Even if we consider only continuous curves, we say that a curve $\Gamma$ is *weakly spacelike* if, for every $\xi \in \Gamma$, $\Gamma$ is contained in the region bounded by the lines through $\xi$ in the left and right ruling which is connected to $\xi$ by spacelike paths. Weakly spacelike curves are precisely graphs $\Gamma = gr(\phi)$, where $\phi$ is an orientation-preserving homeomorphism of $\partial_\infty \mathbb{H}^2$. Moreover, it turns out that a weakly spacelike curve $\Gamma = gr(\phi)$ is the boundary at infinity of a totally geodesic plane if and only if $\phi$ is in $\mathrm{PSL}(2,\mathbb{R})$.

It was proved in [Mes07] that every isometry in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ extends to a diffeomorphism of $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ acting as a projective transformation on every line in the left and right rulings. This gives a natural identification

$$\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \cong \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}\,.$$

### 1.3.1 The geometry of immersed surfaces in Anti-de Sitter space

At this stage, it should be evident to the reader how the definitions given above are extended for the case of Anti-de Sitter space. For instance, the correct form of *Gauss equation* is

$$K_I = -1 - \det B \tag{G-$\mathbb{H}^3$}$$

To make the theory clear, it suffices to state the corresponding fundamental theorem of immersions.

**Theorem 1.3.1** (Fundamental theorem of immersed surfaces in Anti-de Sitter space)**.** *Let $\tilde{S}$ be a simply connected surface. Given any pair $(I, B)$, where $I$ is a Riemannian metric on $\tilde{S}$ and $B$ is a $(1,1)$-tensor self-adjoint for $I$, such that the Gauss-Codazzi equations*

$$\begin{cases} -1 - \det B = K_I \\ d^{\nabla^I} B = 0 \end{cases} \tag{GC-$\mathbb{A}\mathrm{d}\mathbb{S}^3$}$$

*are satisfied, there exists a smooth immersion $\sigma : \tilde{S} \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ such that the first fundamental form is $I$ and the shape operator is $B$. Moreover, given any two such immersions $\sigma$ and $\sigma'$, there exists $\mathcal{R} \in \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ such that $\sigma' = \mathcal{R} \circ \sigma$.*

As stated in Subsection 1.2.1 for $\mathbb{H}^3$, we will be interested in smooth surfaces whose shape operator has zero trace, $\mathrm{tr}B = 0$. Surfaces satisfying this requirement are called *maximal surfaces*. In fact, in the Lorentzian setting a surface with traceless shape operator has the property that small deformations of the surface (supported on a compact subset of $S$) decrease the area.

We are now going to introduce two maps which - in some sense - can play the role of the hyperbolic Gauss maps in the context of Anti-de Sitter geometry. We will call these maps *left projection* and *right projection* from the surface $S$ to a fixed totally geodesic plane $P_0$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Recall that $P_0$ is a copy of hyperbolic plane. Given a point $x \in S$, we define two isometries $\Phi_l^x, \Phi_r^x \in \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ which map the tangent plane $T_xS$ to $P_0$. The first one $\Phi_l^x$ is obtained by following the left ruling of $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$: this means that every point of $\partial_\infty(T_xS)$ is mapped by $\Phi_l^x$ to the point of $\partial_\infty P_0$ which lies on the same line in the left ruling of $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$. Analogously $\Phi_r^x$ is obtained by following the right ruling. This gives the left and right projections $\Phi_l$ and $\Phi_r$ from $S$ to $P_0$, defined by

$$\Phi_l(x) = \Phi_l^x(x)\,; \tag{1.13}$$

$$\Phi_r(x) = \Phi_r^x(x)\,. \tag{1.14}$$

### 1.3.2 Duality for convex surfaces

The construction performed in Subsection 1.2.2 can be repeated for Anti-de Sitter geometry. Given a totally geodesic spacelike plane $P$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, the orthogonal complement $P^\perp$ in $\mathbb{R}^{2,2}$ defines a point which is again in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, called *dual point*. Conversely, for every point $p \in \mathbb{A}\mathrm{d}\mathbb{S}^3$, $p^\perp$ defines a totally geodesic plane in $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

There is an intrinsic geometric characterization of this duality. Indeed, given a point $p \in \mathbb{A}\mathrm{d}\mathbb{S}^3$, by a direct computation one can see that $p^\perp$ is the locus of points in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ which have timelike distance $\pi/2$ from $p$. In other words, $p^\perp$ is composed of the middle-points of all the timelike geodesics leaving from $p$, which we recall are closed and have length $\pi$. Observing that every lightlike geodesic is asymptotic in the future and in the past to the same point of $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$, the boundary at infinity of $p^\perp$ can be described as the set of asymptotic points of lightlike geodesics leaving from $p$.

The duality between points in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and spacelike planes degenerates to a "duality" between points in $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$ and lightlike planes. Given a lightlike plane $Q$, its boundary at infinity is composed of two lines in $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$, one in the left ruling and the other in the right ruling. Hence such two lines intersect in a single point at infinity. When a sequence of spacelike planes degenerates to a lightlike plane $Q$, the dual point tends exactly to the point in the boundary at infinity associated with $Q$.

Again, the *dual surface* of a smooth convex surface $S$ is

$$S^* = \{x \in \mathbb{A}\mathrm{d}\mathbb{S}^3 : x^\perp \text{ is a plane in } \mathbb{A}\mathrm{d}\mathbb{S}^3 \text{ tangent to } S\}\,,$$

and the map

$$d : x \mapsto (T_xS)^\perp$$

pulls back the first fundamental form of $S^*$ to the third fundamental form of $S$:

$$I^*(v,w) = I\!I\!I(v,w) = I(B(v),B(w))\,.$$

It turns out that $S$ and $S^*$ share the same boundary curve at infinity in $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Given a weakly spacelike curve $\Gamma = gr(\varphi)$ in $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$, the notion of *convex hull* $\mathcal{CH}(\Gamma)$ can be again defined, as the smallest convex subset with asymptotic boundary $gr(\varphi)$. Although $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is not convex in $\mathbb{R}\mathrm{P}^3$, the convex hull is always contained in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. See [BS10, Lemma 4.8].

As for hyperbolic space, the connected components of $\partial(\mathcal{CH}(\Gamma)) \setminus \Gamma$ are a convex and a concave surface, with induced metric a complete hyperbolic metric, pleated along a measured geodesic lamination (see Definition 2.3.6 below) for the hyperbolic metric. We call these surfaces *upper boundary* and *lower boundary* of the convex hull.

Another relevant notion is the *domain of dependence* of the curve $\Gamma$. Given a spacelike surface $S$ asymptotic to $\Gamma$, we say that the *past domain of dependence* of $S$ is the set of points $p$ in the past of $S$ such that every future causal path from $p$ intersects $S$. In other words, no future spacelike or lightlike geodesic leaving from $p$ intersects $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ in the past of $S$. Analogously, the *future domain of dependence* of $S$ is the set of points $p$ in the future of $S$ such that every past causal path from $p$ must intersect $S$. If $S$ is contained in the convex hull of $\Gamma = \partial_\infty S$, the reader can convince him/herself that points in the convex hull and in the past (resp. future) of $S$ are contained in the past (resp. future) domain of dependence of $S$.

Given a weakly spacelike curve $\Gamma$ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$, the *domain of dependence* $D(\Gamma)$ of $\Gamma$ is the union of the past and future domains of dependence of any surface asymptotic to $\Gamma$ contained in $\mathcal{CH}(\Gamma)$.

As in the case of surfaces in hyperbolic and de Sitter space, the duality can be extended to non-smooth convex surfaces, defining

$$S^* = \{x \in \mathbb{A}\mathrm{d}\mathbb{S}^3 : x^\perp \text{ is a spacelike support plane of } S \text{ in } \mathbb{A}\mathrm{d}\mathbb{S}^3\}\,.$$

It is interesting to note the dual surface to the upper boundary $\partial_+\mathcal{CH}(\Gamma)$ of the convex hull of a weakly spacelike curve $\Gamma$ is the past boundary of the domain of dependence $D(\Gamma)$. This is very similar to the phenomenon we pointed out in Example 1.2.3 and in the following discussion. To be more precise, the dual $S^*$ of $S = \partial_+\mathcal{CH}(\Gamma)$ is a degenerate set with the above definition, namely the subset of $\partial_- D(\Gamma)$ of points which admit spacelike support planes. Indeed, by the description of duality in Anti-de Sitter space at the beginning of this subsection, a totally geodesic plane $P$ disjoint (in the future) from $\Gamma$ corresponds to a point $p = P^\perp$ such that every future spacelike geodesic from $p$ intersects $P$ at lenght $\pi/2$. The past domain of dependence is obtained as union of such points. By defining the dual surface to $\partial_+\mathcal{CH}(\Gamma)$ to be the future of the degenerate set $S^*$, we obtain the entire past boundary of the domain of dependence. The part which has been added is lightlike, and its lightlike support planes are the duals - by the duality discussed above - to points of the boundary at infinity $\Gamma$.

### 1.3.3 A special class of Anti-de Sitter manifolds containing a closed spacelike surface: maximal globally hyperbolic manifolds

Again, we are going to discuss a special class of Anti-de Sitter manifolds whose topology is $S \times \mathbb{R}$, where $S$ is a closed surface. As in the Minkowski case, we are interested in *maximal globally hyperbolic* manifolds, i.e. (recall Definition 1.1.11) manifolds containing a (closed) Cauchy surface - which intersects every inextensible causal path - and such that every Cauchy embedding into another globally hyperbolic manifold is surjective.

We now sketch the construction of a maximal globally hyperbolic manifold, similar to that explained in Subsection 1.1.4. Given the embedding data $(I, B)$ of a Cauchy surface $S$ in a globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold $M_0$, by lifting to the universal cover $\tilde{S}$ of $S$ we obtain the embedding data $(\tilde{I}, \tilde{B})$, which produce by Theorem 1.3.1 a unique (up to global isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^3$) embedding $\sigma : \tilde{S} \to \mathbb{A}\mathrm{d}\mathbb{S}^3$. Moreover, the embedding $\sigma$ is $\mathcal{R}$-equivariant, with respect to a *holonomy representation*

$$\mathcal{R} : \pi_1(S) \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)\,.$$

We can define the *limit set* $\Lambda(G)$ of a subgroup $G < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ - like for hyperbolic geometry - as the set of accumulation points in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ of $G$-orbits of points in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. The limit set of $\mathcal{R}(\pi_1(S))$ coincides with the asymptotic boundary of $\tilde{S}$, which is a weakly spacelike curve $\Gamma$.

Mess proved that the group $\mathcal{R}(\pi_1(S))$ acts freely and properly discontinuously on the domain of dependence $D$ of the limit set $\Lambda(\mathcal{R}(\pi_1(S)))$. The quotient $M = D/\mathcal{R}(\pi_1(S))$ is then a maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold, the maximal extension of $M_0$.

The convex hull of $\Lambda(G)$ (defined in the previous subsections) is invariant for the action of $\mathcal{R}(\pi_1(S))$. Therefore the quotient $\mathcal{CH}(\Lambda(G))/G$ is a submanifold of $M$, called *convex core* of $M$.

The identification $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \cong \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$ (by means of the choice of a fixed totally geodesic plane $P$) provides the *left* and *right holonomy*,

$$\rho_l, \rho_r : \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$$

by projecting on the left and right factor. In Subsection 1.3.1 we defined the left and right projections

$$\Phi_l, \Phi_r : \tilde{S} \to P\,,$$

where $P$ is a totally geodesic plane. The composition

$$\Phi_l \circ \sigma : \tilde{S} \to P$$

is $\rho_l$-equivariant, where we identify $\mathrm{PSL}_2\mathbb{R} \cong \mathrm{Isom}(\mathbb{H}^2)$ with the subgroup of $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ fixing $P$. Analogously

$$\Phi_r \circ \sigma : \tilde{S} \to P$$

is $\rho_r$-equivariant.

Mess showed that $\rho_l$ and $\rho_r$ are discrete and faithful representations, and therefore endow $S$ with a *left* and *right* hyperbolic metrics on $S$. Indeed $\mathbb{H}^2/\rho_l(\pi_1(S))$ and $\mathbb{H}^2/\rho_r(\pi_1(S))$ are hyperbolic surfaces. We will go back to this point in Subsection 2.2.

*Example* 1.3.2. The obvious example of maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifolds is provided by the *Fuchsian* ones. If $S$ is totally geodesic in a globally hyperbolic manifold $M_0$, then the first fundamental form is a hyperbolic metric and $B \equiv 0$. The fact that $M_0$ contains a totally geodesic plane is equivalent to the fact that the left and right holonomies are conjugate in $\mathrm{PSL}_2\mathbb{R}$. Indeed, in this case the limit set of the holonomy $\mathcal{R}$ is the boundary at infinity of the totally geodesic plane $P = \tilde{S}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. The maximal globally hyperbolic manifold is foliated by the surfaces at timelike distance $t$ from $S$, where $t \in (-\pi/2, \pi/2)$. Let $p = P^\perp$ be the dual point of $P$. The past domain of dependence of $P$ is the region of points of $\mathrm{I}^+(p)$ whose timelike distance from $p$ is in $(0, \pi/2]$. The same description holds for the future domain of dependence. The convex core is the totally geodesic surface $S$.

Finally, the metric of $M$ is of the form

$$-dt^2 + (\cos t)^2 g_\rho \, ,$$

where $\rho = \rho_l = \rho_r$ and $g_\rho$ is the hyperbolic metric of $\mathbb{H}^2/\rho(\pi_1(S))$.

We now sketch the reverse construction, also provided by Mess in [Mes07], avoiding to do not give the details of the arguments. Given two discrete and faithful representations

$$\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}_2\mathbb{R} \, ,$$

there exists an orientation-preserving homeomorphism $\varphi$ of $S^1 \cong \mathbb{R}\mathrm{P}^1$ which conjugates $\rho_1$ and $\rho_2$. The domain of dependence $D$ of the weakly spacelike curve $gr(\phi)$ is - almost by definition - invariant for the representation

$$\mathcal{R} = (\rho_1, \rho_1) : \pi_1(S) \to \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$$

and therefore $D/\mathcal{R}(\pi_1(S))$ is a maximal globally hyperbolic spacetime whose left and right holonomies coincide with $\rho_1$ and $\rho_2$.

## 1.4 Geometric transition

In this section we will consider the notion of *geometric transition*. Two different transition procedures will be described: the first, introduced by Jeffrey Danciger (see [Dan11, Dan13, Dan14]), is a rescaling procedure from $\mathbb{A}\mathrm{d}\mathbb{S}^3$ on one side, and $\mathbb{H}^3$ on the other side having limit in half-pipe geometry $\mathbb{HP}^3$. Although the rescaling procedures can be defined in every dimension, we will focus on three-dimensional manifolds here.

The second is the Lorentzian version of a more classical transition, involving spherical geometry, Euclidean geometry and hyperbolic geometry. Indeed, “zooming in” from a point on the sphere, one obtains in the limit the flat geometry of Euclidean spaces; the same can be obtained on the other side from hyperbolic geometry. Since Minkowski space is the tangent space of both Anti-de Sitter space and de Sitter space, a natural transition is defined from $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and from $\mathrm{d}\mathbb{S}^3$, with limit $\mathbb{R}^{2,1}$. Such transition is described for instance in [DGK13].

We will then show that these two transition procedures are well-behaved with respect to the dualities we defined in Subsections 1.1.2, 1.2.2 and 1.3.2. Some useful conclusions will be drawn from this observation. Geometric transition, in relation with the theory of immersed surfaces introduced in Subsections 1.1.1, 1.2.1 and 1.3.1, will be the subject of Chapter 9.

### 1.4.1 The transition $\mathbb{A}\mathrm{d}\mathbb{S}^3$-$\mathbb{HP}^3$-$\mathbb{H}^3$: blow-up of a plane

Let us consider the linear transformation $\mathfrak{r}_t$ of $\mathbb{R}^4$:

$$\mathfrak{r}_t(x^1, x^2, x^3, x^4) = \left( x^1, x^2, \frac{1}{t}x^3, x^4 \right) \, .$$

For every $t > 0$, $\mathfrak{r}_t$ is an isometry between the quadratic form $\langle \cdot, \cdot \rangle_{3,1}$ we already defined on $\mathbb{R}^{3,1}$ and the following quadratic form:

$$\langle x, y \rangle^t_{3,1} = (x^1)^2 + (x^2)^2 + t^2(x^3)^2 - (x^4)^2 \,.$$

The transformation $\mathfrak{r}_t$ therefore maps the hyperboloid $\mathbb{H}^3$ to

$$\mathfrak{r}_t(\mathbb{H}^3) = \{x : \langle x, x \rangle^t_{3,1} = -1\} \,, \tag{1.15}$$

and induces a projective transformation of $\mathbb{RP}^3$ which maps the projective model of $\mathbb{H}^3$ to the following projective domain:

$$\{x : \langle x, x \rangle^t_{3,1} < 0\} / \sim \,.$$

In the limit as $t \to 0$, $\mathfrak{r}_t(\mathbb{H}^3)$ converges (in the Hausdorff convergence of $\mathbb{R}^4$, for instance) to the half-pipe model

$$\mathbb{HP}^3 = \{x \in \mathbb{R}^{3,1} : \langle x, x \rangle_{2,0,1} < 0\} \,.$$

Observe that $\langle \cdot, \cdot \rangle^t_{2,2}$, for $t > 0$, endows $\mathfrak{r}_t(\mathbb{H}^3)$ with a metric which is isometric to the standard metric of $\mathbb{H}^3$. In the limit, however, we obtain the degenerate metric (1.7) of $\mathbb{HP}^3$. It is not difficult to see that the conjugate copies of $\mathrm{SO}_0(3,1) = \mathrm{Isom}(\mathbb{H}^3)$ acting on $\mathfrak{r}_t(\mathbb{H}^3)$ inside $\mathrm{PGL}_3\mathbb{R}$, namely

$$(\mathfrak{r}_t)\mathrm{SO}_0(3,1)(\mathfrak{r}_t)^{-1} \,,$$

converge to the group $\mathrm{Isom}(\mathbb{HP}^3) < \mathrm{PGL}_3\mathbb{R}$ we defined in Subsection 1.1.2.

This transition can also be regarded as a projective convergence in $\mathbb{RP}^3$ of the domain (1.15) to the projective model of half-pipe geometry (recall Subsection 1.1.2).

A similar rescaled limit is obtained "on the other side" from Anti-de Sitter space to half-pipe geometry. Now $\mathfrak{r}_t$ is an isometry between the quadratic form $\langle \cdot, \cdot \rangle_{2,2}$ of $\mathbb{R}^{2,2}$ and the quadratic form:

$$\langle x, y \rangle^t_{2,2} = (x^1)^2 + (x^2)^2 - t^2(x^3)^2 - (x^4)^2 \,.$$

As $t \to 0$, it turns out that

$$\mathfrak{r}_t(\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}) = \{x : \langle x, x \rangle^t_{2,2} = -1\}$$

converges to $\mathbb{HP}^3$. Recall $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ is the double cover $\{x \in \mathbb{R}^{2,2} : \langle x, x \rangle_{2,2} = -1\}$ of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Again the convergence can also be considered as a projective limit in $\mathbb{RP}^3$. As in the previous case, the rescaled copies $\mathfrak{r}_t(\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3})$ are endowed with a metric isometric to the usual metric of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, which for $t \to 0$ tends to the (degenerate) metric $g_{\mathbb{H}^2} + 0 \cdot dt^2$ of $\mathbb{HP}^3$. The groups of isometries of $\mathfrak{r}_t(\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3})$, namely

$$(\mathfrak{r}_t)\mathrm{SO}_0(2,2)(\mathfrak{r}_t)^{-1} \,,$$

converge again to $\mathrm{Isom}(\mathbb{HP}^3)$.

In [Dan11, Dan13, Dan14], Danciger studied the problem of *degeneration* and *regeneration* of hyperbolic structures (with cone singularities) and Anti-de Sitter structures (with tachyon singularities) on closed three-manifolds. We will discuss here a much simpler phenomenon, namely the degeneration of surfaces.

Suppose $\sigma_t$ is a pointwise-differentiable family of smooth embeddings of a disc in $\mathbb{H}^3$ and/or in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. In the latter case, we assume that the image of $\sigma_t$ is spacelike for every $t$. Suppose $\sigma_0$ is a diffeomorphism to the totally geodesic plane obtained by the condition $x^3 = 0$ (it does not really make a difference whether in $\mathbb{R}^{3,1}$ or $\mathbb{R}^{2,2}$). We identify this totally geodesic plane to $\mathbb{H}^2$. Write

$$\sigma_t(p) = (x_t^1(p), x_t^2(p), x_t^3(p), x_t^4(p))\,,$$

with $\sigma_0(p) = (x^1(p), x^2(p), x^3(p), x^4(p))$. Then $x^3(p) = 0$. Rescaling by $\mathfrak{r}_t$, we obtain

$$\lim_{t\to 0}(\mathfrak{r}_t \circ \sigma_t)(p) = (x^1(p), x^2(p), \left.\frac{d}{dt}\right|_{t=0} x_t^3(p), x^4(p))\,.$$

Hence the limit is a spacelike embedded disc in $\mathbb{HP}^3$, namely the graph of the function $f : \mathbb{H}^2 \to \mathbb{R}$ defined by

$$f = \dot{x}^3 \circ (\sigma_0)^{-1} = \left(\left.\frac{d}{dt}\right|_{t=0} x^3\right) \circ (\sigma_0)^{-1}\,.$$

### 1.4.2 The transition $\mathbb{A}\mathrm{d}\mathbb{S}^3$-$\mathbb{R}^{2,1}$-$\mathrm{d}\mathbb{S}^3$: blow-up of a point

We now describe a second transition procedure which is essentially a blow-up of a point in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ or in $\mathrm{d}\mathbb{S}^3$. We consider again the linear transformation

$$\mathfrak{r}_t^*(x^1, x^2, x^3, x^4) = \left(\frac{1}{t}x^1, \frac{1}{t}x^2, x^3, \frac{1}{t}x^4\right)\,.$$

Observe that $\mathfrak{r}_t^*$ preserves the point $(0, 0, 1, 0)$, which is the dual point - both in the $\mathbb{H}^3$-$\mathrm{d}\mathbb{S}^3$ and in the $\mathbb{A}\mathrm{d}\mathbb{S}^3$-$\mathbb{A}\mathrm{d}\mathbb{S}^3$ duality - of the totally geodesic plane $\{x^3 = 0\}$. We start by considering the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ case. The transformation $\mathfrak{r}_t^*$ maps the double cover of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ to the set

$$\mathfrak{r}_t^*(\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}) = \{t^2(x^1)^2 + t^2(x^2)^2 - (x^3)^2 - t^2(x^4)^2 = -1\}\,,$$

which converges as $t \to 0$ to $\{x^3 = \pm 1\}$. If endowed with the metric induced from $\langle \cdot, \cdot \rangle_{3,1}$, these are two copies of Minkowski space $\mathbb{R}^{2,1}$ (where the metric takes the form $(dx^1)^2 + (dx^2)^2 - (dx^4)^2$). Again it is more natural to consider $\mathfrak{r}_t^*$ as a projective transformation, which rescales $\mathbb{A}\mathrm{d}\mathbb{S}^3 \subset \mathbb{RP}^3$ to the affine chart $\{x^3 \neq 0\}$. Of course a completely analogous procedure holds for $\mathrm{d}\mathbb{S}^3$.

The remarkable difference with the blow-up of a plane we described in the previous section is that in this case, to obtain the Minkowski metric in the limit, the domains $\mathfrak{r}_t^*(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ have to be endowed with the induced metric from the ambient space, not with a rescaled metric.

Let us compute what the limit is in this case. Suppose

$$\varsigma_t(p) = (x_t^1(p), x_t^2(p), x_t^3(p), x_t^4(p))$$

is a differentiable family of smooth maps from a disc with $\varsigma_0(p) = (0,0,1,0)$, defined in a small right interval of $t = 0$. Then we obtain

$$\lim_{t\to 0} \mathfrak{r}_t^*(\varsigma_t(p)) = (\dot{x}^1(p), \dot{x}^2(p), 1, \dot{x}^3(p))\,. \tag{1.16}$$

*Remark* 1.4.1. The reader might consider more natural the procedure to blow-up Anti-de Sitter/de Sitter space to Minkowski space by "homothety", in such a way that the curvature gets rescaled by $t^2$. Although next section should convince that the procedure we consider is very natural for the behaviour of duality, we want to show that the limits obtained in the two cases are the same.

A simple way to blow-up $\mathbb{A}\mathrm{d}\mathbb{S}^3$ to $\mathbb{R}^{2,1}$ by keeping the curvature constant is the following. Consider, for $t > 0$, the map

$$\mathfrak{c}_t(x^1, x^2, x^3, x^4) = \left(\frac{1}{t}x^1, \frac{1}{t}x^2, 1 + \frac{1}{t}(x^3 - 1), \frac{1}{t}x^4\right)\,.$$

Observe that $\mathfrak{c}_t(\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3})$, with the metric induced from $\mathbb{R}^{2,2}$, is a homothetic image of $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ and has constant curvature $-t^2$. For every time $t$, $\mathfrak{c}_t(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ contains the point $(0,0,1,0)$ and is tangent to the plane $\{x^3 = 1\}$. The construction for $\widehat{\mathrm{d}\mathbb{S}^3}$ is completely analoguos. Let us compute the rescaled limit of a smooth path

$$\gamma(t) = (x_t^1, x_t^2, x_t^3, x_t^4) \in \widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3} \text{ or } \widehat{\mathrm{d}\mathbb{S}^3}$$

with $\gamma(0) = (0,0,1,0)$. Using that $(x^1)^2 + (x^2)^2 \pm (x^3)^2 - (x^4)^2$, and thus at first order $\dot{x}^3 = 0$, one shows that

$$\lim_{t\to 0} \mathfrak{c}_t(\gamma(t)) = (\dot{x}^1, \dot{x}^2, 1, \dot{x}^3)\,,$$

which is the same result as obtained in (1.16).

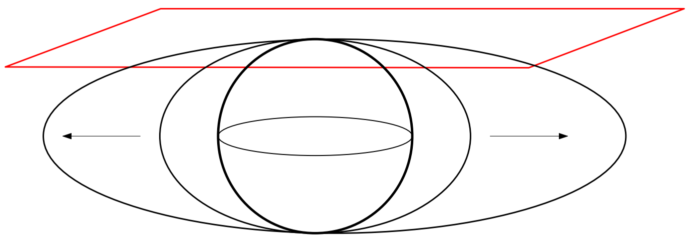


**Figure 1.5:** Heuristically, the rescling of $\mathrm{d}\mathbb{S}^3$ and $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is the analogous of the deformation of a sphere towards its tangent plane at a fixed point.

### 1.4.3 Duality is preserved

In this subsection we will prove that the two transition procedures we described in Subsection 1.4.1 and 1.4.2 preserve the natural dualities $\mathbb{A}\mathrm{d}\mathbb{S}^3$-$\mathbb{A}\mathrm{d}\mathbb{S}^3$ and $\mathbb{H}^3$-$\mathrm{d}\mathbb{S}^3$ we have already introduced. More precisely, in the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ context, we prove the following proposition:

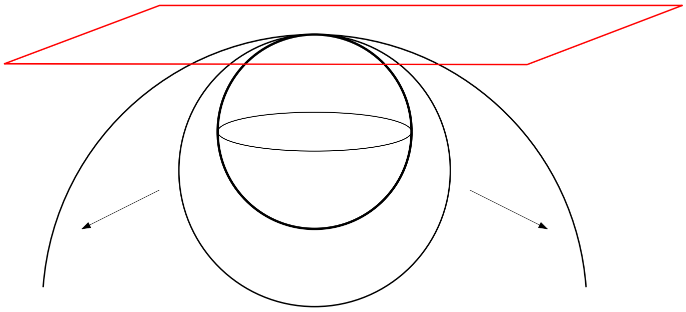


**Figure 1.6:** Another transition procedure of the sphere to Euclidean plane, by constant curvature going to zero.

**Proposition 1.4.2.** *Let $\sigma_t : \mathbb{H}^2 \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ be a $C^2$ family of smooth strictly convex embeddings for $t \geq 0$, with $\sigma_0$ an isometric embedding onto the totally geodesic plane $P_0 = \{x^3 = 0\}$. Let $\sigma_t^* : \mathbb{H}^2 \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ be the family of embeddings (for $t \neq 0$) dual to $\sigma_t$, with $\sigma_0 \equiv P_0^\perp = (0,0,1,0)$. Then the rescaled limits*

$$\sigma := \lim_{t\to 0}(\mathfrak{r}_t \circ \sigma_t) : \mathbb{H}^2 \to \mathbb{HP}^3$$

*and*

$$\sigma^* := \lim_{t\to 0}(\mathfrak{r}_t^* \circ \sigma_t^*) : \mathbb{H}^2 \to \mathbb{R}^{2,1}$$

*are convex surfaces dual to each other in the $\mathbb{R}^{2,1}$-$\mathbb{HP}^3$ duality.*

Before proving Proposition 1.4.2, we consider the case of convergence of totally geodesic planes.

**Lemma 1.4.3.** *Let $P_t : \mathbb{H}^2 \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ be a $C^1$ family of isometric embeddings onto totally geodesic planes of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, with $P_0 = \{x^3 = 0\}$. Let $p_t = P_t^*$ be the dual points in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Then the rescaled limit*

$$\lim_{t\to 0}(\mathfrak{r}_t \circ P_t) : \mathbb{H}^2 \to \mathbb{HP}^3$$

*has image a plane in $\mathbb{HP}^3$ dual to the point*

$$p := \lim_{t\to 0} \mathfrak{r}_t^*(p_t) \in \mathbb{R}^{2,1}\,.$$

*Proof.* Suppose $P_t(q) = (x_t^1(q), x_t^2(q), x_t^3(q), x_t^4(q))$ for $q \in \mathbb{H}^2$, and $p_t = (p_t^1, p_t^2, p_t^3, p_t^4)$. By assumption $x_0^3 = 0$ and therefore $p_0 = (0,0,1,0)$. Then we have

$$\langle P_t(q), p_t\rangle_{2,2} = 0\,.$$

From this relation we obtain

$$x_t^1(q)\frac{p_t^1}{t} + x_t^2(q)\frac{p_t^2}{t} - \frac{x_t^3(q)}{t}p_t^3 - x_t^4(q)\frac{p_t^4}{t} = 0 \tag{1.17}$$

and taking the limit as $t \to 0$ one gets

$$\dot{x}^3(q) = x_0^1(q)\dot{p}^1 + x_0^2(q)\dot{p}^2 - x_0^4(q)\dot{p}^4 = \langle P_0(q), \dot{p}\rangle_{2,1} \tag{1.18}$$

where $p = (\dot{p}^1, \dot{p}^2, \dot{p}^4)$. This shows that the rescaled limit $\lim_{t\to 0}(\mathfrak{r}_t \circ P_t)$ in $\mathbb{HP}^3$ is the graph of the support function of $\mathrm{I}^+(p)$, i.e. the plane in $\mathbb{HP}^3$ dual to the point $p$. $\square$

*Proof of Proposition 1.4.2.* Given the family $\sigma_t$, the rescaled limit $\lim_{t\to 0}(\mathfrak{r}_t \circ \sigma_t)$ defines a convex surface in $\mathbb{HP}^3$. For every point $q \in \mathbb{H}^2$ we consider the $C^1$ family of planes $P_t$ tangent to $\sigma_t(\mathbb{H}^2)$ at $\sigma_t(q)$. The points $\sigma_t^*(q)$ coincide with $p_t = P_t^\perp$ and thus, by Lemma 1.4.3, $\lim_{t\to 0} \mathfrak{r}_t^*(p_t)$ is the point of $\mathbb{R}^{2,1}$ dual to the tangent plane to $\sigma(\mathbb{H}^2)$ at $\sigma(q) \in \mathbb{HP}^3$. $\square$

The analogous proposition for the $\mathbb{H}^3$-$\mathrm{d}\mathbb{S}^3$ duality is proved in the same way.

**Proposition 1.4.4.** *Let $\sigma_t : \mathbb{H}^2 \to \mathbb{H}^3$ be a $C^2$ family of smooth strictly convex embeddings for $t \geq 0$, with $\sigma_0$ an isometric embedding onto the totally geodesic plane $P_0 = \{x^3 = 0\}$. Let $\sigma_t^* : \mathbb{H}^2 \to \mathrm{d}\mathbb{S}^3$ be the family of embeddings (for $t \neq 0$) dual to $\sigma_t$, with $\sigma_0 \equiv P_0^\perp = (0, 0, 1, 0)$. Then the rescaled limits*

$$\sigma := \lim_{t\to 0}(\mathfrak{r}_t \circ \sigma_t) : \mathbb{H}^2 \to \mathbb{HP}^3$$

*and*

$$\sigma^* := -\lim_{t\to 0}(\mathfrak{r}_t^* \circ \sigma_t^*) : \mathbb{H}^2 \to \mathbb{R}^{2,1}$$

*are convex surfaces dual to each other in the $\mathbb{R}^{2,1}$-$\mathbb{HP}^3$ duality.*

The only difference is that here the two rescaled limits coincide up to a change of sign. This is due to the fact that, while in Anti-de Sitter space the dual to a concave surface is a convex surface, this is no longer true in the $\mathbb{H}^3$-$\mathrm{d}\mathbb{S}^3$ duality. For instance, choosing the normal unit vector field $N$ to a convex surface (nearby the $x^3 = 0$ totally geodesic plane) in $\mathbb{H}^3$ to have positive $x^3$-component, the dual surface in $\mathrm{d}\mathbb{S}^3$ is still convex when we choose its normal unit vector field $N'$ coherently with $N$. This means (nearby the point $(0, 0, 1, 0)$) that $N'$ has negative $x^4$-component. When rescaling to Minkowski space, however, we consider convex surface with respect to the future unit normal vector, which is the one with positive $x^4$-component. This explains why the sign has to be switched in Proposition 1.4.4. In the proof, the only change arises from the signature in Equation (1.17) and thus in Equation (1.18).

# Chapter 2

# Teichmüller spaces

## 2.1 Riemann surfaces and hyperbolic geometry

In this section we want to describe the relation between Riemann surfaces and hyperbolic structures on a surface, and the deformation space of such objects. Recall that a Riemann surface is a smooth surface $S$ endowed with a *complex atlas* $\mathcal{A}$ (whose elements are charts $\phi_i : U_i \to \mathbb{C}$, for $\{U_i\}$ is a covering of $S$), such that the changes of coordinates are holomorphic.

It is a known fact that complex structures are equivalent to *conformal structures* on a 2-dimensional surface. Recall that a conformal structure is a Riemannian metric on $S$ up to the equivalence relation of *conformality*: two Riemannian metrics $g$ and $g'$ are *conformal* if there exists a smooth function $\varphi$ such that

$$g' = e^{2\eta} g\,.$$

Given a Riemannian metric $g$ on $S$, for a classical theorem ([Lic16, Che55a]) there exist *isothermal coordinates* in a neighborhood of every point, namely a system of coordinates in which the metric is conformal to the Euclidean metric of $\mathbb{C} \cong \mathbb{R}^2$. In other words, there exist a covering $\{U_i\}$ and charts $\phi_i : U_i \to \mathbb{C}$ so that the pushforward of the metric $g$ in $\phi_i(U_i)$ is of the form

$$e^{2\eta_i}|dz|^2\,.$$

By a direct computation, one checks that a diffeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ is conformal for the Euclidean metrics on $\mathbb{C}$ if and only if it is holomorphic. Hence isothermal coordinates provide a complex atlas for $S$. Vice versa, given a complex structure $\mathcal{A} = \{\phi_i : U_i \to \mathbb{C}\}$ on $S$, one can put a Riemannian metric on every $U_i$ by simply taking the pull-back of the Euclidean metric $(\phi_i)^*(|dz|^2)$. Using a partition of unity, one obtains a Riemannian metric on $S$ whose underlying complex structure is compatible with the original complex atlas $\mathcal{A}$.

In is an even deeper occurrence of dimension 2 that complex structures on a surface $S$ are in bijection with *almost-complex structures*. Given a Riemann surface, one can define a $(1,1)$-tensor $J$ on $S$, which acts on every tangent plane by multiplication by $i$, in every coordinate chart $\phi_i : U_i \to \mathbb{C}$. In other words,

$$J(v) = (\phi_i)^*(i(\phi_i)_*(v))\,,$$

where the tangent vector $v$ is thought as a complex number. This definition does not depend on the choice of charts compatible with the complex structure. The tensor $J$ has the property that

$$J^2 = -E\,.$$

In general, however, not all almost-complex structures (i.e. $(1,1)$-tensors $J$ which square to minus the identity) on a complex manifolds are *integrable*, namely they are obtained from complex structures. The Newlander-Nirenberg Theorem ([NN57]) describes an obstruction for integrability. This obstruction is always trivial in complex dimension 1, meaning that all almost-complex structures are integrable.

There is yet another interpretation of complex structures on a surface, in terms of Riemannian metrics of constant curvature. This is a consequence of the Uniformization Theorem ([Poi08], [Koe09]; see [Jos06] for a modern treatment).

**Uniformization Theorem**. Every simply connected Riemann surface is conformally equivalent to $\mathbb{C}$, $\mathbb{D}$ or the Riemann sphere.

As a Corollary, every closed Riemann surface of genus $g \geq 2$ (which is the case we are interested in this thesis) admits a unique Riemannian metric of constant curvature $-1$. Indeed, the group of conformal transformation of the disc $\mathbb{D}$ coincides with the group of isometries of hyperbolic plane in the Poincaré disc model. The same holds for the positive constant curvature metric on the sphere and the flat metric on the plane. Thus, by Gauss-Bonnet, if the genus is at least two, the universal cover of $S$ is conformally equivalent to $\mathbb{D}$ and the surface $S$ is endowed with a unique metric of constant curvature $-1$, namely a *hyperbolic metric*, compatible with the complex structure.

### 2.1.1 The definition of Fricke space of a closed surface

We will now introduce the notion of Fricke space of a closed surface.

**Definition 2.1.1.** Given a closed surface $S$ of genus $g \geq 2$, we define

$$\mathcal{T}(S) = \{\text{hyperbolic metrics on } S\}/\sim\,,$$

where two metrics $h$ and $h'$ are equivalent for the relation $\sim$ if there exists an isometry $A : (S, h) \to (S, h')$ isotopic to the identity.

Although the original Teichmüller viewpoint will be introduced later in this chapter, we will adopt the denomination of Teichmüller space for $\mathcal{T}(S)$, instead of the more precise Fricke space. By the above discussion, $\mathcal{T}(S)$ is also the space of complex structures on $S$, up to conformal transformations isotopic to the identity.

By a well-known theorem, $\mathcal{T}(S)$ is homeomorphic to a ball of dimension $6g-6$.

### 2.1.2 Holonomy representations

Given a closed surface $S$ with a hyperbolic metric $h$, the universal cover $(\tilde{S}, \tilde{h})$ is isometric to $\mathbb{H}^2$. Indeed, $\mathbb{H}^2$ is the unique complete simply connected surface up to isometres. In a very similar way to what has been explained in the several examples of Chapter 1, one obtains a *developing map*

$$\mathrm{dev} : \tilde{S} \to \mathbb{H}^2 ,$$

which is $\rho$-equivariant, where

$$\rho : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^2)$$

is the *holonomy representation*. It turns out that $\rho(\pi_1(S))$ acts freely and properly discontinuously on $\mathbb{H}^2$ and $\mathbb{H}^2/\rho(\pi_1(S))$ is isometric to the original hyperbolic surface $(S, h)$. The condition that $\rho(\pi_1(S))$ acts freely and properly discontinuously is equivalent to the fact that the representation $\rho$ is discrete and faithful. It turns out that this construction gives a well-defined map

$$\mathbf{hol} : \mathcal{T}(S) \to \mathcal{R}(\pi_1(S), \mathrm{SO}_0(2,1)) /\!\!/ \mathrm{SO}_0(2,1) .$$

The object in the target of **hol** is the space of representations of $\pi_1(S)$ in $\mathrm{SO}(2,1)$, quotiented by the action of $\mathrm{SO}(2,1)$ by conjugation. The space of representations is endowed with a compact-open topology. The double quotient is a standard construction to eliminate singular points, since this space is not a manifold in general (it might not even be Hausdorff).

By a theorem of Goldman, **hol** is a homeomorphism onto a connected component of $\mathcal{R}(\pi_1(S), \mathrm{SO}(2,1)) /\!\!/ \mathrm{SO}(2,1)$. More precisely, to every representation $\rho$ in $\mathcal{R}(\pi_1(S), \mathrm{SO}(2,1))$ one associates a flat bundle over $S$ obtained by the quotient of $\tilde{S} \times \mathbb{H}^2$ by the diagonal action of $\pi_1(S)$:

$$\alpha(x, y) = (\alpha x, \rho(\alpha) y) .$$

The Euler class of flat bundles descends to a continuous function

$$e : \mathcal{R}(\pi_1(S), \mathrm{SO}(2,1)) /\!\!/ \mathrm{SO}(2,1) \to \{2 - 2g, \ldots, 0, \ldots, 2g - 2\} .$$

Goldman ([Gol82, Gol80]) proved that the space of discrete and faithful representation coincides with $e^{-1}(2g-2)$ (or $e^{-1}(2-2g)$, depending on the orientation). Hence $\mathcal{T}(S)$ can be identified to a connected component of $\mathcal{R}(\pi_1(S), \mathrm{SO}(2,1)) /\!\!/ \mathrm{SO}(2,1)$.

### 2.1.3 The tangent space to the space of representations

We can now give a description of the tangent space of $\mathcal{T}(S)$ at a given point $[h]$. We will use the description of Subsection 2.1.2 of Teichmüller space. See [Gol84] for more details. Assume $[h]$ corresponds to the representation $\rho : \pi_1(S) \to \mathrm{SO}(2,1)$.

Given a differentiable path of representations $\rho_t$ such that $\rho_0 = \rho$, we put

$$\dot{\rho}(\alpha)(x) = \left.\frac{d}{dt}\right|_{t=0} \rho_t(\alpha) \circ \rho_0(\alpha)^{-1}(x) \in \mathfrak{so}(2,1) .$$

It turns out that $\dot{\rho}$ defines a *cocycle* for the adjoint action of $\rho$:

$$\dot{\rho}(\alpha\alpha') = \mathrm{Ad}\rho(\alpha)(\dot{\rho}(\alpha')) + \dot{\rho}(\alpha) .$$

Moreover, if two such paths $\rho_t$ and $\rho'_t$ differ by conjugation by a smooth family $\eta_t \in \mathrm{SO}(2,1)$, with $\eta_0 = \mathrm{id}$, then the associated cocycles $\dot\rho$ and $\dot\rho'$ differ by the *coboundary*

$$\dot\rho(\alpha) - \dot\rho'(\alpha) = \mathrm{Ad}\rho(\alpha)\eta_0 - \eta_0\,,$$

for some fixed $\eta_0 \in \mathfrak{so}(2,1)$. Hence the tangent space to the character variety $\mathcal{R}(\pi_1(S), \mathrm{SO}(2,1))/\!\!/\mathrm{SO}(2,1)$ is canonically identified with the cohomology group $H^1_{\mathrm{Ad}\rho}(\pi_1(S), \mathfrak{so}(2,1))$, i.e. the quotient of $\mathfrak{so}(2,1)$-valued cocycles (for the action of $\mathrm{Ad}\rho$) by the subspace of coboundaries.

## 2.2 Moduli spaces of three-manifolds

In this section we will describe a striking relation between some spaces of three-manifolds containing a closed surface, as described in Subsections 1.1.4 (maximal globally hyperbolic flat manifolds), 1.2.3 (quasi-Fuchsian hyperbolic manifolds) and 1.3.3 (maximal globally hyperbolic Anti-de Sitter manifolds), and Teichmüller space of the surface.

### 2.2.1 Maximal globally hyperbolic flat manifolds

Fix a topological surface $S$ of genus $g \geq 2$. We will consider here the space of flat Lorentzian metrics on $S \times \mathbb{R}$ such that the slices $S \times \{*\}$ are Cauchy surfaces, which cannot be embedded isometrically in a larger spacetime with the same properties. As in the case of Fricke/Teichmüller space, the equivalence relation we put on this huge space of metrics is $g \sim g'$ if there exists an isometry between $g$ and $g'$ isotopic to the identity. In this way we obtain a space $\mathcal{MGH}_{\mathbb{R}^{2,1}}(S)$ which classifies all flat maximal globally hyperbolic structures on $S \times \mathbb{R}$.

Mess proved in [Mes07] that $\mathcal{MGH}_{\mathbb{R}^{2,1}}(S)$ is homeomorphic to $T\mathcal{T}(S)$, the tangent bundle of $\mathcal{T}(S)$. Let us give a sketch of how the correspondence is proved.

Given such a spacetime $M$, by analysing the action of the linear part of the holonomy on the bundle of lightlike vectors tangent to $M$, Mess showed that the linear part

$$\rho : \pi_1(S) \to \mathrm{SO}_0(2,1)$$

of the holonomy $\mathcal{R} : \pi_1(S) \to \mathrm{Isom}(\mathbb{R}^{2,1})$ of $M$ has maximal Euler class, in the sense described in the previous section. Hence $\rho$ is a Fuchsian representation. We have already mentioned in Subsection 1.1.4 that the translation part of the holonomy defines a cocycle in

$$H^1_\rho(\pi_1(S), \mathbb{R}^{2,1})\,.$$

However, there is a natural isomorphism for the cohomology groups $H^1_\rho(\pi_1(S), \mathbb{R}^{2,1})$ and $H^1_{\mathrm{Ad}\rho}(\pi_1(S), \mathfrak{so}(2,1))$. Indeed, the Minkowski cross product is defined by $v \boxtimes w = *(v \wedge w)$, where $* : \Lambda^2(\mathbb{R}^{2,1}) \to \mathbb{R}^{2,1}$ is the Hodge operator associated to the Minkowski product. The map which associates to $t \in \mathbb{R}^{2,1}$ the linear map

$$v \mapsto t \boxtimes v$$

is a vector space isomorphism between $\mathbb{R}^{2,1}$ and $\mathfrak{so}(2,1)$, which is equivariant for the action of $\mathrm{SO}(2,1)$ on $\mathbb{R}^{2,1}$ (the natural action) and on $\mathfrak{so}(2,1)$ (the adjoint action). Hence there is a natural identification

$$H^1_{\mathrm{Ad}\rho}(\pi_1(S), \mathfrak{so}(2,1)) \cong H^1_\rho(\pi_1(S), \mathbb{R}^{2,1}) .$$

Since the LHS - by the results in [Gol84] discussed in Subsection 2.1.3 - is the tangent space to Teichmüller space at the point corresponding to the Fuchsian representation $\rho$, this construction uniquely associates an element in $T\mathcal{T}(S)$ to every class of maximal globally hyperbolic flat spacetimes (up to isotopy). Mess proved (and in Chapter 5 we will basically give another proof of this well-known fact) that pairs of a Fuchsian representation $\rho$ and a cocycle in $H^1_\rho(\pi_1(S), \mathbb{R}^{2,1})$ are in 1-1 correspondence with the holonomies of maximal globally hyperbolic spacetimes, thus concluding the claim on the structure of $\mathcal{MGH}_{\mathbb{R}^{2,1}}(S)$.

### 2.2.2 Maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifolds and quasi-Fuchsian manifolds

We now give a brief description of analogous results for the spaces $\mathcal{MGH}_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(S)$ of maximal globally hyperbolic Anti-de Sitter structures on $S \times \mathbb{R}$, up to isometries isotopic to the identity. In Subsection 1.3.3 we have already mentioned that the left and right holonomy of a maximal globally hyperbolic flat spacetimes are Fuchsian representations in $\mathrm{SO}(2,1)$. This was proved by Mess by showing again that the Euler class is maximal. As sketched at the end of Subsection 1.3.3, by using the fact that any two Fuchsian representations are topologically conjugate, Mess showed that for every pair $(\rho_l, \rho_r)$ of Fuchsian representations it is possible to construct a maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$ spacetimes having the $\rho_l$ and $\rho_r$ as left and right holonomy. In this was it was proved in [Mes07] that

$$\mathcal{MGH}_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(S) \cong \mathcal{T}(S) \times \mathcal{T}(S) .$$

This parametrization is analogous to the classical parametrization of the space of quasi-Fuchsian manifolds (again, up to isotopy), which we denote $\mathcal{QF}(S)$, by means of two copies of Teichmüller space:

$$\mathcal{QF}(S) \cong \mathcal{T}(S) \times \mathcal{T}(S) .$$

Given a quasi-Fuchsian, the limit set $\Lambda$ of its holonomy representation is a Jordan curve in $\partial_\infty \mathbb{H}^3$. Moreover, we have already remarked in Section 1.2 that $\partial_\infty \mathbb{H}^3$ is endowed with a conformal structure and the group of isometries $\mathrm{Isom}(\mathbb{H}^3)$ acts on $\partial_\infty \mathbb{H}^3$ by conformal transformations. Hence each connected component of $\partial_\infty \mathbb{H}^3 \setminus \Lambda$ is invariant for the holonomy representation and provides two complex structures on $S$, by taking the quotient. In this way, two elements in Teichmüller space are associated to every quasi-Fuchsian manifolds. The viceversa is a consequence of the following famous theorem ([Ber60]).

**Bers' Simultaneous Uniformization Theorem.** Given every pair of complex structures $\mathcal{A}_1$ and $\mathcal{A}_2$ on a closed surface $S$ of genus $g \geq 2$, there exists a quasi-Fuchsian group $\Gamma \in \mathrm{Isom}(\mathbb{H}^3)$ such that

$$\Omega_+/\Gamma \cong (S, \mathcal{A}_1) \quad \text{and} \quad \Omega_-/\Gamma \cong (S, \mathcal{A}_2)$$

where $\Omega_+$ and $\Omega_-$ are the two connected components of the complement of the limit set of $\Gamma$ in $\partial_\infty \mathbb{H}^3$.

## 2.3 Quasiconformal mappings and universal Teichmüller space

The aim of this section is to introduce the theory of quasiconformal mappings and universal Teichmüller space. We will give a brief account of the very rich and developed theory. Useful references are [Gar87, GL00, Ahl06, FM07] and the nice survey [Sug07].

We start by recalling the definition of quasiconformal map.

**Definition 2.3.1.** Given a domain $\Omega \subset \mathbb{C}$, an orientation-preserving homeomorphism $f : \Omega \to f(\Omega) \subset \mathbb{C}$ is *quasiconformal* if $f$ is absolutely continuous on lines and there exists a constant $k < 1$ such that

$$|\partial_{\bar{z}} f| \leq k |\partial_z f| \, .$$

Let us denote $\mu_f = \partial_{\bar{z}} f / \partial_z f$, which is called *complex dilatation* of $f$. This is well-defined almost everywhere, hence it makes sense to take the $L_\infty$ norm. Thus a homeomorphism $f : \Omega \to f(\Omega) \subset \mathbb{C}$ is quasiconformal if $||\mu_f||_\infty < 1$. Moreover, a quasiconformal map as in Definition 2.3.1 is called $K$-*quasiconformal*, where

$$K = \frac{1+k}{1-k} \, .$$

It turns out that the best such constant $K \in [1, +\infty)$ represents the *maximal dilatation* of $f$, i.e. the supremum over all $z \in \Omega$ of the ratio between the major axis and the minor axis of the ellipse which is the image of a unit circle under the differential $d_z f$.

It is known that a 1-quasiconformal map is conformal, and that the composition of a $K_1$-quasiconformal map and a $K_2$-quasiconformal map is $K_1 K_2$-quasiconformal. Hence composing with conformal maps does not change the maximal dilatation.

Actually, there is an explicit formula for the complex dilatation of the composition of two quasiconformal maps $f, g$ on $\Omega$:

$$\mu_{g \circ f^{-1}} = \frac{\partial_z f}{\overline{\partial_z f}} \frac{\mu_g - \mu_f}{1 - \overline{\mu_f} \mu_g} \, . \tag{2.1}$$

Using Equation (2.1), one can see that $f$ and $g$ differ by post-composition with a conformal map if and only if $\mu_f = \mu_g$ almost everywhere. We now mention the classical and important result of existence of quasiconformal maps with given complex dilatation.

**Measurable Riemann mapping Theorem**. Given any measurable function $\mu$ on $\mathbb{C}$ there exists a unique quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 0$, $f(1) = 1$ and $\mu_f = \mu$ almost everywhere in $\mathbb{C}$.

The uniqueness part of Measurable Riemann mapping Theorem means that every two solutions (which can be thought as maps on the sphere $\widehat{\mathbb{C}}$) of the equation

$$(\partial_z f)\mu = \partial_{\bar{z}} f$$

differ by post-composition with a Möbius transformation of $\widehat{\mathbb{C}}$. Indeed, post-composing with a Möbius transformation allows the freedom to choose the image of a triple of points, and the statement of Measurable Riemann mapping Theorem above implicitely assumes the normalization $f(\infty) = \infty$.

Given any fixed $K \geq 1$, $K$-quasiconformal mappings have an important compactness property. See [Gar87] or [Leh87].

**Theorem 2.3.2.** *Let $K > 1$ and $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a sequence of $K$-quasiconformal mappings such that, for three fixed points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$, the mutual spherical distances are bounded from below: there exists a constant $C_0 > 0$ such that*

$$d_{\mathbb{S}^2}(f_n(z_i), f_n(z_j)) > C_0$$

*for every $n$ and for every choice of $i, j = 1, 2, 3$, $i \neq j$. Then there exists a subsequence $f_{n_k}$ which converges uniformly to a $K$-quasiconformal map $f_\infty : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

### 2.3.1 Quasiconformal deformations of the disc

We are now ready to introduce the first model of universal Teichmüller space. It turns out that every quasiconformal homeomorphisms of $\mathbb{D}$ to itself extends to the boundary $\partial\mathbb{D} = S^1$. Let us consider the space:

$$QC(\mathbb{D}) = \{\Phi : \mathbb{D} \to \mathbb{D} \text{ quasiconformal}\} / \sim$$

where $\Phi \sim \Phi'$ if and only if $\Phi|_{S^1} = \Phi'|_{S^1}$. Universal Teichmüller space is then defined as

$$\mathcal{T}(\mathbb{D}) = QC(\mathbb{D})/\text{Möb}(\mathbb{D}) \,,$$

where Möb$(\mathbb{D})$ is the subgroup of Möbius transformations of $\mathbb{D}$. Equivalently, $\mathcal{T}(\mathbb{D})$ is the space of quasiconformal maps $\Phi : \mathbb{D} \to \mathbb{D}$ which fix $1$, $i$ and $-1$ up to the same relation $\sim$.

Such quasiconformal homeomorphisms of the disc can be obtained in the following way. Given a domain $\Omega$, elements in the unit ball of the (complex-valued) Banach space $L^\infty(\mathbb{D})$ are called *Beltrami differentials* on $\Omega$. Let us denote Belt$(\mathbb{D})$ this unit ball. Given any $\mu$ in Belt$(\mathbb{D})$, let us define $\hat{\mu}$ on $\mathbb{C}$ by extending $\mu$ on $\mathbb{C} \setminus \mathbb{D}$ so that

$$\hat{\mu}(z) = \overline{\mu(1/\bar{z})} \,.$$

Now let $f : \mathbb{C} \to \mathbb{C}$ be the quasiconformal map with Beltrami differential $\mu$, whose existence is provided by Measurable Riemann mapping Theorem. We assume $f$ fixes the three points 1, $i$ and $-1$ on $\partial\mathbb{D}$. It is not difficult to check that $f$ and $g(z) = 1/\overline{f(1/\overline{z})}$ both have $\mu$ as Beltrami coefficient (i.e. $\mu = \mu_f = \mu_g$) and satisfy the same normalization (i.e. fixing 1, $i$ and $-1$). Hence by the uniqueness part of Measurable Riemann mapping Theorem, $f = g$. This implies that $f$ maps $\partial\mathbb{D}$ to itself, and thus $f$ restricts to a homeomorphism of $\mathbb{D}$ to itself.

The function $f$ obtained in this way from $\mu \in \mathrm{Belt}(\mathbb{D})$ will be usually denoted by $f^\mu$.

The Teichmüller distance on $\mathcal{T}(\mathbb{D})$ is defined as

$$d_{\mathcal{T}(\mathbb{D})}([\Phi],[\Phi']) = \frac{1}{2}\inf\log K(\Phi_1^{-1}\circ\Phi_1')\,,$$

where the infimum is taken over all quasiconformal maps $\Phi_1 \in [\Phi]$ and $\Phi_1' \in [\Phi']$. It can be shown that $d_{\mathcal{T}(\mathbb{D})}$ is a well-defined distance on Teichmüller space, and $(\mathcal{T}(\mathbb{D}), d_{\mathcal{T}(\mathbb{D})})$ is a complete metric space.

### 2.3.2 Quasisymmetric homeomorphisms of the circle

We will introduce here another model of Teichmüller space, namely, the space of quasisymmetric homeomorphism of the circle.

We think here at $S^1$ as the boundary of $\mathbb{H}^2$, which is identified to $\mathbb{D}$ by means of the Poincaré disc model. Given a homeomorphism $\phi : S^1 \to S^1$, we define the cross-ratio norm of $\phi$ as

$$||\phi||_{cr} = \sup_{cr(Q)=-1} |\ln|cr(\phi(Q))||\ ,$$

where $Q = (z_1, z_2, z_3, z_4)$ is any quadruple of points on $S^1$ and we use the following definition of cross-ratio:

$$cr(z_1,z_2,z_3,z_4) = \frac{(z_4-z_1)(z_3-z_2)}{(z_2-z_1)(z_3-z_4)}.$$

According to this definition, a quadruple $Q = (z_1, z_2, z_3, z_4)$ is symmetric (i.e. the hyperbolic geodesics connecting $z_1$ to $z_3$ and $z_2$ to $z_4$ intersect orthogonally) if and only if $cr(Q) = -1$.

**Definition 2.3.3.** An orientation-preserving homeomorphism $\phi : S^1 \to S^1$ is quasisymmetric if and only if $||\phi||_{cr} < +\infty$.

*Remark* 2.3.4. If we consider the symmetric quadruple $Q = (x-t, x, x+t, \infty)$ on $\mathbb{R}\cup\{\infty\}$ (which can be mapped to $S^1$ by a Möbius transformation) and we assume $\phi(\infty) = \infty$, we have

$$cr(\phi(Q)) = -\frac{\phi(x+t)-\phi(x)}{\phi(x)-\phi(x-t)}\,. \tag{2.2}$$

Hence, considering a lift $\tilde{\phi}$ of $\phi$ to the universal cover $\mathbb{R}$, so that $\phi(e^{i\theta}) = e^{i\tilde{\phi}(\theta)}$, one readily sees that the condition that $\phi$ is quasisymmetric is equivalent to the existence of a constant $C$ such that

$$\frac{1}{C} < \left| \frac{\tilde{\phi}(\theta+h) - \tilde{\phi}(\theta)}{\tilde{\phi}(\theta) - \tilde{\phi}(\theta-h)} \right| < C\,, \tag{2.3}$$

for all $\theta, h \in \mathbb{R}$. Indeed, the boundedness of the expression in Equation 2.2 is a condition invariant under smooth changes of coordinates.

The connection between quasiconformal homeomorphisms of $\mathbb{D}$ and quasisymmetric homeomorphisms of the boundary of $\mathbb{D}$ is made evident by the following classical theorem (see [BA56]).

**Ahlfors-Beuring Theorem**. Every quasiconformal map $\Phi : \mathbb{D} \to \mathbb{D}$ extends to a quasisymmetric homeomorphism of $S^1$. Conversely, an orientation-preserving homeomorphism $\phi : S^1 \to S^1$ is quasisymmetric if it admits a quasiconformal extension to $\mathbb{D}$.

Universal Teichmüller space is then equivalently defined as the space of quasisymmetric homeomorphisms of the circle up to post-composition with Möbius transformations:

$$\mathcal{T}(\mathbb{D}) = \left\{\phi : S^1 \to S^1 \text{ quasisymmetric}\right\} / \text{Möb}(S^1)\,.$$

Again, $\mathcal{T}(\mathbb{D})$ can be identified to the space of quasisymmetric homeomorphisms of $S^1$ fixing 1, $i$ and $-1$.

The topology of $\mathcal{T}(\mathbb{D})$ coincides with the topology induced by the distance induced on the space of quasisymmetric homeomorphisms by the cross-ratio norm. Namely, one can define the distance between two quasisymmetric homeomorphisms $\phi, \phi'$ as $||\phi^{-1} \circ \phi'||_{cr}$.

As we will see later, $\mathcal{T}(\mathbb{D})$ is also endowed with a smooth structure using the Bers embedding. In the model discussed in this subsection, it turns out that the differentiability of a continuous path $\phi_t$ of quasisymmetric homeomorphisms implies that $\phi_t(z)$ is smooth for every $z \in \partial\mathbb{D}$ and $||\phi_t \circ \varphi_0||_{cr} \le Mt$ for some constant $M$.

**Theorem 2.3.5.** *Let $k > 0$ and $\phi_n : S^1 \to S^1$ be a family of orientation-preserving quasisymmetric homeomorphisms of the circle, with $||\phi_n||_{cr} \le k$. Then there exists a subsequence $\phi_{n_k}$ for which one of the following hold:*

- *The homeomorphisms $\phi_{n_k}$ converge to a quasisymmetric homeomorphism $\phi : S^1 \to S^1$, with $||\phi||_{cr} \le k$;*
- *The homeomorphisms $\phi_{n_k}$ converge on the complement of any open neighborhood of a point of $S^1$ to a constant map $c : S^1 \to S^1$.*

### 2.3.3 Relation with earthquakes and the infinitesimal theory

In this subsection we will discuss the relation of Teichmüller space introduced in the previous subsection with the theory of earthquakes of $\mathbb{H}^2$. Moreover, we introduce the notion of Zygmund field, which is a vector field with the regularity which corresponds to an infinitesimal deformation of quasisymmetric homeomorphisms in Teichmüller space.

First, at this point we need to introduce the precise definition of measured geodesic lamination. The definition we give here is different with the most usual one, but is more suitable for our purposes, especially for Chapter 4. The equivalence with the most common definition is discussed for instance in [MŠ12].

Let $\mathcal{G}$ be the set of (unoriented) geodesics of $\mathbb{H}^2$. The space $\mathcal{G}$ is identified to $((S^1 \times S^1) \setminus diag)/\sim$ where the equivalence relation is defined by $(a,b) \sim (b,a)$. Note that $\mathcal{G}$ has the topology of an open Möbius strip. Given a subset $B \subset \mathbb{H}^2$, we denote by $\mathcal{G}_B$ the set of geodesics of $\mathbb{H}^2$ which intersect $B$.

**Definition 2.3.6.** A geodesic lamination on $\mathbb{H}^2$ is a closed subset of $\mathcal{G}$ such that its elements are pairwise disjoint geodesics of $\mathbb{H}^2$. A measured geodesic lamination is a locally finite Borel measure on $\mathcal{G}$ such that its support is a geodesic lamination.

Elements of a geodesic lamination are called *leaves*. *Strata* of the geodesic lamination are either leaves or connected components of the complement of the geodesic lamination in $\mathbb{H}^2$.

**Definition 2.3.7.** A surjective map $E : \mathbb{H}^2 \to \mathbb{H}^2$ is a left earthquake if it is an isometry on the strata of a geodesic lamination of $\mathbb{H}^2$ and, for every pair of strata $S$ and $S'$, the composition

$$(E|_S)^{-1} \circ (E|_{S'})$$

is a hyperbolic translation whose axis weakly separates $S$ and $S'$ and such that $S'$ is translated on the left as seen from $S$.

In general $E$ is clearly not continuous. A measured geodesic lamination is called *discrete* if its support is a discrete set of geodesics. A measured geodesic lamination $\mu$ is *bounded* if

$$\sup_I \mu(\mathcal{G}_I) < +\infty \,,$$

where the supremum is taken over all geodesic segments $I$ of lenght at most 1 transverse to the support of the lamination. The *Thurston norm* of a bounded measured geodesic lamination is

$$||\mu||_{Th} = \sup_I \mu(\mathcal{G}_I) \,.$$

Given an earthquake $E$, there is a measured geodesic lamination associated to $E$, called the *earthquake measure*. See [Thu86]. The earthquake measure $\mu$ determines $E$ up to post-composition with an hyperbolic isometry (in other words, up to the choice of the image of one stratum), and up to the ambiguity on the weighted leaves of the lamination. Hence an earthquake whose earthquake measure is $\mu$ will be denoted by $E^\mu$. An earthquake is *bounded* if its earthquake measure is bounded.

The following theorem was proved by Thurston [Thu86].

**Theorem 2.3.8.** *Any earthquake $E : \mathbb{H}^2 \to \mathbb{H}^2$ extends to an orientation-preserving homeomorphism of $S^1 = \partial_\infty \mathbb{H}^2$. Conversely, every orientation-preserving homeomorphism of $S^1$ is induced by a unique earthquake of $\mathbb{H}^2$.*

Moreover, in [Thu86] Thurston suggested the following characterization of quasisymmetric homeomorphisms, which was later proved independently by Gardiner ([GHL02]) and Šarić ([Šar06]). See also [Šar08].

**Theorem 2.3.9.** *Given an earthquake $E^\mu$ with earthquake measure $\mu$, $\mu$ is bounded if and only if the extension of $E^\mu$ to $S^1$ is a quasisymmetric homeomorphism.*

We are now going to discuss briefly the tangent space to Teichmüller space. We will use the notation $\hat{\varphi}$ for a vector field on $S^1$ and $\varphi$ for the function from $S^1$ to $\mathbb{R}$ which corresponds to $\hat{\varphi}$ under the standard trivialization of $TS^1$. In the following definition, we regard tangent vectors to $S^1$ as elements of $\mathbb{C}$. Hence, $\hat{\varphi}(z) = iz\varphi(z)$ for every $z \in \partial\mathbb{D}$.

**Definition 2.3.10.** A vector field $\hat{\varphi}$ on $S^1$ is a *Zygmund field* if there is a constant $C$ such that

$$\sup_Q \hat{\varphi}[Q] \leq +\infty\,, \tag{2.4}$$

where the supremum is taken over all quadruples $Q = (a, b, c, d)$ with $cr(Q) = -1$ and

$$\hat{\varphi}[Q] = \left| \frac{\hat{\varphi}(a) - \hat{\varphi}(d)}{a - d} + \frac{\hat{\varphi}(b) - \hat{\varphi}(c)}{b - c} - \frac{\hat{\varphi}(a) - \hat{\varphi}(b)}{a - b} - \frac{\hat{\varphi}(d) - \hat{\varphi}(c)}{d - c} \right| .$$

We say the associated function $\varphi : S^1 \to \mathbb{R}$ such that $\hat{\varphi}(z) = iz\varphi(z)$ is in the *Zygmund class.*

It has been proved that a function $\varphi : S^1 \to \mathbb{R}$ is in the Zygmund class if and only if there exists a constant $C$ such that

$$|\varphi(e^{i(\theta+h)}) + \varphi(e^{i(\theta-h)}) - 2\varphi(e^{i\theta})| < C|h| \tag{2.5}$$

for all $\theta, h \in \mathbb{R}$. Equation (2.5) is basically the infinitesimal version of the condition in Equation (2.3).

Functions in the Zygmund class are $\alpha$-Hölder for any $\alpha \in (0, 1)$, but in general they are not Lipschitz. Given a Zygmund field $\hat{\varphi}$, we define the *infinitesimal cross-ratio norm* of $\hat{\varphi}$ by

$$||\hat{\varphi}||_{cr} = \sup_Q \hat{\varphi}[Q]\,.$$

We say that a vector field on $\partial\mathbb{D}$ is a *quadratic polynomial* if it is of the form

$$\hat{\varphi}(z) = p(z) = \alpha z^2 + \beta z + \gamma\,.$$

Clearly $p$ has to satisfy the condition that $\hat{\varphi}(z) = iz\varphi(z)$ for some $\varphi : S^1 \to \mathbb{R}$. By a direct computation, vector fields on $S^1$ for which $\varphi(z)$ is a quadratic polynomial of $z$ correspond precisely to the derivative of a family of Möbius transformations of $S^1$, of the form

$$z \mapsto \frac{a_t z + b_t}{c_t z + d_t}\,,$$

which is the identity at time $t = 0$. In other words, these are the traces on $\partial\mathbb{H}^2$ of Killing fields of $\mathbb{H}^2$. The tangent space of Teichmüller space is isomorphic to the quotient of the space of vector fields by the subspace of quadratic polynomials, which will be denoted by $\mathfrak{Mob}$ to emphasize that it is the subspace of infinitesimal Möbius transformations. This fact can be expressed by the following isomorphism (see [GL00]):

$$T_{\mathrm{id}}\mathcal{T}(\mathbb{D}) \cong \{\text{Zygmund fields on } \partial\mathbb{D}\}/\mathfrak{Mob}\,.$$

On the other hand, it is easy to show that for a Zygmund field $\hat{\varphi}$, $||\hat{\varphi}||_{cr}$ vanishes if and only if $\hat{\varphi} = p$ is a quadratic polynomial. Hence $||\cdot||_{cr}$ defines a norm on the tangent space at the identity of $\mathcal{T}(\mathbb{D})$.

We now state a result first proved by Gardiner-Hu-Lakic, which can be regarded as the infinitesimal version of Theorem 2.3.9. Given a measured geodesic lamination $\mu$, the *infinitesimal earthquake* along $\mu$ is a vector field on $\partial\mathbb{D}$ obtained as

$$(\dot{E}^\mu)(z) = \left.\frac{d}{dt}\right|_{t=0} E^{t\mu}(z)\,.$$

In particular, when $\mu$ is composed by a single geodesic $l$ with weight 1, $\dot{E}^\mu$ has an easy expression. Indeed (up to choosing a suitable normalization) $\dot{E}^\mu$ vanishes on one stratum of $\mu$, and coincides on the other stratum with the Killing vector field in $\mathfrak{sl}(2,\mathbb{R})$ whose exponential is a one-parameter family of hyperbolic isometries fixing $l$, which at time 1 translates (on the left) by unit lenght. We will denote this special case of infinitesimal earthquake by $\dot{E}_l$. See Figure 2.1.

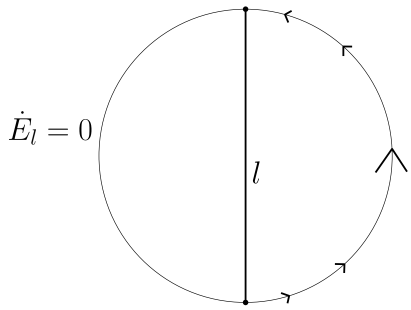



**Figure 2.1:** The infinitesimal earthquake along a single geodesic $l$.

**Theorem 2.3.11.** *Given a bounded measured geodesic lamination $\mu$ and a fixed point $x_0$ which does not lie on any weighted leaf of $\mu$, the integral*

$$\dot{E}^\mu(z) = \int_{\mathcal{G}} \dot{E}^l(z)d\mu(l) \tag{2.6}$$

*converges for every $\eta \in \partial\mathbb{D}$ and defines a Zygmund field $\hat{\varphi}$ on $S^1$, which corresponds to the infinitesimal earthquake*

$$\dot{E}^\mu = \left.\frac{d}{dt}\right|_{t=0} E^{t\mu}\,.$$

*Conversely, for every Zygmund field $\hat{\varphi}$ on $S^1 = \partial\mathbb{D}$, there exists a bounded measured geodesic lamination $\mu$ such that $\hat{\varphi}$ is the infinitesimal earthquake along $\mu$, namely*

$$\hat{\varphi} = \left.\frac{d}{dt}\right|_{t=0} E^{t\mu}\,,$$

*up to an infinitesimal Möbius transformation.*

See [GHL02] or [MŠ12, Appendix] for a proof. Analogously to the case of earthquakes, although the infinitesimal earthquake $\dot{E}^\mu$ is not continuous in $\mathbb{H}^2$, its boundary value is a continuous field.

### 2.3.4 Quasicircles and Bers embedding

We now want to discuss another interpretation of Teichmüller space, as the space of quasidiscs, and the relation with the Schwartzian derivative and the Bers embedding.

**Definition 2.3.12.** A *quasicircle* is a simple closed curve $\Gamma$ in $\widehat{\mathbb{C}}$ such that $\Gamma = \Psi(S^1)$ for a quasiconformal map $\Psi$. Analogously, a *quasidisc* is a domain $\Omega$ in $\widehat{\mathbb{C}}$ such that $\Omega = \Psi(\mathbb{D})$ for a quasiconformal map $\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Let us remark that in the definition of quasicircle, it would be equivalent to say that $\Gamma$ is the image of $S^1$ by a $K'$-quasiconformal map of $\widehat{\mathbb{C}}$ (not necessarily conformal on $\mathbb{D}^*$). However, the maximal dilatation $K'$ might be different, with $K \leq K' \leq 2K$. Hence we consider the space of quasidiscs:

$$QD(\mathbb{D}) = \{\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} : \Psi|_{\mathbb{D}} \text{ is quasiconformal and } \Psi|_{\mathbb{D}^*} \text{ is conformal}\}/\sim,$$

where the equivalence relation is $\Psi \sim \Psi'$ if and only if $\Psi|_{\mathbb{D}^*} = \Psi'|_{\mathbb{D}^*}$. We will again consider the quotient of $QD(\mathbb{D})$ by Möbius transformation.

Given a Beltrami differential $\mu \in \mathrm{Belt}(\mathbb{D})$, one can construct a quasiconformal map on $\widehat{\mathbb{C}}$, by applying Measurable Riemann mapping Theorem to the Beltrami differential obtained by extending $\mu$ to 0 on $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$. The quasiconformal map obtained in this way (fixing the three points 0,1 and $\infty$) is denoted by $f_\mu$. The key fact to show the equivalence of $\mathcal{T}(\mathbb{D})$ with this new model is the following standard Lemma (see [Gar87, §5.4, Lemma 3]):

**Lemma 2.3.13.** *Given* $\mu, \mu' \in \mathrm{Belt}(\mathbb{D})$*, the following are equivalent:*

- $f^\mu|_{S^1} = f^{\mu'}|_{S^1}$*;*
- $f_\mu|_{S^1} = f_{\mu'}|_{S^1}$*;*
- $f_\mu|_{\mathbb{D}^*} = f_{\mu'}|_{\mathbb{D}^*}$*.*

Hence it can be shown that $\mathcal{T}(\mathbb{D})$ is identified to $QD(\mathbb{D})/\mathrm{M\ddot{o}b}(\widehat{\mathbb{C}})$, or equivalently to the subset of those $[\Psi] \in QD(\mathbb{D})$ which fix 0, 1 and $\infty$.

We will say that a quasicircle $\Gamma$ is a $K$-quasicircle if

$$K = \inf_{\substack{\Gamma=\Psi(S^1)\\ \Psi\in QD(\mathbb{D})}} K(\Psi)\,.$$

This is equivalent to saying that the element $[\Phi]$ of the first model of universal Teichmüller space, namely $\mathcal{T}(\mathbb{D}) = QC(\mathbb{D})/\mathrm{M\ddot{o}b}(\mathbb{D})$, which corresponds to $[\Psi]$ has Teichmüller distance from the identity $d_{\mathcal{T}(\mathbb{D})}([\Phi], [\mathrm{id}]) = (\log K)/2$.

By using the model of quasidiscs for Teichmüller space, we now introduce another norm on $\mathcal{T}(\mathbb{D})$. Given a quasidisc, which corresponds to a map $\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is conformal on $\mathbb{D}^*$, the idea is to measure how far $\Psi|_{\mathbb{D}^*}$ is from being a Möbius transformation. The necessary tool is therefore the Schwarzian derivative, of which we give a brief account here.

Given a holomorphic function $f : \Omega \to \mathbb{C}$ with $f' \neq 0$ in $\Omega$, the *Schwarzian derivative* of $f$ is the holomorphic function

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

It can be easily checked that $S_{1/f} = S_f$, hence the Schwarzian derivative can be defined also for meromorphic functions at simple poles, and that the Schwarzian derivative of a Möbius transformation vanishes. Moreover, the following transformation rule holds:

$$S_{f\circ g} = (S_f \circ g)(g')^2 + S_g \,. \tag{2.7}$$

Equation (2.7) will be important for several reasons. A first consequence is that $S_f$ satisfies the transformation rule of a quadratic differential. Indeed, by choosing $A$ a Möbius transformation, we see that

$$S_{f\circ A} = (S_f \circ A)(A') \,, \tag{2.8}$$

since $S_A = 0$. As a byproduct, the relation (2.8) enables to define the Schwarzian derivative for any locally injective holomorphic function of the Riemann sphere (for instance choosing a change of coordinates $w = 1/z$). Analogously, one obtains that

$$S_{A\circ f} = S_f \,.$$

The basic fact for this theory is the following.

**Proposition 2.3.14.** *If $A$ is a Möbius transformation of $\widehat{\mathbb{C}}$, then $S_A \equiv 0$. Conversely, if a locally injective holomorphic function $f : \Omega \to \widehat{\mathbb{C}}$ has $S_f \equiv 0$, then $f$ is the restriction of a Möbius transformation of $\widehat{\mathbb{C}}$.*

Let us now consider the space of holomorphic quadratic differentials on $\mathbb{D}$. We will consider the following norm, for a holomorphic quadratic differential $q = h(z)dz^2$:

$$||q||_\infty = \sup_{z\in\mathbb{D}} e^{-2\eta(z)}|h(z)| \,,$$

where $e^{2\eta(z)}|dz|^2$ is the Poincaré metric on $\mathbb{D}$. Observe that $||q||_\infty$ behaves like a function, in the sense that it is invariant by pre-composition with Möbius transformations of $\mathbb{D}$, which are isometries for the Poincaré metric.

We now define the *Bers embedding* of universal Teichmüller space. This is the map $\beta_{\mathbb{D}}$ which associates to $[\Psi] \in \mathcal{T}(\mathbb{D}) = QD(\mathbb{D})/\text{Möb}(\widehat{\mathbb{C}})$ the Schwarzian derivative $S_\Psi$. Let us denote by $||\cdot||_{\mathcal{Q}(\mathbb{D}^*)}$ the norm on holomorphic quadratic differentials on $\mathbb{D}^*$ obtained from the $||\cdot||_\infty$ norm on $\mathbb{D}$, by identifying $\mathbb{D}$ with $\mathbb{D}^*$ by an inversion. Then

$$\beta_{\mathbb{D}} : \mathcal{T}(\mathbb{D}) \to \mathcal{Q}(\mathbb{D}^*)$$

is an embedding of $\mathcal{T}(\mathbb{D})$ in the Banach space $(\mathcal{Q}(\mathbb{D}^*), ||\cdot||_{\mathcal{Q}(\mathbb{D}^*)})$ of bounded holomorphic quadratic differentials (i.e. for which $||q||_{\mathcal{Q}(\mathbb{D}^*)} < +\infty$). By means of the

Bers embedding, universal Teichmüller space is endowed with a smooth and complex structure. Finally, the Bers norm of en element $\Psi \in \mathcal{T}(\mathbb{D})$ is

$$||\Psi||_{\mathcal{B}} = ||\beta_{\mathbb{D}}[\Psi]||_\infty = ||S_\Psi||_{\mathcal{Q}(\mathbb{D}^*)}\,.$$

The fact that the Bers embedding is locally bi-Lipschitz will be used in the following. See for instance [FKM13, Theorem 4.3]. In the statement, we again implicitly identify the model of universal Teichmüller space by quasiconformal homeomorphisms of the disc (denoted by $[\Phi]$) and by quasicircles (denoted by $[\Psi]$).

**Theorem 2.3.15.** *Let $r > 0$. There exist constants $b_1$ and $b_2 = b_2(r)$ such that, for every $[\Psi], [\Psi']$ in the ball of radius $r$ for the Teichmüller distance centered at the origin (i.e. $d_{\mathcal{T}}([\Psi], [\mathrm{id}]), d_{\mathcal{T}}([\Psi'], [\mathrm{id}]) < R$),*

$$b_1||\beta_{\mathbb{D}}[\Psi] - \beta_{\mathbb{D}}[\Psi]||_\infty \leq d_{\mathcal{T}}([\Psi], [\Psi']) \leq b_2||\beta_{\mathbb{D}}[\Psi] - \beta_{\mathbb{D}}[\Psi]||_\infty\,.$$

We conclude this preliminary part by mentioning a theorem by Nehari, see for instance [Leh87] or [FM07].

**Nehari Theorem**. The image of the Bers embedding is contained in the ball of radius 3/2 in $\mathcal{Q}(\mathbb{D}^*)$, and contains the ball of radius 1/2.

## 2.4 Teichmüller spaces of closed and punctured surfaces

In this section we will discuss the relation of the theory of quasiconformal mappings with Teichmüller spaces of Riemann surfaces. We are most interested in the case of *closed* surfaces (i.e. compact and without boundary) and *punctured* surfaces (i.e. a closed surface to which a finite number of points has been removed).

The original definition of Teichmüller space is the following.

**Definition 2.4.1.** Given a Riemann surface $S$, the Teichmüller space of $S$ is:

$$\mathcal{T}(S) = \{(S_0, f : S \to S_0) : S_0 \text{ is a Riemann surface}, f \text{ is quasiconformal}\}/\sim$$

where the equivalence relation $\sim$ is, for $f : S \to S_0$ and $f' : S \to S_0'$, $f \sim f'$ if and only if there exists a conformal map $g : S_0 \to S_0'$ which is homotopic to $f' \circ f^{-1}$.

Suppose $S$ is a Riemann surface whose universal cover is conformal to $\mathbb{D}$. Let us fix holomorphic covering projections $\pi : \mathbb{D} \to S$ and $\pi_0 : \mathbb{D} \to S_0$. Observe that they are determined up to Möbius transformations of $\mathbb{D}$. It is not difficult to see that a map $g_0$ from $S_0$ to $S_0$ is homotopic to the identity if and only if it lifts to a map $\tilde{g}_0 : \mathbb{D} \to \mathbb{D}$, which commutes with the projection $\pi_0$, whose extension to $\partial\mathbb{D}$ is the identity. Hence there is a well-defined embedding $\mathcal{T}(S) \hookrightarrow \mathcal{T}(\mathbb{D})$ which maps a pair $f : S \to S_0$, where $f$ is quasiconformal, to the class of the lift $\tilde{f}$ (which is still quasiconformal) to the holomorphic universal cover. This shows that for a Riemann surface $S$ which is either closed with genus $g \geq 2$ or punctured, satisfying the condition that there are at least three punctures if $g = 0$, Teichmüller space $\mathcal{T}(S)$ is embedded in universal Teichmüller space $\mathcal{T}(\mathbb{D})$. However, the embedding depends on the choice of the basepoint $S$, which is a surface endowed with a reference complex structure.

By the Uniformization Theorem, when $S$ is a closed surface, this definition of Teichmüller space is equivalent to the definition of Fricke space given in Subsection 2.1.1. In Chapter 6 we will be interested in the analogous of Fricke space for hyperbolic metrics with cone singularities. Although the precise definition of metric with cone singularities will be given at the beginning of that chapter (see Example 6.1.1), we remark here that the space of hyperbolic metrics on s surface $S$ with fixed cone singularities at $n$ points is equivalent to the Teichmüller space of $S$ with $n$ punctures. Again $n$ has to be at least 3 if the genus of $S$ is zero; there is also a condition on the prescribed cone angles due to the Gauss-Bonnet theorem, which will be explained below. This fact is a consequence of the following Uniformization Theorem for hyperbolic cone metrics, see [Tro91].

**Uniformization Theorem for hyperbolic cone surfaces**. Given a closed Riemann surface $S$, $n$ points $x_1, \dots, x_n$ on $S$ and positive numbers $\theta_1, \dots, \theta_n$ such that

$$2\pi\chi(S) + \sum_{i=1}^{n}(\theta_i - 2\pi) < 0\,,$$

there exists a unique conformal metric on $S$ of constant curvature $-1$ having cone points of angle $\theta_i$ at $x_i$.

### 2.4.1 The tangent space to Teichmüller space of a surface

We now give an interpretation of the tangent space of $\mathcal{T}(S)$, where $S$ is a closed or punctured surface as in the hypothesis above.

As we have just explained, $\mathcal{T}(S)$ can be embedded in universal Teichmüller space, by considering quasiconformal homeomorphisms of the disc. Recall from Subsection 2.3.1 that such objects are determined by Beltrami differentials on the disc $\mathbb{D}$, up to some equivalence relation. It turns out from the transformation rule in Equation (2.1) that a Beltrami differential $\mu$ which corresponds to the lift of a quasiconformal map defined on $S$ provides a section of the bundle $K^{-1} \otimes \bar{K}$ over $S$, where $K$ is the canonical bundle of $S$. This basically means that a Beltrami differential can be regarded as a $(0,1)$-form with value in the holomorphic tangent bundle of $S$. We denote by $\mathcal{B}(S)$ the space of Beltrami differentials on the Riemann surface $S$.

Moreover, there is a natural pairing between quadratic differentials and Beltrami differentials, given by the integration of the $(1,1)$ form obtained by contraction

$$\langle q, \mu \rangle = \int_S q \bullet \mu\,, \tag{2.9}$$

where in complex chart $q \bullet \mu := q(z)\mu(z)dz \wedge d\bar{z}$, if $\mu = \mu(z)d\bar{z}/dz$ and $q = q(z)dz^2$.

We say that a Beltrami differential $\mu$ is trivial if $\langle q, \mu \rangle = 0$ for any holomorphic quadratic differential. We will denote by $\mathcal{B}(S)^{\perp}$ the subspace of trivial Beltrami differentials.

The tangent space to the Teichmüller space of $S$ at the reference complex structure is naturally identified with $\mathcal{B}(S)/\mathcal{B}(S)^{\perp}$ as a complex vector space. See [Ahl06] for more details. We want to remark here that an analogous construction holds for a punctured surface, where one defines a Beltrami differential $\mu$ to be trivial when $\langle q, \mu \rangle = 0$ for every holomorphic quadratic differential with at most simple poles at the punctures of $S$. This is indeed the condition that ensures that the integral in Equation (2.9) is finite.

### 2.4.2 The Weil-Petersson form

We conclude the section by observing that the characterization of the tangent space of Teichmüller space given in Subsection 2.4.1 and the expression in Equation (2.9) enable to identity the cotangent bundle of Teichmüller space of $S$ (possibly with punctures) with the space of holomorphic quadratic differentials on $S$ (with at most simple poles at the punctures). By means of this technology, the Weil-Petersson symplectic form $g_{WP}$ is defined by

$$g_{WP}(q, q') = \int_S \frac{f\bar{g}}{e^{2\eta}} dx \wedge dy \, ,$$

where one observes that in conformal coordinates, if $q(z) = f(z)dz^2$, $q'(z) = g(z)dz^2$ and $h(z) = e^{2\eta}|dz|^2$, the 2-form

$$\frac{f\bar{g}}{e^{2\eta}} dx \wedge dy$$

is independent of the coordinates. Again, in the case of punctured surfaces, the condition that $q$ and $q'$ have at most simple poles at the punctures ensures that the integral is well-defined.

# Chapter 3

# Partial differential equations related to the curvature of surfaces

The purpose of this chapter is to give a brief overview of theorems on two types of partial differential equations, which are related to the curvature of surfaces. Heuristically, the curvature of a surface in a three-dimensional manifold is encoded in the Hessian of some function. The simplest example of this fact is obtained from the parametrizes a surface in $\mathbb{R}^3$ which is a graph over $\mathbb{R}^2$, or analogously a spacelike graph in $\mathbb{R}^{2,1}$.

In this thesis, we are mostly concerned with the properties of the *mean curvature* and the *Gaussian curvature* of surfaces. The former is the trace of the shape operator, hence in general it will be related to the trace of the Hessian of some function, i.e. the Laplacian. The latter is in general more complicated, since it deals with the determinant of the Hessian, which is a non-linear expression.

## 3.1 Linear elliptic PDEs

The first case we consider is the case of linear elliptic partial differential equations, which in general are of the form

$$L(u) = f(x) \tag{3.1}$$

for $x \in \Omega$ and $L$ is a linear operator, where $\Omega \subset \mathbb{R}^n$. We will be interested in the case $n = 2$. The linear operator is of the form

$$L(u) = \sum_{i,j} a_{ij}(x)\partial_{ij}u(x) + \sum_i b_i(x)\partial_i u(x) + (x)u(x)\,, \tag{3.2}$$

where $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ are such that there exists a constant $\lambda > 0$ for which

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi_i||\xi_j| \tag{3.3}$$

for any $\xi_i, \xi_j$.

We will present a collection of results, without any pretension of completeness. We first focus on the most basic example of this equation, in which the linear operator $L$ is the Euclidean Laplace operator $\Delta_0$, and collect some interior estimates, which are called *Schauder estimates*, which are very important for the study of the regularity of solutions. The proofs can be found in [GT83].

**Theorem 3.1.1.** *Let $\Omega \subset \mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be any solution, $C^2$ up to the boundary of $\Omega$, of the equation*

$$\Delta_0 u = f$$

*for $f \in L^\infty(\Omega)$. Let $\alpha \in [0,1)$. For any compact subdomain $\Omega'$ compactly contained in $\Omega$, there exists a constant $C$ only depending on $\Omega$ and $\Omega'$ such that*

$$||u||_{C^{1,\alpha}(\Omega')} \leq C \left( ||u||_{C^0(\Omega)} + ||f||_{L^\infty(\Omega)} \right) .$$

**Theorem 3.1.2.** *Let $\Omega \subset \mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be any $C^{2,\alpha}$ bounded solution on $\Omega$ of the equation*

$$\Delta_0 u = f$$

*for $f \in C^\alpha(\Omega)$. Let $\alpha \in [0,1)$. For any compact subdomain $\Omega'$ compactly contained in $\Omega$, there exists a constant $C$ only depending on $\Omega$ and $\Omega'$ such that*

$$||u||_{C^{2,\alpha}(\Omega')} \leq C \left( ||u||_{C^0(\Omega)} + ||f||_{C^{0,\alpha}(\Omega)} \right) .$$

Theorem 3.1.1 can be generalized for general linear operators $L$, thus obtaining a uniform bound which does not depend on $L$, provided the operators $L$ are uniformly strictly elliptic and have uniformly bounded coefficients.

**Theorem 3.1.3.** *Let $\Omega \subset \mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be any $C^{2,\alpha}$ bounded solution on $\Omega$ of the equation*

$$L(u) = f$$

*where $L$ is a linear operator of the form* (3.2) *and there exist constants $\lambda, \Lambda > 0$ such that* (3.3) *holds, and*

$$||a_{ij}||_{C^{0,\alpha}(\Omega)}, ||b_i||_{C^{0,\alpha}(\Omega)}, ||c||_{C^{0,\alpha}(\Omega)} \leq \Lambda .$$

*Let $\alpha \in [0,1)$. For any compact subdomain $\Omega'$ compactly contained in $\Omega$, there exists a constant $C$ only depending on $\Omega$,$\Omega'$,$\lambda$ and $\Lambda$ such that*

$$||u||_{C^{2,\alpha}(\Omega')} \leq C \left( ||u||_{C^0(\Omega)} + ||f||_{C^{0,\alpha}(\Omega)} \right) .$$

## 3.2 Monge-Ampère equations and the definition of generalized solution

We now move to Monge-Ampère equations, which will be important especially in Chapter 4 for the formulation of Minkowski problem. As a reference, see [Gut01], [TW08].

In general, a *Monge-Ampère equation* is a partial differential equation of the form

$$\det D^2u(x) = f(x, u, Du)\,,$$

where $u$ is a function defined on a domain $\Omega \subset \mathbb{R}^n$, and $D^2u(x)$ denotes the Euclidean Hessian of $u$ at the point $x$. We are mostly interested in the case $n = 2$.

Given a convex function $u : \Omega \to \mathbb{R}$, for $\Omega$ a convex domain in $\mathbb{R}^2$, we define the *normal mapping* or *subdifferential* of $u$ as the set-valued function $N_u$ whose value at a point $\bar{w} \in \Omega$ is:

$$N_u(\bar{w}) = \{Df : f \text{ affine; } gr(f) \text{ is a support plane for } gr(u), (\bar{w}, u(\bar{w})) \in gr(f)\}\ .$$

Here $Df$ denotes the Euclidean gradient of $f$ (which is independent of the point in $\mathbb{R}^n$ is $f$ is affine). In general $N_u(\bar{w})$ is a convex set; if $u$ is differentiable at $\bar{w}$, then $N_u(\bar{w}) = \{Du(\bar{w})\}$. We define the *Monge-Ampère measure* on the collection of Borel subsets $\omega$ of $\mathbb{R}^2$ as:

$$MA_u(\omega) = \mathcal{L}(N_u(\omega))$$

where $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}^n$. Roughly speaking, when $u$ is differentiable, $N_u(\omega)$ is the Lebesgue measure of the image of $\omega$ by the gradient mapping.

It can be proved that, when $u$ is $C^2$, the Monge-Ampère measure corresponds precisely to integration of the function obtained by the determinant of the Hessian of $u$.

**Lemma 3.2.1** ([Gut01, Theorem 1.1.5])**.** *If $u$ is a $C^2$ convex function, then*

$$MA_u(\omega) = \mathcal{L}(Du(\omega)) = \int_\omega (\det D^2u)d\mathcal{L}\,. \tag{3.4}$$

Indeed, it can be shown that, when restricted to the subset

$$\Omega_0 = \{x \in \Omega : \det D^2u(x) > 0\}\,,$$

$u$ is a diffeomorphism onto its image. Then by a simple change of variables, one obtains

$$MA_u(\omega \cap S_0) = \int_{\omega \cap S_0} (\det D^2u)d\mathcal{L}\,.$$

But, by Sard's Lemma, the image under the gradient mapping of the subset $S_0$ has zero measure. Hence the equality in Equation (3.4) holds. We now state a more precise characterization of the Monge-Ampère measure.

**Lemma 3.2.2** ([TW08, Lemma 2.3])**.** *Given a convex function $u : \Omega \to \mathbb{R}$, the regular part of the Lebesgue decomposition of $MA_u(\omega)$ is $\int_\omega (\partial^2 u)d\mathcal{L}$, where we set*

$$\partial^2 u(\bar{w}) = \begin{cases} \det D^2u(\bar{w}) & \textit{if } u \textit{ is twice-differentiable at } \bar{w} \\ 0 & \textit{otherwise} \end{cases}\,.$$

Recall that, by Alexandrov Theorem, a convex function $u$ on $\Omega$ is twice-differentiable almost everywhere. We have now given sufficient motivation to state the definition of generalized solution to Monge-Ampère equations.

**Definition 3.2.3.** Given a nonnegative measure $\nu$ on $\Omega$, we say a convex function $u : \Omega \to \mathbb{R}$ is a generalized solution to the Monge-Ampère equation

$$\det D^2 u = \nu \tag{3.5}$$

if $MA_u(\omega) = \nu(\omega)$ for all Borel subsets $\omega$.

In particular, given an integrable function $f : \Omega \to \mathbb{R}$, $u$ is a generalized solution to the equation $\det D^2 u = f$ if and only if, for all $\omega$,

$$MA_u(\omega) = \int_\omega f d\mathcal{L}\,.$$

## 3.3 Some known results and peculiarities of dimension 2

We collect here, without proofs, some facts which will be used in the following. Unless explicitly stated, the results hold in $\mathbb{R}^n$, although we are only interested in $n = 2$. The first result shows the continuity of the Monge-Ampère measure with respect to uniform convergence on compact sets on the space of convex functions, and weak convergence of measures.

**Lemma 3.3.1** ([Gut01, Lemma 1.2.3], [TW08, Lemma 2.2])**.** *Given a sequence of convex functions $u_n$ which converges uniformly on compact sets to $u$, the Monge-Ampère measure $MA_{u_n}$ converges weakly to $MA_u$. Namely, for every continuous function $f$ with compact support in $\Omega$,*

$$\lim_{n\to\infty} \int_\Omega f dMA_{u_n} = \int_\Omega f dMA_u\,.$$

We will be interested in solutions of Monge-Ampère equations with some boundary condition on $\partial\Omega$. We state a very important principle to compare functions solving different Monge-Ampère generalized equations with different boundary data.

**Theorem 3.3.2** (Comparison principle, [TW08, Gut01])**.** *Given a bounded convex domain $\Omega$ and two convex functions $u, v$ defined on $\overline{\Omega}$, if $MA_u(\omega) \leq MA_v(\omega)$ for every Borel subset $\omega$, then*

$$\min_{\overline{\Omega}}(u - v) = \min_{\partial\Omega}(u - v)\,.$$

A direct Corollary, obtained by applying Theorem 3.3.2 twice (reversing the roles of $u$ and $v$), is the following result about uniqueness:

**Corollary 3.3.3.** *Given two generalized solutions $u_1, u_2 \in C^0(\overline{\Omega})$ to the Monge-Ampère equation* $\det D^2(u_i) = \nu$ *on a bounded convex domain $\Omega$, if $u_1 \equiv u_2$ on $\partial\Omega$, then $u_1 \equiv u_2$ on $\Omega$.*

The following Theorem gives an *a priori* bound on the second derivative of any solution of Monge-Ampère equations with constant boundary value.

**Theorem 3.3.4** ([CY77, Lemma 3], [Pog71]). *Given a bounded convex domain $\Omega$, let $u$ be a $C^4$ solution to $\det D^2u = f$ defined on $\overline{\Omega}$ which is constant on $\partial\Omega$. There is an estimate on the second derivatives of $u$ at $x \in \Omega$ which depends only on*

$$\max_{\Omega} \left\{ |u|, ||Du||^2, ||D\log(f)||^2, \sum_{i,j} \partial_{ij}(\log(f))^2 \right\}$$

*and on the distance of $x$ to $\partial\Omega$.*

The following property will be used repeatedly in the paper, and is a peculiar property of dimension $n = 2$.

**Theorem 3.3.5** (Aleksandrov-Heinz). *A generalized solution to $\det D^2u = f$ on a domain $\Omega \subset \mathbb{R}^2$ with $f > 0$ must be strictly convex.*

Theorem 3.3.4 and Theorem 3.3.5 can be used to prove the following property of regularity of strictly convex solutions, which again holds in dimension $n = 2$.

**Theorem 3.3.6** ([TW08, Theorem 3.1]). *Let $u$ be a strictly convex generalized solution to $\det D^2u = f$ on a bounded convex domain $\Omega$ with smooth boundary. If $f > 0$ and $f$ is smooth, then $u$ is smooth.*

We conclude by giving a characterization of generalized solutions of the equation

$$\det D^2u = 0\,.$$

**Theorem 3.3.7** ([Gut01, Theorem 1.5.2]). *Given any bounded convex domain $\Omega \subset \mathbb{R}^2$ and a continuous function $\varphi : \partial\Omega \to \mathbb{R}$, the unique convex generalized solution of the equation*

$$\det D^2u = 0$$

*with boundary condition*

$$u|_{\partial\mathbb{D}} = \varphi$$

*is obtained as the convex envelope of $\varphi$, namely*

$$u(z) = \sup\{f(z) : f \text{ is an affine function on } \Omega, f|_{\partial\Omega} \leq \varphi\}\,.$$

This is in some sense the opposite situation from the Theorem 3.3.6. Indeed, in dimension 2, Theorem 3.3.5 and Theorem 3.3.6 ensure that a solution of the equation $\det D^2u = f$ with $f > 0$ is automatically smooth and strictly convex. On the other hand, the prototype of a solution of $\det D^2u = 0$, in light of Theorem 3.3.7, is a piecewise affine function. Namely, it is not strictly convex, and even not differentiable at the bending locus.

# Part II

# Flat Lorentzian geometry

# Chapter 4

# Convex surfaces in Minkowski space

The aim of this chapter is to study convex surfaces in Minkowski space which are spacelike entire graphs, in relation with the curvature function. We say that a convex surface in $\mathbb{R}^{2,1}$ is a *spacelike entire graph* if $S = \{(p, f(p)) \,|\, p \in \mathbb{R}^2\}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a $C^1$ function on the horizontal plane such that $||Df(p)|| < 1$ for all $p \in \mathbb{R}^2$.

The first result of the chapter concerns the Minkowski problem in Minkowski 2+1 space. Given a smooth spacelike strictly convex surface $S$ in $\mathbb{R}^{2,1}$, the *curvature function* is defined as $\psi : G(S) \to \mathbb{R}$, $\psi(x) = -K_S(G^{-1}(x))$, where $G : S \to \mathbb{H}^2$ is the Gauss map and $K_S$ is the scalar intrinsic curvature on $S$. Minkowski problem consists in finding a convex surface in Minkowski space whose curvature function is a prescribed positive function $\psi$, which is a Cauchy surface in a prescribed domain of dependence. Using the support function technology, the problem turns out to be equivalent to solving the equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2}\,. \tag{MA}$$

with the boundary condition $u|_{\partial\mathbb{D}} = \varphi$. Li proved in [Li95] the existence and uniqueness of solutions of the Minkowski problem (also in dimension higher than $2+1$) for a $C^\infty$ boundary function $\varphi$. This was later improved by [GJS06] for $\varphi$ Lipschitz. On the other hand, in [BBZ11] the Minkowski problem was solved for maximal globally hyperbolic flat spacetimes (which have been discussed in Subsection 1.1.4).

In this chapter we will also deal with the Minkowski problem for Cauchy surfaces in domains of dependence with weaker assumptions on the regularity of $\varphi$. Namely, we consider domains of dependence contained in the future cone over a point. The following will be proved:

**Theorem 4.A.** *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a lower semicontinuous and bounded function and $\psi : \mathbb{D} \to [a, b]$ for some $0 < a < b < +\infty$. Then there exists a unique spacelike graph $S$ in $\mathbb{R}^{2,1}$ whose support function $u$ extends $\varphi$ and whose curvature function is $\psi$.*

The proof of Theorem 4.A is given in Section 4.3, and relies on several preliminary constructions (introduced in Section 4.1) and on the existence of some explicit constant curvature surfaces to be used as barriers. The latter are presented in Section 4.2, which can be also read in order to have a family of explicit examples of the previous theory.

The second goal of the chapter is to determine the exact regularity class of the extension on $\partial\mathbb{D}$ of the support functions $u : \mathbb{D} \to \mathbb{R}$ corresponding to surfaces with principal curvatures bounded from below. Recall that there is a direct relation between the support function at infinity of a convex surface in $\mathbb{R}^{2,1}$ and the geometry of the surface. Indeed, a lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ uniquely determines a domain of dependence $D$ (this notion has been heuristically introduced in Subsection 1.1.4 and will be discussed in detail in Subsection 4.1 below). In this context, surfaces with support function at infinity equal to $\varphi$ are precisely Cauchy surfaces for $D$.

**Theorem 4.B.** *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. There exists a spacelike entire graph in $\mathbb{R}^{2,1}$ whose principal curvatures are bounded from below by a positive constant and whose support function at infinity is $\varphi$ if and only if $\varphi$ is in the Zygmund class.*

This is an improvement - in dimension (2+1) - of results obtained in [Li95], where it is showed that the continuity of the support function $\varphi = u|_{\partial\mathbb{D}}$ is a necessary condition for the existence of a Cauchy surface with principal curvatures bounded from below by a positive constant. On the other hand, the existence of such a Cauchy surface is guaranteed under the assumption that $\varphi$ is smooth.

We will then restrict the study to the case of constant curvature surfaces. Namely, we will prove the following:

**Theorem 4.C.** *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a function in the Zygmund class. For every $K < 0$ there is a unique spacelike entire graph $S$ in $\mathbb{R}^{2,1}$ of constant curvature $K$ and with bounded principal curvatures whose corresponding function $u$ extends $\varphi$.*

This makes also more precise the statement of Theorem 4.B. Indeed we prove that if $\varphi$ is in the Zygmund class, then the constant curvature surfaces in the domain of dependence described by $\varphi$ (whose existence is guaranteed by Theorem 4.A) have principal curvatures bounded from below. Finally, we show that the constant curvature surfaces always foliate the domain of dependence:

**Theorem 4.D.** *If $D$ is a domain of dependence contained in the future cone of a point, then $D$ is foliated by surfaces of constant curvature $K \in (-\infty, 0)$.*

Theorem 4.D is proved in Section 4.4. Theorem 4.B and Theorem 4.C are discussed in Section 4.5.

## 4.1 Constructions for convex surfaces in Minkowski space

Recall from Subsection 1.1.3 that we defined the support function $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$ of a future-convex domain $D$ in $\mathbb{R}^{2,1}$ as

$$U(x) = \sup_{p\in D} \langle p, x\rangle_{2,1} \,.$$

We will mostly consider the restriction of $U$ to the Klein projective model of hyperbolic space, which is the disc

$$\mathbb{D} = \{(z,1) \in \mathbb{R}^{2,1} : |z| < 1\}\,.$$

This restriction will be denoted by lower case letters, $u = U|_{\mathbb{D}}$, and uniquely determines the 1-homogeneous extension $U$. We will generally write $u(z)$ instead of $u(z,1)$. Analogously, also the restriction of $U$ to the hyperboloid, denoted $\bar{u} = U|_{\mathbb{H}^2}$, can be uniquely extended to a 1-homogeneous function, and will be often used in the following.

*Remark* 4.1.1. It is easy to relate the restrictions $u$ and $\bar{u}$ of the support function to $\mathbb{D}$ and $\mathbb{H}^2$ respectively. Let us consider the radial projection $\pi : \mathbb{H}^2 \to \mathbb{D}$ defined by

$$\pi(x^1, x^2, x^3) = (x^1/x^3, x^2/x^3, 1)\,.$$

Its inverse is given by

$$\pi^{-1}(z,1) = \left( \frac{z}{\sqrt{1-|z|^2}}, \frac{1}{\sqrt{1-|z|^2}} \right)\,.$$

Since $U$ is 1-homogeneous, we obtain

$$u(z) = \sqrt{1-|z|^2}\,\bar{u}(\pi^{-1}(z))\,.$$

A 1-homogeneous convex function is called *sublinear*. We will now discuss in more detail some important properties of support functions.

**Lemma 4.1.2** ([FV13, Lemma 2.21])**.** *Given a future-convex domain $D$ in $\mathbb{R}^{2,1}$, the support function $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$ is sublinear and lower semicontinuous. Conversely, given a sublinear function $\hat{U}$ on $\mathrm{I}^+(0)$ (or equivalently every convex function $u$ on $\mathbb{D}$), consider the lower semicontinuous extension $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$, which is defined on $\partial \mathrm{I}^+(0)$ as*

$$U(x) = \liminf_{y \to x} \hat{U}(y)\,.$$

*Then $U$ is the support function of a future-convex domain, defined by*

$$D = \{p \in \mathbb{R}^{2,1} : \langle p, x \rangle \leq U(x) \text{ for every } x \in \mathrm{I}^+(0)\}\,.$$

The support function of a future-convex domain $D$ is finite on the image of the Gauss map of $\partial_s D$, since for every point $x$ in $G(\partial_s D)$ there exists a support plane with normal vector $x$. Observe that $\bar{u}(x) = \infty$ if $x \in \mathbb{H}^2 \setminus \overline{G(\partial_s D)}$. We will call *support function at infinity* the restriction of $U$ to $\partial\mathbb{D} = \{(z,1) : |z| = 1\}$. Given $z \in \partial\mathbb{D}$, $u(z) < +\infty$ if and only if there exists a lightlike support plane $P$ orthogonal to the lightlike vector $(z,1)$. In this case $-u(z)$ is the intercept of $P$ on the $x^3$-axis.

Some explicit examples have been considered in Example 1.1.10. Some less elementary examples will be descibed in Section 4.2. See also Remark 4.3.10.

In this chapter, we are mostly concerned with domains of dependence for which the support function at infinity is finite, and is actually bounded. Geometrically, this means that the domain is contained in the future cone over some point.

The following lemma will be useful to compute the value of support functions at infinity.

**Lemma 4.1.3** ([Roc70, Theorem 7.4,7.5]). *Let $U : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$ be a sublinear and lower semincontinuous function. Let $c : [0,1] \to \overline{\mathrm{I}^+(0)}$ be a spacelike line such that $x = c(1) \in \partial \mathrm{I}^+(0)$. Then $U(x) = \lim_{t\to 1} U(c(t))$.*

We define the *hyperbolic Hessian* of a function $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$ as the $(1,1)$-tensor

$$\mathrm{Hess}\bar{u}(v) = \nabla^{\mathbb{H}^2}_v \operatorname{grad} \bar{u} \,,$$

where $\nabla^{\mathbb{H}^2}$ is the Levi-Civita connection of $\mathbb{H}^2$. We denote by $D^2 u$ the Euclidean Hessian of a function $u$ defined on an open subset of $\mathbb{R}^2$. In the following, the identity operator is denoted by $E$.

Some of the geometric invariants of $\partial_s D$ can be directly recovered from the support function. This is the content of next lemma.

**Lemma 4.1.4** ([FV13, §2.10, 2.13]). *Let $D$ be a future-convex domain in $\mathbb{R}^{2,1}$ and let $G : \partial_s D \to \mathbb{H}^2$ be its Gauss map.*

- *If the support function is $C^1$, then the intersection of $\partial_s D$ with any spacelike support plane consists of exactly one point. The inverse of the Gauss map $G$ is well defined and is related to the support function of $D$ by the formula*

$$G^{-1}(x) = \operatorname{grad} \bar{u}(x) - \bar{u}(x) x \,, \tag{4.1}$$

*where $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$ is the support function restricted to $\mathbb{H}^2$.*

- *If the support function is $C^2$ and the operator $\mathrm{Hess}\bar{u} - \bar{u}\mathrm{I}$ is positive definite, then $\partial_s D$ is a convex $C^2$-surface. The inverse of its shape operator and its curvature are*

$$B^{-1} = \mathrm{Hess}\, \bar{u} - \bar{u}\, E \,, \tag{4.2}$$

$$-\frac{1}{K(G^{-1}(x))} = \det(\mathrm{Hess}\, \bar{u} - \bar{u}\, E)(x) = (1 - |z|^2)^2 \det D^2 u(z) \,, \tag{4.3}$$

*where $z = \pi(x)$ is the point of $\mathbb{D}$ obtained from $x$ by radial projection.*

We will often abuse notation and write $G^{-1}(z)$ in place of $G^{-1}(\pi^{-1}(z))$ for $z \in \mathbb{D}$.

### 4.1.1 The boundary value of the support function of an entire graph

In this subsection we will give another geometric interpretation of the support function at infinity of a future-convex entire graph in Minkowski space. This interpretation is much in the spirit of the theory of constant mean curvature surfaces in $\mathbb{R}^{2,1}$, and is taken as a definition in several articles, for instance [Tre82] and [CT90]. Hence the following proposition will clarify that the asymptotic conditions defined for instance in [Tre82, CT90] coincide with those considered here and in [GJS06, Li95].

**Proposition 4.1.5.** *Let $S$ be the boundary of a future-convex domain in $\mathbb{R}^{2,1}$. Denote by $f : \mathbb{R}^2 \to \mathbb{R}$ the function defining $S$ as a graph and $u : \mathbb{D} \to \mathbb{R}$ the support function. Then*

$$\lim_{r\to\infty} (f(rz) - r) = -u(z)$$

*for every unitary vector $z \in \partial\mathbb{D}$.*

*Proof.* First consider the case where $D = \mathrm{I}^+(p) = graph(f_p)$ where $p = (w_0, a_0)$ and $f_p(z) = |z - w_0| + a_0$. In this case the support function is $u_p(z) = \langle z, w_0\rangle - a_0$. A simple computation shows

$$\begin{aligned} f_p(rz) - r =& \sqrt{|w_0|^2 - 2r\langle z, w_0\rangle + r^2} - r + a_0 \\ =& \frac{-2r\langle z, w_0\rangle + |w_0|^2}{\sqrt{|w_0|^2 - 2r\langle z, w_0\rangle + r^2} + r} + a_0 \longrightarrow -\langle z, w_0\rangle + a_0 = -u_p(z)\,. \end{aligned}$$

Now consider the general case. Imposing that the point $(rz, f(rz))$ lies in the future of the support plane $\{q \in \mathbb{R}^{2,1} \mid \langle q, (z,1)\rangle = u(z)\}$ we get $f(rz) - r \geq -u(z)$. So it is sufficient to prove that $\limsup(f(rz) - r) \leq -u(z)$.

Fix $\epsilon > 0$ and consider the lightlike plane $P = \{q \in \mathbb{R}^{2,1} \mid \langle q, (z,1)\rangle = u(z) - \epsilon\}$. This plane must intersect the future of $S$. Let $p = (w_0, a_0)$ be a point in this intersection. The cone $\mathrm{I}^+(p)$ is contained in the future of $S$, hence $f_p \geq f$, where $f_p$ is the graph function for $\mathrm{I}^+(p)$ as above.

In particular, using the computation above for $\mathrm{I}^+(p)$,

$$\limsup_{r\to+\infty}(f(rz) - r) \leq \lim_{r\to+\infty}(f_p(rz) - r) = -\langle z, w_0\rangle + a_0\,.$$

Imposing that $p$ lies on the plane $P$,

$$-\langle z, w_0\rangle + a_0 = -\langle (z,1), p\rangle = -u(z) + \epsilon\,.$$

Therefore, for any $\epsilon > 0$,

$$\limsup_{r\to+\infty}(f(rz) - r) \leq -u(z) + \epsilon$$

and this concludes our claim, since $\epsilon$ is arbitrary. $\square$

### 4.1.2 Homogeneous functions and vector fields on $S^1$

We have defined support functions as 1-homogeneous functions. However, Theorem 4.B and Theorem 4.C express a geometric property of convex surfaces in $\mathbb{R}^{2,1}$ in terms of Zygmund regularity, which is a well-defined regularity for vector fields on $S^1$. The purpose of this section is to fill this gap, by showing that a vector field on $S^1$ defines a 1-homogeneous function on the boundary of the null-cone in a natural way.

Let us consider the boundary at infinity $\partial_\infty\mathbb{H}^2$ of $\mathbb{H}^2$, as $\mathbb{P}(N) \cong S^1$, where $N = \partial\mathrm{I}^+(0) \setminus \{0\}$. In particular, we will use vector fields on $S^1$ to define 1-homogeneous functions on $\partial\mathrm{I}^+(0)$. We want to show that this is well-defined, i.e. does not depend on the choice of a section $S^1 \to N$.

**Lemma 4.1.6.** *There is a* 1*-to-*1 *correspondence between vector fields* $X$ *on* $S^1$ *and* 1*-homogeneous functions* $H : N \to \mathbb{R}$ *satisfying the following property: if* $\gamma : S^1 \to N$ *is any* $C^1$ *spacelike section of the projection* $N \to S^1$ *and* $v$ *is the unit tangent vector field to* $\gamma$*, then*

$$\gamma_*(X(\xi)) = H(\gamma(\xi))v(\gamma(\xi))\,. \tag{4.4}$$

*Proof.* Consider coordinates $(x^1, x^2, x^3)$ on $\mathbb{R}^{2,1}$ and $z, \theta$ on $N$ given by

$$\phi : (r, \theta) \to (r\cos\theta, r\sin\theta, r) \in N.$$

In these coordinates, the restriction of the Minkowski metric to $N$ takes the (degenerate) form

$$g = r^2 d\theta^2 \tag{4.5}$$

We take $\gamma_1$ to be the section $\gamma_1(\theta) = (1, \theta)$. Namely, the image of $\gamma_1$ is $N \cap \{x^3 = 1\}$. Any other section $\gamma_2$ is of the form $\gamma_2(\theta) = (r(\theta), \theta)$ and is obtained as $\gamma_2 = f \circ \gamma_1$ by a radial map $f(1, \theta) = (r(\theta), \theta)$. Let $X$ be a vector field on $S^1$. We define a 1-homogeneous function $H$ such that $(\gamma_1)_*(X(\theta)) = H(1, \theta)v_1$ and compute

$$(\gamma_2)_*(X(\theta)) = f_*(H(1, \theta)v_1) = H(1, \theta)f_*(v_1).$$

Now $f_*(v_1)$ is a tangent vector to $\gamma_2(S^1)$ whose norm (recall the form (4.5) of the metric) is $r(\theta)$. Therefore

$$(\gamma_2)_*(X(\theta)) = H(1, \theta)r(\theta)v_2 = H(r(\theta), \theta)v_2$$

where $v_2$ is the unit tangent vector. Conversely, given any 1-homogeneous function, (4.4) defines a vector field on $S^1$ which does not depend on the choice of $\gamma$. $\square$

In light of the theory of infinitesimal earthquakes explained in Subsection 2.3.3, we give an explicit example of this relation between vector fields on $S^1$ and 1-homogeneous functions on $N$, the boundary of the null-cone. This is obtained by computing the vector field which is an infinitesimal earthquake along a single geodesic.

*Example* 4.1.7. Let $\mu$ be the measured geodesic lamination whose support consists of a single geodesic $l$, with weight 1. Then, once a point $x_0 \in \mathbb{H}^2 \setminus l$ is fixed, it is easy to describe the earthquake along $\mu$:

$$E_l([\eta]) = \begin{cases} [\eta] & \text{if } x_0 \text{ and } [\eta] \text{ are in the same component of } (\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \setminus \bar{l} \\ [A^l(1)(\eta)] & \text{otherwise} \end{cases} \tag{4.6}$$

for any $\eta \in N$, where $A^l(t) \in \mathrm{SO}(2,1)$ induces the hyperbolic isometry of $\mathbb{H}^2$ which translates on the left (as seen from $x_0$) along the geodesic $l$ by lenght $t$. Hence the 1-homogeneous function $H$ associated to the Zygmund field

$$\dot{E}_l = \left.\frac{d}{dt}\right|_{t=0} E_{tl}$$

(as in Lemma 4.1.6) has the following expression, for any section $\gamma : S^1 \to N$:

$$H(\gamma(\xi)) = \begin{cases} 0 & \text{if } x_0 \text{ and } [\gamma(\xi)] \text{ are in the same} \\ & \text{component of } (\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \setminus \bar{l} \\ \langle \dot{A}^l(\gamma(\xi)), v(\gamma(\xi)) \rangle & \text{otherwise} \end{cases} \tag{4.7}$$

where $v$ is any unit spacelike tangent vector field to $N$, in the counterclockwise orientation. Under the standard identification of $S^1$ with $\partial\mathbb{D}$, we obtain, for $\eta \in \partial\mathbb{D}$

$$\dot{E}_l(\eta) = \langle \dot{A}^l(\eta), v\rangle v \,.$$

where $v$ is now the unit tangent vector to $\partial\mathbb{D}$.

The above explicit expression will be useful in the proof of Proposition 4.5.2, given in Subsection 4.5.1.

### 4.1.3 Cauchy surfaces and domains of dependence

Given a future-convex domain $D$ in $\mathbb{R}^{2,1}$, a *Cauchy surface* for $D$ is a spacelike embedded surface $S \subseteq D$ such that every differentiable inextensible causal path in $D$ (namely, such that its tangent vector is either timelike or lightlike at every point) intersects $S$ in exactly one point. Given an embedded surface $S$ in $\mathbb{R}^{2,1}$, the maximal future-convex domain $D(S)$ such that $S$ is a Cauchy surface for $D(S)$ is the *domain of dependence* of $S$. It turns out that $D(S)$ is obtained as intersection of future half-spaces bounded by lightlike planes which do not disconnect $S$.

It is easy to prove the following lemma.

**Lemma 4.1.8.** *Let $h : \overline{\mathbb{D}} \to \mathbb{R}$ be the support function of a future-convex domain $D$, with $h|_{\partial\mathbb{D}} < \infty$. Let $S \subseteq D$ be a convex embedded surface and let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be the support function of $S$. Then $S$ is a Cauchy surface for $D$ if and only if $h|_{\partial\mathbb{D}} = u|_{\partial\mathbb{D}}$.*

Domains of dependence can be characterized in terms of the support function, see [BF14, Proposition 2.21].

**Lemma 4.1.9.** *Let $D$ be a domain of dependence in $\mathbb{R}^{2,1}$, whose lightlike support planes are determined by the function $\varphi : \partial\mathbb{D} \to \mathbb{R}\cup\{\infty\}$. Then the support function $h : \overline{\mathbb{D}} \to \mathbb{R}$ of $D$ is the convex envelope $h = \mathrm{co}\,(\varphi)$, namely:*

$$h(z) = \sup\{f(z) : f \text{ is an affine function on } \mathbb{D}, f|_{\partial\mathbb{D}} \leq \varphi\} \,.$$

An example of this phenomenon can be obtained by looking at the leaves of the cosmological time. Observe that a *timelike distance* (compare the similar definitions given in Section 1.3) can be defined for two points $x_1$ and $x_2 \in \mathrm{I}^+(x)$ in $\mathbb{R}^{2,1}$, by means of the definition

$$d(x_1, x_2) = \sup_\gamma \int_\gamma \sqrt{|\langle\gamma'(t), \gamma'(t)\rangle|}dt \,,$$

where the supremum is taken over all causal paths $\gamma$ from $x_1$ to $x_2$. This is not a distance though, because it satisfies a reverse triangle inequality; however, $d(x_1, x_2)$ is achieved along the geodesic from $x_1$ to $x_2$. Given an embedded spacelike surface $S$, consider the equidistant surface

$$S_d = \{x \in \mathbb{R}^{2,1} : x \in \mathrm{I}^+(S), d(x, S) = d\} \,,$$

where of course $d(x, S) = \sup_{x' \in S} d(x, x')$. If the support function of $S$ restricted to $\mathbb{H}^2$ is $\bar{u}$, then $S_d$ has support function (see for instance [FV13])

$$\bar{u}_d(x) = \bar{u}(x) - d \,.$$

This can be applied also for $\partial_s D$, instead of an embedded surface. In this way, we obtain the level sets of the *cosmological time*, namely the function $T : D \to \mathbb{R}$ defined by

$$T(x) = \sup_\gamma \int_\gamma \sqrt{|\langle \gamma'(t), \gamma'(t) \rangle|} dt \,,$$

where the supremum is taken over all causal paths $\gamma$ in $D$ with future endpoint $x$. If $\bar{h} : \mathbb{H}^2 \to \mathbb{R}$ is the support function of $D$, the level sets $L_d = \{T = d\}$ of the cosmological time have support function on the disc $\bar{h}_d(x) = \bar{h}(x) - d$. It can be easily seen that all leaves of the cosmological time of $D$ are Cauchy surfaces for $D$ (although only $C^{1,1}$). Indeed, the support functions $h_d : \mathbb{D} \to \mathbb{R}$ can be computed:

$$h_d(z) = h(z) - d\sqrt{1 - |z|^2} \,.$$

Therefore they all agree with $h$ on $\partial\mathbb{D}$.

### 4.1.4 Dual lamination

In his groundbreaking work, Mess associated a domain of dependence $D$ to every measured geodesic lamination $\mu$, in such a way that the support function $h : \mathbb{D} \to \mathbb{R}$ of $D$ is linear on every stratum of $\mu$. Although we do not enter into details here, the measure of $\mu$ determines the bending of $h$. (Recall $h$ is the convex envelope of some lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.) The domain $D$ is determined up to translation in $\mathbb{R}^{2,1}$, and $\mu$ is called *dual lamination* of $D$.

Given $y_0 \in \mathbb{R}^{2,1}$ and $x_0 \in \mathbb{H}^2$ , we will denote by $D(\mu, x_0, y_0)$ the domain of dependence having $\mu$ as dual laminations and $P = y_0 + x_0^\perp$ as a support plane tangent to the boundary at $y_0$.

We sketch here the explicit construction of $D(\mu, x_0, y_0)$. In the following, given the oriented geodesic interval $[x_0, x]$ in $\mathbb{H}^2$, $\boldsymbol{\sigma} : \mathcal{G}[x_0, x] \to \mathbb{R}^{2,1}$ is the function which assigns to a geodesic $l$ (intersecting $[x_0, x]$) the corresponding point in $\mathrm{d}\mathbb{S}^2$, namely, the spacelike unit vector in $\mathbb{R}^{2,1}$ orthogonal to $l$ for the Minkowski product, pointing outward with respect to the direction from $x_0$ to $x$. Then

$$y(x) = y_0 + \int_{\mathcal{G}[x_0,x]} \boldsymbol{\sigma} d\mu \tag{4.8}$$

is a point of the regular boundary of $D(\mu, x_0, y_0)$ such that $y(x) + x^\perp$ is a support plane for $D(\mu, x_0, y_0)$, for every $x \in \mathbb{H}^2$ such that the expression in Equation (4.8) is integrable. The image of the Gauss map of the regular boundary of $D(\mu, x_0, y_0)$ is composed precisely of those $x \in \mathbb{H}^2$ which satifsy this integrability condition.

In the following proposition, we give an explicit expression for the support function of the domain of dependence $D(\mu, x_0, y_0)$ we constructed. By an abuse of notation, given two points $x_0, x \in \mathrm{I}^+(0)$, we will denote by $[x_0, x]$ the geodesic interval of $\mathbb{H}^2$ obtained by projecting to the hyperboloid $\mathbb{H}^2 \subset \mathbb{R}^{2,1}$ the line segment from $x_0$ to $x$.

**Proposition 4.1.10.** *Suppose $D$ is a domain of dependence in $\mathbb{R}^{2,1}$ with dual lamination $\mu$ and such that the plane $P = y_0 + (x_0)^\perp$ is a support plane for $D$, for*

*$y_0 \in \partial_s D$ and $x_0 \in \mathbb{H}^2$. Then the support function $H : \overline{\mathrm{I}^+(0)} \to \mathbb{R}$ of $D$ is:*

$$H(x) = \langle x, y_0 \rangle + \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu \,. \tag{4.9}$$

Indeed, the expression in Equation (4.9) holds for $x \in \mathbb{H}^2$ by Equation (4.8). Since the expression is 1-homogeneous, it is clear that it holds for every $x \in \mathrm{I}^+(0)$. Using Lemma 4.1.3, the formula holds also for the lower semicontinuous extension to $\partial \mathrm{I}^+(0)$.

It is easily seen from Equation (4.9) that the support function $h : \mathbb{D} \to \mathbb{R}$ (which is the restriction of $H$ to $\mathbb{D}$) is affine on each stratum of $\mu$. In [Mes07] it was proved that every domain of dependence can be obtained by the above construction. Hence a dual lamination is uniquely associated to every domain of dependence.

The work of Mess ([Mes07]) mostly dealt with domains of dependence which are invariant under a discrete group of isometries $\Gamma < \mathrm{Isom}(\mathbb{R}^{2,1})$, whose linear part is a cocompact Fuchsian group. We resume here some results.

**Proposition 4.1.11.** *Let $D$ be a domain of dependence in $\mathbb{R}^{2,1}$ with dual lamination $\mu$. The measured geodesic lamination $\mu$ is invariant under a cocompact Fuchsian group $G$ if and only if $D$ is invariant under a discrete group $\Gamma < \mathrm{Isom}(\mathbb{R}^{2,1})$ such that the projection of $\Gamma$ to $\mathrm{SO}(2,1)$ is an isomorphism onto $G$. In this case, assuming $P = y_0 + (x_0)^\perp$ is a support plane for $D$, for $y_0 \in \partial_s D$ and $x_0 \in \mathbb{H}^2$, the translation part of an element $g \in G$ is:*

$$t_g = \int_{\mathcal{G}[x_0,g(x_0)]} \boldsymbol{\sigma} d\mu \,.$$

### 4.1.5 Relation to Monge-Ampère equations

Given a smooth strictly convex spacelike surface $S$ in $\mathbb{R}^{2,1}$, let $U : \mathrm{I}^+(0) \to \mathbb{R}$ be the support function of $S$ and let $u$ be its restriction to $\mathbb{D} = \mathrm{I}^+(0) \cap \left\{x^3 = 1\right\}$. Given a point $z \in D$, let $x = \pi^{-1}(z) \in \mathbb{H}^2$. The curvature of $S$ is given by (see Lemma 4.1.4)

$$-\frac{1}{K(G^{-1}(x))} = (1 - |z|^2)^2 \det D^2 u(z)\,,$$

where $G : S \to \mathbb{H}^2$ is the Gauss map, which is a diffeomorphism. For $K$-surfaces, namely surfaces with constant curvature equal to $K \in (-\infty, 0)$, the support function satisfies the Monge-Ampère equation

$$\det D^2 u(z) = \frac{1}{|K|}(1 - |z|^2)^{-2}\,. \tag{4.10}$$

More generally, the Minkowski problem consists of finding a convex surface with prescribed curvature function on the image of the Gauss map. Given a smooth function $\psi : \mathbb{D} \to \mathbb{R}$, the support function of a surface with curvature $K(G^{-1}(z)) = -\psi(z)$ solves the Monge-Ampère equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2}\,. \tag{MA}$$

It will be useful for us to obtain an a priori estimate of the $C^2$-norm of support functions of surfaces of constant curvature in Minkowski space in terms of the $C^0$-norm. Although this result is well-known we sketch a proof here. The main tools have been introduced in Chapter 3, especially Theorem 3.3.4 and Theorem 3.3.5.

**Lemma 4.1.12.** *Let $u_n : \mathbb{D} \to \mathbb{R}$ be a sequence of smooth solutions of the Monge-Ampère equation*

$$\det D^2(u_n) = \frac{1}{|K|}(1-|z|^2)^{-2}$$

*uniformly bounded on $\mathbb{D}$. Then $||u_n||_{C^2(\Omega)}$ is uniformly bounded on any compact domain $\Omega$ contained in $\mathbb{D}$.*

*Proof.* Assume that the conclusion is false and that there is a subsequence (which we still denote by $u_n$ by a slight abuse of notation) for which the $C^2$-norm goes to infinity. Hence it suffices to show that there exists a further subsequence $u_{n_k}$ for which $||u_{n_k}||_{C^2(\Omega)}$ is bounded. Take $\Omega'$ such that $\Omega \subset\subset \Omega' \subset\subset \mathbb{D}$. Using the uniform bound on $||u_n||_{C^0(\mathbb{D})}$ and the convexity, one can derive that the $C^1$-norms $||u_n||_{C^1(\overline{\Omega}')}$ are uniformly bounded by a constant $C$. By Ascoli-Arzelà theorem, we can extract a subsequence which converges uniformly on compact subsets of $\Omega'$. Let $u_\infty$ be the limit function. By Lemma 3.3.1, $u_\infty$ is a generalized solution to

$$\det D^2(u_\infty) = \frac{1}{|K|}(1-|z|^2)^{-2}$$

and $u_\infty$ is strictly convex, by Theorem 3.3.5.

For any $z \in \Omega$ and $n \geq 0$ we can fix an affine function $f_{n,z}$ such that $v_{n,z} = u_n + f_{n,z}$ takes its minimum at $z$ and $v_{n,z}(z) = 0$. We claim that there are $\epsilon_0 > 0$ and $r_0 > 0$ such that

- $\min_{\partial\Omega'} v_{n,z} \geq 2\epsilon_0$ for any $n \geq 0$ and $z \in \Omega$.
- $\max_{B(z,r_0)} v_{n,z} \leq \epsilon_0$ for any $n \geq 0$ and $z \in \Omega$.

First let us show how the claim implies the statement. Indeed for any $z$ and $n$ consider the domain $U_{n,z} = \{z \in \Omega' \,|\, v_{n,p}(z) \leq \epsilon_0\}$. We have that $U_{n,z} \subset\subset \Omega'$ by the first point of the claim. In particular, $v_{n,z}$ is constant equal to $\epsilon_0$ along the boundary of $U_{n,z}$. On the other hand the second point of the claim implies that the distance of $z$ from $\partial U_{n,z}$ is at least $r_0$. So by Theorem 3.3.4 there is a constant $C'$ depending on $C$ and $r_0$ such that $||D^2 u_n(z)|| = ||D^2 v_{n,z}(z)|| < C'$ for all $z \in \Omega'$ and $n \geq 0$.

To prove the claim we argue by contradiction. Suppose there exist sequences $z_n, z'_n \in \Omega$ such that, defining $2\epsilon_n = \min_{\partial\Omega'} v_{n,z_n}$,

- $||z_n - z'_n|| \to 0$;
- $v_{n,z_n}(z'_n) > \epsilon_n$.

Up to passing to a subsequence we may suppose that $z_n \to z_\infty$, so that $z'_n \to z_\infty$ as well. As the $C^1$-norm of $u_n$ is bounded, the $C^1$-norm of $f_{n,z}$ is uniformly bounded for any $z \in \Omega$ and $n \geq 0$, so we may suppose that $f_{n,z_n}$ converges to an affine function $f_\infty$. Therefore $v_{n,z_n}$ converges to $v_\infty = u_\infty + f_\infty$.

As $\lim v_{n,z_n}(z'_n) = v_\infty(z_\infty) = \lim v_{n,z_n}(z_n) = 0$ we conclude that $\epsilon_n \to 0$, so that $\min_{\partial\Omega'} v_\infty = 0$, but this contradicts the strict convexity of $v_\infty$. $\square$

On the other hand, by considering a spacelike convex entire graph in $\mathbb{R}^{2,1}$, namely a surface $S$ obtained as

$$S = \{(z, f(z)) : z \in \mathbb{R}^2\}$$

for a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $|Df| < 1$, it turns out that the curvature $K(z)$ of $S$ at the point $(z, f(z))$ is expressed by the following relation ([Li95]):

$$K(z) = -\frac{1}{(1 - ||Df||^2)^2} \det D^2 f \,,$$

which shows again that the problem of existence of surfaces of constant curvature in Minkowski space is related to an equation of Monge-Ampére type. Indeed, when we restrict to the case of constant Gaussian curvature $K < 0$, we obtain the following equation:

$$\det D^2 f_K = |K|(1 - ||Df||^2)^2 \,,$$

## 4.2 Some explicit solutions: surfaces with a 1-parameter family of symmetries

In this subsection we construct some explicit solutions to the Monge-Ampère equation associated to surfaces with constant curvature $K < 0$, namely

$$\det D^2 u(z) = \frac{1}{|K|}(1 - |z|^2)^{-2} \,. \tag{4.11}$$

We study constant curvature surfaces invariant under a one-parameter subgroup of $\mathrm{Isom}(\mathbb{R}^{2,1})$.

### 4.2.1 Linear parabolic one-parameter group

Let us consider surfaces invariant for one-parameter parabolic subgroup. In order to have such a surface, the subgroup must necessarily fix the origin.

Hence, let us denote by $A_\bullet : \mathbb{R} \to \mathrm{Isom}(\mathbb{R}^{2,1})$ the representation associated to the linear parabolic subgroup. Let us choose a basis $\{v_0, v_1, v_2\}$ of $\mathbb{R}^{2,1}$ such that $v_0$ is the null vector fixed by the parabolic group, $v_1$ is a null vector with $\langle v_0, v_1\rangle = -1$, and $v_2$ is a spacelike unit vector orthogonal to both $v_0$ and $v_1$.

The parabolic group is acting by

$$\begin{aligned} A_t(v_0) =& v_0 \,; \\ A_t(v_1) =& (t^2/2)v_0 + v_1 + tv_2 \,; \\ A_t(v_2) =& v_2 + tv_0 \,. \end{aligned}$$

Let $\gamma_0(s) = \frac{\sqrt{2}}{2}(e^s v_0 + e^{-s} v_1)$ be the unit speed geodesic of $\mathbb{H}^2$ with endpoints $[v_1]$ (for $s \to -\infty$) and $[v_0]$ (for $s \to +\infty$). Let us consider the following parametrization of $\mathbb{H}^2$:

$$\sigma(t, s) = A_t(\gamma_0(s)) \,,$$

namely, the levels $\{s = c\}$ are horocycles, while the levels $\{t = c\}$ are geodesics asymptotic to $[v_0]$. In these coordinates, the metric of $\mathbb{H}^2$ takes the form $ds^2 + (e^{-2s}/2)dt^2$.

We consider support functions restricted to $\mathbb{H}^2$, which we denote as usual by $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$, corresponding to surfaces of constant curvature. Hence we want to find solutions of the equation

$$\det(\mathrm{Hess}\,\bar{u} - \bar{u}\,E) = \frac{1}{|K|}\,, \tag{4.12}$$

where $K$ is a negative constant. Since are imposing that the surface dual to $\bar{u}$ is invariant for the parabolic group (with no translation), recalling Equation (1.8) for the transformation of support functions under isometries of $\mathbb{R}^{2,1}$, we look for a solution which only depends on $s$, namely a solution of the form $\bar{u}(t, s) = f(s)$.

By a direct computation, one can see that the gradient and the Hessian of $\bar{u}$ for the hyperbolic metric in this coordinate frame has the form:

$$\begin{aligned} \mathrm{grad}\,\bar{u} =& f'(s)\partial_s \\ \mathrm{Hess}(\bar{u})(\partial_s) =& f''(s)\partial_s \\ \mathrm{Hess}(\bar{u})(\partial_t) =& -f'(s)\partial_t\,, \end{aligned}$$

therefore the constant curvature condition (4.12) gives

$$(f''(s) - f(s))(-f'(s) - f(s)) = 1/|K|\,. \tag{4.13}$$

We now solve Equation (4.13). By convexity, we impose that both eigenvalues $(f''(s)-f(s))$ and $(-f'(s)-f(s))$ are positive. Let us perform the change of variables

$$g(s) = -f'(s) - f(s)\,, \tag{4.14}$$

so that Equation (4.13) becomes

$$g(s)(g(s) - g'(s)) = 1/|K|\,, \tag{4.15}$$

whose general positive solution is, as $C$ varies in $\mathbb{R}$,

$$g(s) = \sqrt{|K|^{-1} + Ce^{2s}}\,. \tag{4.16}$$

**Solutions for $C = 0$**

We observe that the case $C = 0$ gives the trivial solution, namely the hyperboloid. Indeed $f$ can be recovered by integrating (4.14), hence obtaining

$$f(s) = e^{-s}\left(D - \int_0^s e^x g(x)dx\right)\,. \tag{4.17}$$

Observe that the term $e^{-s}D$ corresponds to a translation in the direction $-\sqrt{2}Dv_0$. Hence, as the parameter $D$ varies over $\mathbb{R}$, the corresponding surface varies by a translation in the line spanned by $v_0$. If $C = 0$, we have $g \equiv 1/\sqrt{|K|}$. By choosing $D$ suitably, we obtain the solution

$$f_{0,K}(s) = -\frac{1}{\sqrt{|K|}}\,, \tag{4.18}$$

which is the support function of a hyperboloid of curvature $K$ centered at the origin.
**Solutions for $C > 0$**

If $C > 0$, from Equation (4.17) we obtain the solution (for a constant $D$ which we will fix later)

$$f_{C,K}(s) = -\frac{1}{2}\sqrt{|K|^{-1} + Ce^{2s}} - \frac{1}{2|K|\sqrt{C}} e^{-s} \log\left(\sqrt{C}\sqrt{|K|^{-1} + Ce^{2s}} + Ce^{s}\right) + e^{-s}D\,. \tag{4.19}$$

We now describe some of the properties of the surface $S_{C,v_0}(K)$ whose support function is $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$, for any fixed curvature $K < 0$.

First, we want to determine the value on $\partial\mathbb{D}$ of the support function $u_{C,K}$, namely the restriction to $\mathbb{D}$ of the 1-homogeneous extension of $\bar{u}_{C,K}$. Let us denote by $\sigma(t,s)_z$ the vertical component of $\sigma(t,s)$. Then we have

$$u_{C,K}(t,s) = \frac{f_{C,K}(s)}{\sigma(t,s)_z}\,.$$

Using the invariance for the parabolic group in $\mathrm{Isom}(\mathbb{R}^{2,1})$, it suffices to consider the case $t = 0$. We have

$$\sigma(0,s)_z = -\langle\sigma(0,s), \frac{v_0 + v_1}{\sqrt{2}}\rangle = \cosh(s)\,.$$

Observe that the term $e^{-s}D/\cosh(s)$ tends to 0 when $s \to \infty$ and to $2D$ when $s \to -\infty$. By an explicit computation, choosing $D$ suitably, we can obtain

$$\lim_{s\to-\infty} u_{C,K}(t,s) = \lim_{s\to-\infty} \frac{f_{C,K}(s)}{\sigma(t,s)_z} = 0\,,$$

while

$$\lim_{s\to-\infty} u_{C,K}(t,s) = \lim_{s\to+\infty} \frac{f_{C,K}(s)}{\sigma(t,s)_z} = -\sqrt{C}\,,$$

hence the support function at infinity is

$$u_{C,K}|_{\partial\mathbb{D}}(z) = \begin{cases} -\sqrt{C} & [z] = [v_0] \\ 0 & [z] \neq [v_0] \end{cases}.$$

Geometrically, this means that the domain of dependence of the surface is the future of a parabola, obtained as the intersection of the cone centered at the origin (whose support function is identically 0) with a lightlike plane with normal vector $v_0$ (whose intercept on the $z$-axis is $\sqrt{C}$). See Figure 4.1.

It is easy to see that the dual lamination is the measured geodesic lamination of $\mathbb{H}^2$ whose leaves are all geodesics asymptotic to $[v_0]$, with a measure invariant under the parabolic group (a multiple of the Lebesgue measure, in the upper half-space model, pictured in Figures 4.2 and 4.3).

The surface $S_{C,v_0}(K)$ dual to $u_{C,K}$ can be described by an explicit parametrization, recalling that the inverse of the Gauss map of the surface is

$$G^{-1}(x) = \mathrm{grad}\,\bar{u}_{C,K}(x) - \bar{u}_{C,K}(x)x\,.$$

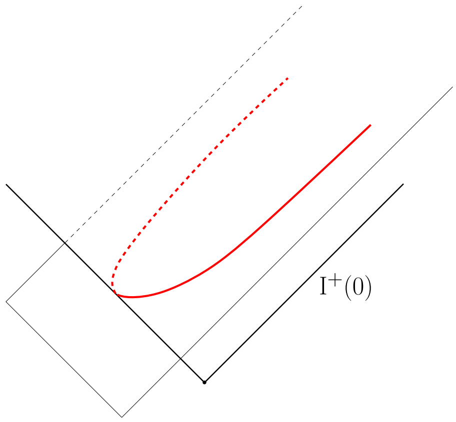



**Figure 4.1:** The future of a parabola is a domain of dependence invariant for the parabolic group. Its support function at infinity is lower-semicontinuous and is affine on the complement of a point of $\partial\mathbb{D}$.

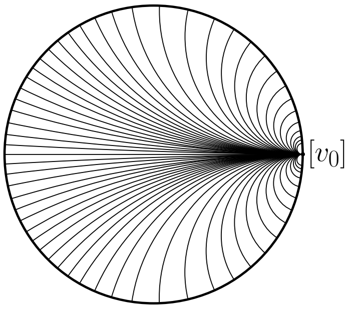



**Figure 4.2:** The dual lamination to the boundary of the domain of dependence, which is the future of a parabola.

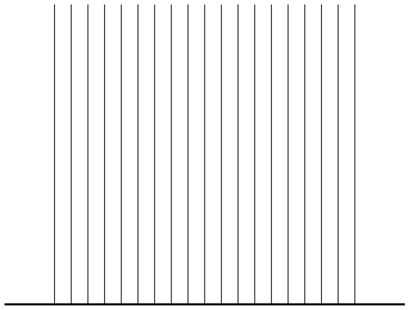


**Figure 4.3:** In the upper half space model, with fixed point at infinity, the measure is the Lebesgue measure.

In these coordinates,

$$G^{-1}(\sigma(t,s)) = \operatorname{grad} \bar{u}_{C,K}(t,s) - \bar{u}_{C,K}(t,s)\sigma(t,s) \tag{4.20}$$

$$= \frac{\sqrt{2}}{2} f'(s)(e^s v_0 - e^{-s} A_t(v_1)) - \frac{\sqrt{2}}{2} f(s)(e^s v_0 + e^{-s} A_t(v_1)) \tag{4.21}$$

$$= \frac{\sqrt{2}}{2} \big((f'(s) - f(s))e^s v_0 - (f'(s) + f(s))e^{-s} A_t(v_1)\big) \tag{4.22}$$

$$= \frac{\sqrt{2}}{2} \big(-(g(s) + 2f(s))e^s v_0 + g(s)e^{-s} A_t(v_1)\big) \ . \tag{4.23}$$

We now want to show that the surface is an entire graph. For this purpose, we will use the following criterion.

**Lemma 4.2.1.** *Let $u : \mathbb{D} \to \mathbb{R}$ be a $C^2$ support function with positive Hessian, and suppose that the inverse of the Gauss map $G^{-1} : \mathbb{D} \to \mathbb{R}^{2,1}$ is proper. Then the boundary of the future-convex domain $D$ defined by $u$ is a spacelike entire graph.*

*Proof.* As $u$ is $C^1$, we can use Equation (4.1) and get $G^{-1}(x) = \operatorname{grad} \bar{u}(x) - \bar{u}(x)x$ for every $x \in \mathbb{H}^2$. It can be readily shown that this implies that the vertical projection of $G^{-1}(x)$ is $Du(z)$, where $z = \pi(x) \in \mathbb{D}$, see Lemma 2.8 of [BF14]. As the Hessian of $u$ is positive, the image of the gradient map of $u$ is an open subset of $\mathbb{R}^2$, so it follows that $\partial_s D$ is open in $\partial D$. Since $G^{-1}$ is proper, $\partial_s D$ is also closed in $\partial D$, and this concludes the proof. $\square$

We will actually show that the height function given by

$$z(t,s) = -\langle G^{-1}(\sigma(t,s)), \frac{v_0 + v_1}{\sqrt{2}}\rangle$$

is proper. By a direct computation,

$$\begin{aligned}
2z(t,s) &= -(g(s) + 2f(s))e^s + g(s)e^{-s}\left(\frac{t^2}{2} + 1\right) \\
&= \frac{1}{|K|\sqrt{C}} \log\left(\sqrt{C}\sqrt{|K|^{-1} + Ce^{2s}} + Ce^s\right) + e^{-s}\sqrt{|K|^{-1} + Ce^{2s}}\left(\frac{t^2}{2} + 1\right) \\
&\geq \frac{1}{|K|\sqrt{C}} s + e^{-s}|K|^{-1/2}\left(\frac{t^2}{2} + 1\right) - C_0\,,
\end{aligned}$$

for some constant $C_0$. Observe that $z(t,s)$ tends to infinity for $s \to \pm\infty$. It is easily checked that on a sequence $\sigma(t_n, s_n)$ which escapes from every compact, $t_n^2 + s_n^2 \to \infty$, and thus $z(t_n, s_n) \to \infty$. This concludes the claim that $S_{C,v_0}(K)$ is a spacelike entire graph, by Lemma 4.2.1.

Finally, we briefly discuss the isometry type of the induced metric. By an explicit computation using the expression in Equation (4.20), we find the pull-back of the induced metric via $G^{-1}$:

$$(G^{-1} \circ \sigma)^*(g_{\mathbb{R}^{2,1}}) = (f''(s) - f(s))^2 ds^2 + \frac{1}{2}e^{-2s}g(s)^2 dt^2\,, \tag{4.24}$$

where it turns out that

$$f''(s) - f(s) = \frac{1}{|K|\sqrt{|K|^{-1} + Ce^{2s}}}$$

By an explicit change of variables

$$r(s) = \frac{1}{\sqrt{|K|}} \operatorname{arctanh}\left(\frac{1}{\sqrt{1 + |K|Ce^{2s}}}\right)\,,$$

so that $(r'(s))^2 = (f''(s) - f(s))^2$, by computing

$$g(s)^2 = \frac{1}{|K|}(1 + |K|Ce^{2s}) = \frac{1}{|K|\tanh^2(r\sqrt{|K|})}$$

and

$$e^{-2s} = |K|C\sinh^2\left(r\sqrt{|K|}\right)$$

one obtains that the induced metric is

$$dr^2 + \left(\frac{C}{2}\right)\cosh^2\left(r\sqrt{|K|}\right) dt^2\,.$$

Rescaling $t$, one obtains

$$dr^2 + \cosh^2\left(r\sqrt{|K|}\right) dt^2 ,$$

that is, the first fundamental form of $S_{C,v_0}(K)$ is isometric to a half-plane of constant curvature $K$, namely, to the region of a hyperboloid bounded by a geodesic $l$.

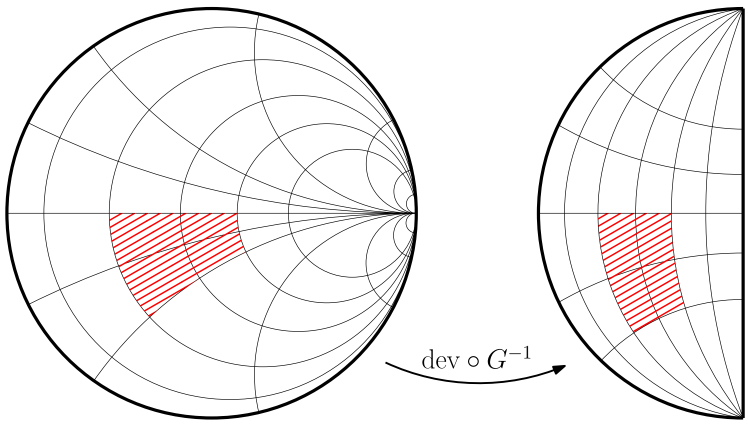



**Figure 4.4:** A developing map dev : $S_{C,v_0}(K) \to \mathbb{H}^2$ for the induced metric, composed with the inverse of the Gauss map $G^{-1} : \mathbb{H}^2 \to S_{C,v_0}(K)$, in the Poincaré disc model of $\mathbb{H}^2$.

We resume the content of this subsection in the following proposition. Let us denote by $\mathbb{H}^2(K)$ the rescaled hyperbolic plane, of curvature $K < 0$, and by $\mathbb{H}^2(K)_+$ a half-plane in $\mathbb{H}^2(K)$. Observe that $\mathbb{H}^2(K)_+$ has a one-parameter group of isometries $T(l)$ which consists of (the restriction of) hyperbolic translations along the geodesic $l$ which bounds $\mathbb{H}^2(K)_+$.

**Proposition 4.2.2.** *For every $K < 0$, $C > 0$ and every null vector $v_0 \in \mathbb{R}^{2,1}$ there exists an isometric embedding*

$$i_{K,C,v_0} : \mathbb{H}^2(K)_+ \to \mathbb{R}^{2,1}$$

*with image a Cauchy surface $S_{C,v_0}(K)$ in the domain of dependence whose support function at infinity is*

$$\varphi(z) = \begin{cases} -\sqrt{C} & [z] = [v_0] \\ 0 & [z] \neq [v_0] \end{cases} .$$

*The surface $S_{C,v_0}(K)$ is a spacelike entire graph and $i_{K,C,v_0}$ is equivariant with respect to the group of isometries $T(l)$ of $\mathbb{H}^2(K)_+$ and the parabolic linear subgroup of* $\mathrm{Isom}(\mathbb{R}^{2,1})$ *fixing $v_0$.*

**Solutions for $C < 0$**

If $C < 0$, the function $g(s) = \sqrt{|K|^{-1} + Ce^{2s}}$ is only defined for

$$s \leq \frac{1}{2}\log\left(\frac{1}{|CK|}\right) .$$

From Equation (4.17) we can explicitly write the solution (again $D$ is a constant to be fixed):

$$f_{C,K}(s) = -\frac{1}{2}\sqrt{|K|^{-1} + Ce^{2s}} - \frac{1}{2|K|\sqrt{|C|}}e^{-s}\arctan\left(\frac{\sqrt{|C|}e^s}{\sqrt{|K|^{-1} + Ce^{2s}}}\right) + e^{-s}D\,. \tag{4.25}$$

Again, we study briefly the properties of the surface $S_{C,v_0}(K)$ whose support function is $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$. Observe that the solution (4.25) is only defined in the range $s \leq \frac{1}{2}\log\left(\frac{1}{|CK|}\right)$, namely, in the complement of a horoball. Let us notice that, in the same notation as before, the limit of the support function (which only makes sense for $s \to -\infty$) is

$$\lim_{s\to-\infty} u_{C,K}(t,s) = \lim_{s\to-\infty}\frac{f_{C,K}(s)}{\sigma(t,s)_z} = 0\,,$$

provided we choose $D = 0$. On the other hand, as $s \to \frac{1}{2}\log\left(\frac{1}{|CK|}\right)$, the function $f_{C,K}(s)$ has the finite limit $-(\pi/4)\sqrt{|K|^{-1}}$. We observe that $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$ can be extended to a convex function defined on the whole $\mathbb{H}^2$ by declaring

$$f_{C,K}(s) = -\frac{\pi}{4}\frac{1}{|K|\sqrt{|C|}}e^s$$

for $s \geq \frac{1}{2}\log\left(\frac{1}{|CK|}\right)$. We will now denote by $\bar{u}_{C,K}(t,s) = f_{C,K}(s)$ the function extended in this way. The surface $S_{C,v_0}(K)$ is thus a constant curvature surface which develops a singular point, namely it intersects the boundary of the domain of dependence, which in this case is just $\mathrm{I}^+(0)$. The inverse of the Gauss map sends the whole horoball $\{s \geq \frac{1}{2}\log\left(\frac{1}{|CK|}\right)\}$ to the point $\frac{\sqrt{2}}{2}\frac{1}{|K|\sqrt{|C|}}\frac{\pi}{4}v_0$.

We remark that $u_{C,K}$ is a generalized solution to the Monge-Ampère equation on the disc $\det D^2 u = \nu$, where $\nu$ in this case is a measure which coincides with $(1/|K|)(1-|z|^2)^{-2}\mathcal{L}$ on the complement of the horoball (where $\mathcal{L}$ is the Lebesgue measure), and is 0 inside the horoball.

By a computation analogous to the previous case, to compute the induced metric we manipulate Equation (4.24): setting

$$r(s) = \frac{1}{\sqrt{|K|}}\operatorname{arctanh}(\sqrt{1+|K|Ce^{2s}})\,,$$

and replacing

$$g(s)^2 = \frac{1}{|K|}(1+|K|Ce^{2s}) = \frac{1}{|K|}\tanh^2(r\sqrt{|K|})$$

and

$$e^{-2s} = |K|C\cosh^2(r\sqrt{|K|})$$

we obtain the expression for the metric

$$dr^2 + \left(\frac{C}{2}\right)\sinh^2\left(r\sqrt{|K|}\right)dt^2\,,$$

or, after rescaling of $t$,

$$dr^2 + \sinh^2\left(r\sqrt{|K|}\right) dt^2 \,.$$

This shows that the first fundamental form of $S_{C,v_0}(K)$ is isometric to the universal cover of the complement of a point in $\mathbb{H}^2(K)$, which we will denote by $\widetilde{\mathbb{H}^2(K) \setminus p}$. Let $R(p)$ the group of rotations of $\mathbb{H}^2(K)$ fixing a point $p$ and let $\widetilde{R(p)}$ be its universal cover. We conclude by including all the information in the following proposition.

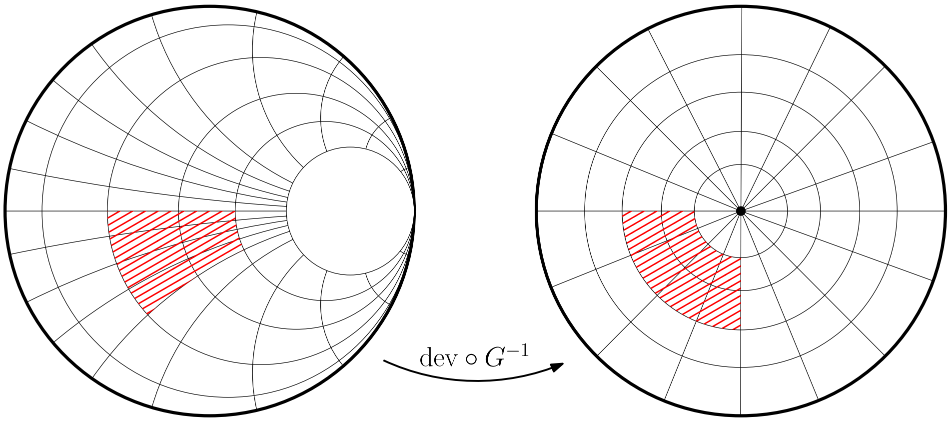



**Figure 4.5:** Again, the developing map in the Poincaré disc model. In the case $C < 0$, the inverse of the Gauss map $G^{-1} : \mathbb{H}^2 \to S_{C,v_0}(K)$ shrinks a horoball to a point.

**Proposition 4.2.3.** *For every $K < 0$, $C < 0$ and every null vector $v_0 \in \mathbb{R}^{2,1}$ there exists an isometric embedding*

$$i_{K,C,v_0} : \widetilde{\mathbb{H}^2(K) \setminus p} \to \mathbb{R}^{2,1}$$

*with image a Cauchy surface $S_{C,v_0}(K)$ for $\mathrm{I}^+(0)$. The closure of the surface $S_{C,v_0}(K)$ intersects the null cone $\partial\mathrm{I}^+(0)$ in the point $\beta v_0$, where*

$$\beta = \frac{\sqrt{2}}{2}\frac{1}{|K|\sqrt{|C|}}\frac{\pi}{4}\,.$$

*The inverse of the Gauss map of the closure of $S_{C,v_0}(K)$ maps a horoball of $\mathbb{H}^2$ to $\beta v_0$. Finally $i_{K,C,v_0}$ is equivariant with respect to the group of isometries $\widetilde{R(p)}$ of $\widetilde{\mathbb{H}^2(K) \setminus p}$ and the parabolic linear subgroup of* $\mathrm{Isom}(\mathbb{R}^{2,1})$ *fixing $v_0$.*

## 4.3 The Minkowski problem in Minkowski space

The aim of this section is to prove that, for every domain of dependence in $\mathbb{R}^{2,1}$ contained in the cone over a point, there exists a unique smooth Cauchy surface with prescribed (à la Minkowski) negative curvature, which is an entire graph. Equivalently, the main statement is the following.

**Theorem 4.A** (refined version)**.** *Given a bounded lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ and a smooth function $\psi : \mathbb{D} \to [a,b]$ for some $0 < a < b < +\infty$, there exists a unique smooth spacelike surface $S$ in $\mathbb{R}^{2,1}$ with support function at infinity $\varphi$ and curvature $K(G^{-1}(x)) = -\psi(x)$. Moreover, $S$ is an entire graph and is contained in the past of the $(1/\sqrt{\inf\psi})$-level surface of the cosmological time of the domain of dependence with support function $h = \mathrm{co}\,(\varphi)$.*

The proof will be split in several steps. In Subsection 4.3.1 we construct a solution to the Monge-Ampère equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1-|z|^2)^{-2} \tag{MA}$$

with the prescribed boundary condition at infinity

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

In Subsection 4.3.2 we study the behavior of Cauchy surfaces in terms of the support functions, and we use this condition to prove uniqueness by applying the theory of Monge-Ampère equations. Finally, in Subsection 4.3.3 we prove that the surface is not tangent to the boundary of the domain of dependence, and hence is a spacelike entire graph.

### 4.3.1 Existence of solutions

The surface $S$ will be obtained as a limit of surfaces $S_\Gamma$ invariant under the action of discrete groups $\Gamma < \mathrm{Isom}(\mathbb{R}^{2,1})$, isomorphic to the fundamental group of a closed surface, acting freely and properly discontinuously on some future-convex domain in $\mathbb{R}^{2,1}$ for which $S_\Gamma$ is a Cauchy surface. Indeed, such a surface $S_\Gamma$ can be obtained as the lift to the universal cover of a closed Cauchy surface $S_\Gamma/\Gamma$ in a maximal globally hyperbolic spacetime $D(S_\Gamma)/\Gamma$, and the existence of surfaces with prescribed curvature in such spacetimes is guaranteed by results of Barbot-Béguin-Zeghib in [BBZ11].

In this subsection we prove the following existence result for the Monge-Ampère equation (MA).

**Theorem 4.3.1.** *Given a bounded lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ and a smooth function $\psi : \mathbb{D} \to [a,b]$ for some $0 < a < b < +\infty$, there exists a smooth solution $u : \mathbb{D} \to \mathbb{R}$ to the equation*

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1-|z|^2)^{-2} \tag{MA}$$

*such that $u$ extends to a lower semicontinuous function on $\overline{\mathbb{D}}$ with*

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

*Moreover, $u$ satisfies the inequality*

$$h(z) - C\sqrt{1-|z|^2} \le u(z) \le h(z)\,, \tag{CT}$$

*where $h$ is the convex envelope of $\varphi$ and $C = 1/\sqrt{\inf\psi}$.*

There are several notions of convergence of measured geodesic laminations, as discussed for instance in [MŠ12]. Recall in Definition 2.3.6 we defined a measured geodesic lamination as a locally finite Borel measure on the set of (unoriented) geodesics $\mathcal{G}$ of $\mathbb{H}^2$, with support a closed set of pairwise disjoint geodesics. We denote by $\mathcal{G}_B$ the set of geodesics of $\mathbb{H}^2$ which intersect the subset $B \subseteq \mathbb{H}^2$.

**Definition 4.3.2.** A sequence $\{\mu_n\}_n$ of measured geodesics laminations converges in the weak* topology to a measured geodesic lamination, $\mu_n \rightharpoonup \mu$, if

$$\lim_{n\to\infty} \int_{\mathcal{G}} f d\mu_n = \int_{\mathcal{G}} f d\mu$$

for every $f \in C_0^0(\mathcal{G})$.

We are going to approximate a measured geodesic lamination in the weak* topology by measured geodesic laminations which are invariant under the action of a cocompact Fuchsian group. A stronger notion of convergence is given by the Fréchet topology on the space of measured geodesic laminations, see [MŠ12].

**Lemma 4.3.3.** *Given a measured geodesic lamination $\mu$, there exists a sequence of measured geodesic laminations $\mu_n$ such that $\mu_n$ is invariant under a torsion-free cocompact Fuchsian group $G_n < \mathrm{Isom}(\mathbb{H}^2)$ and $\mu_n \rightharpoonup \mu$.*

*Proof.* We construct the approximating sequence in several steps.

*Step 1.* We show there is a sequence of discrete measured geodesic laminations $\mu_n$ which converge to $\mu$ in the weak* topology. In [MŠ12, §7] it was proved that, if $\mu$ is bounded, there exists a sequence of discrete measured geodesic laminations which converge to $\mu$ in the Fréchet topology, which implies weak* convergence. So, assume $\mu$ is not bounded. We define $\nu_n$ by $\nu_n(A) = \mu(A \cap \mathcal{G}_{B(0,n)})$, i.e. the support of $\nu_n$ consists of the geodesics of $\mu$ which intersect $B(0,n)$. By the results in [MŠ12, §7], for every $n$, there exists a sequence $(\nu_{n,m})_m$ which converges to $\nu_n$ in the Fréchet sense. As a consequence of Fréchet convergence, for every $n$ we can find $m = m(n)$ such that

$$\sup_{f\in C_0^0(\mathcal{G}_{B(0,n)})} \left| \int_{\mathcal{G}} f d\nu_n - \int_{\mathcal{G}} f d\nu_{n,m(n)} \right| \leq \frac{1}{n}\,.$$

It follows that, for every $f$ compactly supported in $\mathcal{G}$, if *supp*$(f) \subset \mathcal{G}_{B(0,n_0)}$, then for $n \geq n_0$

$$\left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\nu_{n,m(n)} \right| = \left| \int_{\mathcal{G}} f d\nu_n - \int_{\mathcal{G}} f d\nu_{n,m(n)} \right| \xrightarrow{n\to\infty} 0\,.$$

Hence $\mu_n := \nu_{n,m(n)}$ gives the required approximation.

*Step 2.* We now modify the sequence $\mu_n$ to obtain a sequence $\mu'_n \rightharpoonup \mu$ of finite measured laminations with ultraparallel geodesics. We can assume the discrete laminations $\mu_n$ construncted in Step 1 are finite (namely they consist of a finite number of weighted geodesics), by taking the intersection with $\mathcal{G}_{B(0,n)}$. Let $d_{\mathcal{G}}$ be a Riemannian metric on $\mathcal{G}$. Suppose the leaves of $\mu_n$ are $l_1^n, \ldots, l_{p(n)}^n$ with weights $a_1^n, \ldots, a_{p(n)}^n$. Then we construct a finite lamination $\nu'_{n,m}$ by replacing $l_1^n, \ldots, l_{p(n)}^n$ by leaves $k_1^n, \ldots, k_{p(n)}^n$ so that

- $k_i^n$ and $k_j^n$ are ultraparallel for every $i \neq j$;
- $d_{\mathcal{G}}(l_i^n, k_i^n) \leq 1/m$;
- The weight of $k_i^n$ is $a_i^n$.

Let us show that, defining $\mu'_n = \nu'_{n,n}$, $(\mu'_n)_n$ converges weak* to $\mu$. For this purpose, fix a function $f$ with $supp(f) \subset \mathcal{G}_{B(0,n_0)}$ and fix $\epsilon > 0$. Since $f$ is uniformly continuous, there exists $n_1$ such that if $d_{\mathcal{G}}(l,k) < 1/n_1$, then $|f(l) - f(k)| < \epsilon$. We have

$$\left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\mu'_n \right| \leq \left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\mu_n \right| + \left| \int_{\mathcal{G}} f d\mu_n - \int_{\mathcal{G}} f d\nu'_{n,n} \right| .$$

By construction, there exists $n_2$ such that the first term in the RHS is smaller than $\epsilon$ provided $n \geq n_2$. Now for every $n$, if $m \geq \max\{n_0, n_1\}$,

$$\left| \int_{\mathcal{G}} f d\mu_n - \int_{\mathcal{G}} f d\nu'_{n,m} \right| = \sum_{i=1}^{p(n)} \left( f(l_i^n) - f(k_i^n) \right) a_i^n \leq \epsilon \mu_n(\mathcal{G}_{(B(0,n_0))}) .$$

Since $\mu_n \rightharpoonup \mu$, there exists a constant $C$ such that $\mu_n(\mathcal{G}_{(B(0,n_0))}) \leq C$ for $n \geq n_3$. In conclusion, if $n \geq \max\{n_0, n_1, n_2, n_3\}$, then

$$\left| \int_{\mathcal{G}} f d\mu - \int_{\mathcal{G}} f d\mu'_n \right| \leq (1 + C)\epsilon .$$

*Step 3.* We claim it is possible to find a polygon $P_n$ with the following properties:

- $P_n$ contains the ball $B(0,n)$;
- The angles of $P_n$ are $\pi/2$;
- $P_n$ intersects the leaves of $\mu'_n$ orthogonally.

We construct the polygon $P_n$ in the following way. For every point $z \in \partial\mathbb{D}$ which is limit of a leaf $k_i^n$ of $\mu'_n$, we pick a geodesic orthogonal to $k_i^n$ which separates $z$ from $B(0,n)$ and from all the other limit points of $\mu'_n$. Let $\{g_1, \ldots, g_p\}$ be the geodesics obtained in this way. Replacing the $g_i$ by other geodesics further from $B(0,n)$, we can assume the geodesics $g_1, \ldots, g_p$ are pairwise ultraparallel. See Figures 4.6 and 4.7.

We extend the family of geodesics $\{g_1, \ldots, g_p\}$ to a larger family $\{g'_1, \ldots, g'_{p'}\}$ satisfying:

- $\{g_1, \ldots, g_p\} \subset \{g'_1, \ldots, g'_{p'}\}$;
- $g'_1, \ldots, g'_{p'}$ are contained in $\mathbb{H}^2 \setminus B(0,n)$
- $g'_1, \ldots, g'_{p'}$ are pairwise ultraparallel;
- No geodesic $g'_i$ separates two geodesics $g'_j$ and $g'_k$ in the family (so we can assume the indices in $\{g'_1, \ldots, g'_{p'}\}$ are ordered counterclockwise);

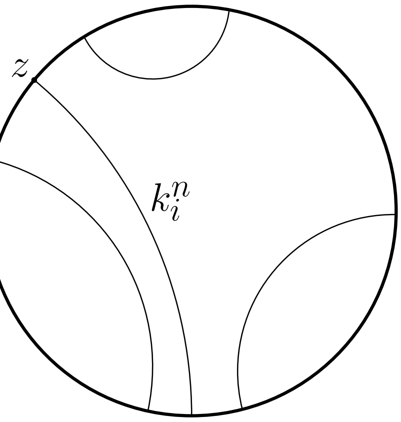



**Figure 4.6:** Start from a finite geodesic laminations with leaves $k_i^n$.

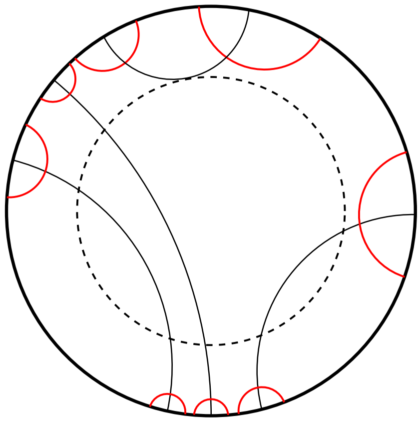


**Figure 4.7:** The construction of the geodesics $g_1, \ldots, g_p$.

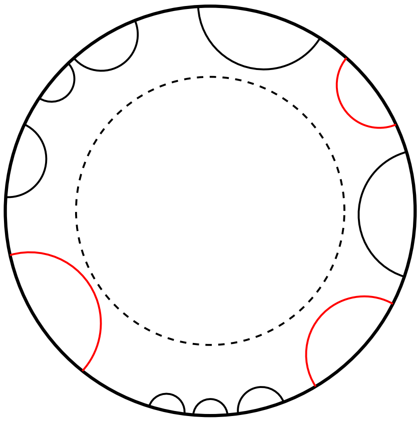


**Figure 4.8:** The geodesics in $\{g'_1, \ldots, g'_{p'}\} \setminus \{g_1, \ldots, g_p\}$.

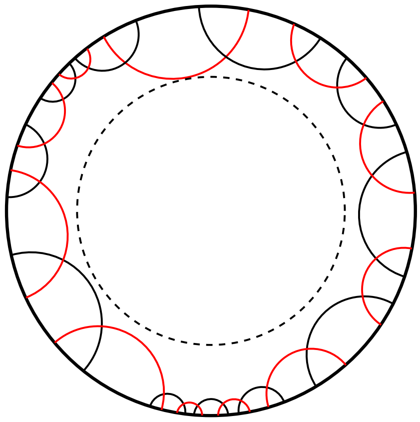


**Figure 4.9:** The geodesics $h_1, \ldots, h_{p'}$ orthogonal to $g'_1, \ldots, g'_{p'}$.

- The geodesics $h_i$ orthogonal to $g'_i$ and $g'_{i+1}$ (if the indices $i$ are considered $\bmod p'$) are contained in $\mathbb{H}^2 \setminus B(0,n)$.

The reader can compare with Figures 4.8 and 4.9. It is clear that the polygon $P_n$ (Figure 4.10) given by the connected component of $\mathbb{H}^2 \setminus \{g'_1, \ldots, g'_{p'}, h_1, \ldots, h_{p'}\}$ containing $B(0,n)$ satisfies the given properties.

*Step 4.* We finally construct a sequence $\mu''_n \rightharpoonup \mu$ with $\mu''_n$ invariant under the action of a torsion-free cocompact Fuchsian group. We consider the discrete group of isometries of $\mathbb{H}^2$ generated by reflections in the sides of the polygon $P_n$ constructed in Step 3. The index 2 subgroup $\overline{G}_n$ of orientation-preserving isometries is a discrete cocompact group and it is well-known that $\overline{G}_n$ contains a finite index torsion-free cocompact Fuchsian group $G_n$. We define $\mu''_n$ as the $G_n$-orbit of $\mu'_n$. $\mu''_n$ is a measured geodesic lamination since the leaves of $\mu'_n$ intersect the sides of $P_n$ orthogonally. Note that $\mu''_n$ is obtained by modifying $\mu'_n$ only in the complement of $B(0,n)$, since by

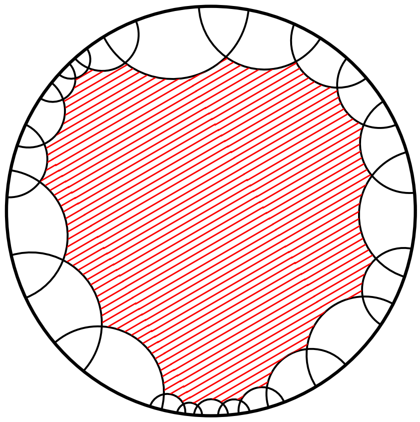


**Figure 4.10:** The polygon $P_n$.

construction $B(0,n) \subset P_n$. Hence it is clear that, if *supp*$(f) \subset \mathcal{G}_{B(0,n_0)}$, then

$$\int_{\mathcal{G}} f d\mu_n' = \int_{\mathcal{G}} f d\mu_n''$$

for $n > n_0$, and thus $\mu_n'' \rightharpoonup \mu$. □

**Lemma 4.3.4.** *Given a sequence $\mu_n$ of measured geodesic laminations converging to $\mu$ in the weak* sense, let $D_n = D(\mu_n, x_0, y_0)$ and $D = D(\mu, x_0, y_0)$ be the domains of dependence having $\mu_n$ and $\mu$ respectively as dual laminations, and $P = y_0 + x_0^\perp$ as a support plane tangent to the boundary at $y_0$, where $x_0$ does not belong to a weighted leaf of $\mu$. Let $h_n$ and $h$ be the support functions of $D_n$ and $D$. Then $h_n$ converges uniformly on compact sets of $\mathbb{D}$ to $h$.*

*Proof.* It suffices to prove that the convergence is pointwise, since the functions $h_n$ and $h$ are convex on $\mathbb{D}$, and therefore pointwise convergence implies uniform convergence on compact sets. In fact, it suffices to prove pointwise convergence for almost every point.

We will actually prove that the support functions $\bar{h}_n$ restricted to $\mathbb{H}^2$ converge pointwise to $\bar{h}$ almost everywhere, which is clearly equivalent to the claim. Let us assume $x_0$ and $x$ are points which do not lie on weighted leaves of the lamination $\mu$. Let $\mathcal{G}[x_0, x]$ be the set of geodesics which intersect the closed geodesic segment $[x_0, x]$ and $\boldsymbol{\sigma} : \mathcal{G}[x_0, x] \to \mathbb{R}^{2,1}$ be the function which assigns to a geodesic $l$ the corresponding point in $\mathrm{d}\mathbb{S}^2$, namely, the spacelike unit vector in $\mathbb{R}^{2,1}$ orthogonal to $l$ with respect to Minkowski product, pointing outward with respect to the direction from $x_0$ to $x$. The support function of $D$ can be written as (compare expression (4.9) in Proposition 4.1.10):

$$\bar{h}_n(x) = \langle x, y_0 \rangle + \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu_n \,.$$

We thus want to show that

$$\int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu_n \xrightarrow{n\to\infty} \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu \,.$$

Note that $\mathcal{G}[x_0, x]$ is compact in $\mathcal{G}$; define $\varphi_i$ a smooth function such that $\varphi_i(l) = 1$ for every $l \in \mathcal{G}[x_0, x]$ and $supp(\varphi_i) \subset \mathcal{G}(x_0 - \frac{1}{i}, x + \frac{1}{i})$. Here $(x_0 - \frac{1}{i}, x + \frac{1}{i})$ denotes the open geodesic interval which extends $[x_0, x]$ of a lenght $1/i$ on both sides. Hence we have:

$$\left| \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \chi_{\mathcal{G}[x_0,x]} d\mu_n - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \chi_{\mathcal{G}[x_0,x]} d\mu \right| \leq \left| \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \chi_{\mathcal{G}[x_0,x]} d\mu_n - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \varphi_i d\mu_n \right| \qquad (\star)$$

$$+ \left| \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \varphi_i d\mu_n - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \varphi_i d\mu \right| \qquad (\star\star)$$

$$+ \left| \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \varphi_i d\mu - \int_{\mathcal{G}} \langle x, \boldsymbol{\sigma} \rangle \chi_{\mathcal{G}[x_0,x]} d\mu \right| . \qquad (\star\star\star)$$

Let $\mathcal{F} = \overline{\mathcal{G}(x_0 - \frac{1}{i}, x + \frac{1}{i}) \setminus \mathcal{G}[x_0, x]}$. Since we have assumed $x$ and $x_0$ are not on weighted leaves of $\mu$, we have $(\star\star\star) \leq K\mu(\mathcal{F}) \leq K\epsilon$ if $i \geq i_0$, for some fixed $i_0$. By definition of weak* convergence, the term numbered $(\star\star)$ converges to zero as $n \to \infty$ for $i = i_0$ fixed, so $(\star\star) \leq \epsilon$ for $n \geq n_0$. Finally, $\limsup_{n\to\infty} \mu_n(\mathcal{F}) \leq \mu(\mathcal{F})$ by Portmanteau Theorem (which in this case can be easily proved again by an argument of enlarging the interval and approximating by bump functions). Hence there exists $n_0'$ such that $(\star) \leq K\mu_n(\mathcal{F}) \leq 2K\epsilon$ if $n \geq n_0'$ and $i \geq i_0$. Choosing $n \geq \max\{n_0, n_0'\}$, the proof is concluded. $\square$

Let us now consider an arbitrary measured geodesic lamination $\mu$ and take the sequence $\mu_n \rightharpoonup \mu$ constructed as in Lemma 4.3.3. Let $D(\mu_n, x_0, y_0)$ be the domain of dependence having dual lamination $\mu_n$ and $P = y_0 + x_0^\perp$ as a support plane tangent at $y_0$. Since $\mu_n$ is invariant under the action of a Fuchsian cocompact group $G_n$, $D(\mu_n, x_0, y_0)$ is a domain of dependence invariant under a discrete group $\Gamma_n$. The linear part of $\Gamma_n$ is $G_n$ and the translation part is determined (up to conjugacy) by $\mu_n$ (see Proposition 4.1.11). This means that $D(\mu_n)/\Gamma_n$ is a maximal globally hyperbolic flat spacetime.

**Theorem 4.3.5** ([BBZ11]). *Let $G_0$ be a Fuchsian cocompact group and $\mu_0$ be a $G_0$-invariant measured geodesic lamination. Let $\psi_0 : \mathbb{H}^2 \to (0, \infty)$ be a $G_0$-invariant smooth function. Let $\Gamma_0$ be a subgroup of $\mathrm{Isom}(\mathbb{R}^{2,1})$ whose linear part is $G_0$ and whose translation part is determined by $\mu_0$. Then there exists a unique smooth Cauchy surface $S_0$ of curvature $K(G^{-1}(x)) = -\psi_0(x)$ in the maximal future-convex domain of dependence $D_0$ invariant under the action of the group $\Gamma_0$.*

We will construct a Cauchy surface $S$ for $D = D(\mu, x_0, y_0)$ with prescribed curvature as a limit of Cauchy surfaces $S_n$ in $D_n = D(\mu_n, x_0, y_0)$. Let $h_n$ be the support function of $D_n$. By Lemma 4.3.4, $h_n$ converges uniformly on compact sets of $\mathbb{D}$ to $h$, the support function of $D$. Let $u_n$ be the support function of $S_n$.

**Lemma 4.3.6.** *Let $S_0$ be a smooth strictly convex Cauchy surface in a domain of dependence $D_0$ invariant under the action of a discrete group $\Gamma_0 < \mathrm{Isom}(\mathbb{R}^{2,1})$, such that $D_0/\Gamma_0$ is a maximal globally hyperbolic flat spacetime. Let $K : S_0 \to (-\infty, 0)$*

*be the curvature function of $S_0$. Then the support functions $u_0$ of $S_0$ and $h_0$ of $\partial_s D_0$ satisfy*
$$h_0(z) - C\sqrt{1-|z|^2} \leq u_0(z) \leq h_0(z) \tag{4.26}$$
*for every $z \in \mathbb{D}$. Moreover, one can take $C = 1/\sqrt{\inf |K|}$.*

*Proof.* Since $S_0 \subset \mathrm{I}^+(\partial_s D_0)$, it is clear that $u_0 \leq h_0$. For the converse inequality, recall that we denote by $\bar{u}_0$ the restriction to $\mathbb{H}^2$ of the 1-homogeneous extension $U_0$ of $u_0$, and analogously for $\bar{h}_0$. Let us consider $x \in \mathbb{H}^2$ and show that $\bar{u}_0(x) \geq \bar{h}_0(x) - C$, for $C = 1/\sqrt{\inf |K|}$. The inequality (4.26) then follows, since $u_0(z) = U_0(z,1) = \sqrt{1-|z|^2}\bar{u}_0(x)$ and $h_0(z) = H_0(z,1) = \sqrt{1-|z|^2}\bar{h}_0(x)$, for $x \in \mathbb{H}^2$ which projects to $(z,1)$. Let us consider the foliation of $D_0$ by leaves of the cosmological time, namely the surfaces $L_T$ whose support function is $\bar{h}_T(x) = \bar{h}_0(x) - T$ if $x \in \mathbb{H}^2$, with $T \in (0,\infty)$. The surface $S_0$ descends to a compact surface in $D_0/\Gamma_0$ and therefore the time function $T$ on $S_0$ achieves a maximum $T_{\max}$ at a point $p$ (actually, a full discrete $\Gamma_0$−orbit) on $S_0$. It follows that the level surface $L_{T_{\max}}$ is entirely contained in the future of $S_0$ and tangent to $S_0$ at $p$.

The level surface $L_T$ is obtained by grafting a hyperboloid of constant curvature $-1/T^2$ according to the lamination $\mu_0$. More precisely, $L_T$ is obtained by inserting a flat piece (whose principal curvatures are 0 and $1/T$) on the leaves of the lamination, where the lenght of each flat piece is determined by the measure of the lamination. By this construction, it is clear that $L_T$ might fail to be smooth; however for any point $p \in L_T$ there exists a hyperboloid of curvature $-1/T^2$ which is tangent to $L_T$ and contained in the future of $L_T$.

Hence, the surface $S_0$ is contained in the past of a hyperboloid of curvature $-1/T^2$, and tangent to such hyperboloid at some point, which implies that $\inf |K| \leq 1/T^2$. Therefore $T \leq C$ for $C = 1/\sqrt{\inf |K|}$. This shows that the surface $S_0$ is contained in the past of the level surface $L_C$ and its support function on $\mathbb{H}^2$ satisfies $\bar{h}_0(x) - C \leq \bar{u}_0(x)$. □

We are now ready to conclude the proof.

*Proof of Theorem 4.3.1.* Given the lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$, let us consider the dual lamination $\mu$ of the domain of dependence $D$ defined by $\varphi$. Hence $D = D(\mu, x_0, y_0)$ for some $x_0, y_0$ and the support function $h$ of $D$ is the convex envelope of $\varphi$.

By Lemma 4.3.3, there exist measured geodesic laminations $\mu_n$, invariant under the action of torsion-free cocompact Fuchsian groups $G_n$, which converge weakly to $\mu$. Recall from the proof of Lemma 4.3.3 that the Fuchsian group $G_n$ has a fundamental domain $P'_n$ which contains the ball $B(0,n)$ for the hyperbolic metric. Let us define a $G_n$-invariant function $\psi_n : \mathbb{H}^2 \to \mathbb{R}$, which approximates $\psi$. We take a partition of unity $\{\rho_n, \varrho_n\}$ subordinate to the covering $\{B(0,n), P'_n \setminus B(0,n/2)\}$ of $P'_n$. We define
$$\psi_n(x) = \rho_n(x)\psi(x) + \varrho_n(x)(\inf \psi)$$
and we extend $\psi_n$ to $\mathbb{H}^2$ by invariance under the isometries in $G_n$. It is clear that the sequence $\psi_n$ converges to $\psi$ uniformly on compact sets of $\mathbb{H}^2$, since $\psi_n$ agrees with $\psi$ on $B(0,n/2)$. Since $\psi_n$ is constant on $P'_n \setminus B(0,n/2)$, the $G_n$-invariant extension

is smooth. Finally, $\inf \psi_n = \inf \psi$. Applying Theorem 4.3.5, we obtain a solution $u_n$ to the equation

$$\det D^2 u_n(z) = \frac{1}{\psi_n(z)}(1-|z|^2)^{-2}$$

for every $n$.

Let $D_n = D(\mu_n, x_0, y_0)$ be the domain of dependence associated with $\mu_n$, so that $y_0 + x_0^\perp$ is a support plane of the domain $D_n$. We can assume $x_0$ is a point of $\mathbb{H}^2$ which does not belong to a weighted leaf of any $\mu_n$. Let $H_n$ be the extended support function of $D_n$ and let $h_n$ be its usual restriction to $\mathbb{D}$. By Lemma 4.3.4, $h_n$ converges uniformly on compact sets of $\mathbb{D}$ to $h$. Moreover by inequality (4.26) of Lemma 4.3.6, the convex functions $u_n$ are uniformly bounded on every compact set of $\mathbb{D}$. Hence, by convexity, the $u_n$ are equicontinuous on compact sets of $\mathbb{D}$ and therefore, by the Ascoli-Arzelà Theorem, there exists a subsequence converging uniformly on compact sets to a convex function $u$. The limit function $u$ is a generalized solution of the equation

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1-|z|^2)^{-2}\,. \tag{MA}$$

By Theorem 3.3.5, $u$ is strictly convex and therefore is smooth by Theorem 3.3.6. Moreover, the functions $u_n$ satisfy the inequality in (4.26) for every $z \in \mathbb{D}$:

$$h_n(z) - C\sqrt{1-|z|^2} \leq u_n(z) \leq h_n(z)\,,$$

hence for the limit function $u$ we have

$$h(z) - C\sqrt{1-|z|^2} \leq u(z) \leq h(z)\,, \tag{CT}$$

where $h$ is the limit of the support functions $h_n$ of $D_n$ and is the support function of $D$ by Lemma 4.3.4. Since both $u$ and $h$, extended to $\overline{\mathbb{D}}$, are lower semicontinuous and convex functions, and the value on a point $z \in \partial\mathbb{D}$ coincides with the limit along a radial geodesic (see Lemma 4.1.3), we have the boundary condition $u|_{\partial\mathbb{D}} = h|_{\partial\mathbb{D}} = \varphi$. This shows that the condition (BC) holds, and concludes the proof. □

### 4.3.2 Uniqueness of solutions

In this subsection, we discuss the uniqueness of the solution of Equation (MA), for which the existence was proved in Theorem 4.3.1. More precisely, we prove the following:

**Proposition 4.3.7.** *Given a bounded lower semicontinuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ and a smooth function $\psi : \mathbb{H}^2 \to [a,b]$ for some $0 < a < b < +\infty$, the smooth solution $u : \mathbb{D} \to \mathbb{R}$ to the equation*

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1-|z|^2)^{-2} \tag{MA}$$

*satisfying*

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

*is unique.*

The claim in Proposition 4.3.7 holds if $\varphi$ is continuous, by a direct application of Theorem 3.3.2. The rest of this subsection will be devoted to the proof of the claim when $\varphi$ is only assumed to be lower semicontinuous. The key property is that every solution of (MA) with boundary value $\varphi$, for $\psi > a > 0$, satisfies the condition (CT). Geometrically, this means that every Cauchy surface with curvature bounded away from zero has bounded cosmological time.

**Proposition 4.3.8.** *Given a smooth function $\psi : \mathbb{D} \to [a, b]$ for some $0 < a < b < +\infty$, any smooth solution $u : \mathbb{D} \to \mathbb{R}$ to the equation*

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1 - |z|^2)^{-2} \tag{MA}$$

*with*

$$u|_{\partial\mathbb{D}} = \varphi\,. \tag{BC}$$

*satisfies*

$$h(z) - C\sqrt{1 - |z|^2} \leq u(z) \leq h(z)\,, \tag{CT}$$

*for some constant $C > 0$, where $h = \mathrm{co}\,\varphi$.*

*Proof.* The statement is true if $\varphi$ is continuous, as we have already observed that in that case the solution is unique, and in Theorem 4.3.1 we have proved the existence of a solution satisfying the required condition. Let us now consider the general case. Since $u$ is convex, it is clear that $u \leq h$. We show the other inequality. Let $r \in (0, 1]$ and $u_r : \mathbb{D} \to \mathbb{R}$ be defined as

$$u_r(z) = u(rz)\,.$$

Since $u$ is continuous (actually, smooth) on $\mathbb{D}$, $u_r$ converges uniformly on compact sets of $\mathbb{D}$ to $u$ as $r \to 1$. Let $\psi_r$ be such that

$$\det D^2 u_r(z) = \frac{1}{\psi_r(z)}(1 - |z|^2)^{-2}\,.$$

We have

$$\det D^2 u_r(z) = r^4 \det D^2 u(rz) = \frac{r^4}{\psi(rz)}(1 - r^2|z|^2)^{-2} \leq \frac{1}{\inf \psi}(1 - |z|^2)^{-2}$$

and therefore $\psi_r(z) \geq \inf \psi$. Since $u_r$ is continuous on $\overline{\mathbb{D}}$, by the continuous case and the above inequality we obtain

$$h_r(z) - \frac{1}{\sqrt{\inf \psi}}\sqrt{1 - |z|^2} \leq u_r(z)\,, \tag{4.27}$$

where $h_r = \mathrm{co}\,(u_r|_{\partial\mathbb{D}})$.

Fix a point $z_0 \in \mathbb{D}$. We claim that $h(z_0) \leq \liminf_r h_r(z_0)$. Indeed, given an arbitrary affine function $f : \mathbb{D} \to \mathbb{R}$ such that $f|_{\partial\mathbb{D}} < \varphi$, the set $\{z : u(z) \leq f(z)\}$ is compact in $\mathbb{D}$. Since $u_r$ converges to $u$ uniformly on compact sets, $u_r(z) \geq f(z)$ outside of a compact set, for $r$ close to 1. Hence $h_r(z_0)$ is definitevely larger than $f(z_0)$, namely $f(z_0) \leq \liminf_r h_r(z_0)$, and the claim follows since $f$ is arbitrary. Taking limits in Equation (4.27), we conclude that $h(z_0) - C\sqrt{1 - |z_0|^2} \leq u(z_0)$ for $C = 1/\sqrt{\inf \psi}$. As the point $z_0$ is arbitrary, we conclude the proof. $\square$

*Proof of Proposition 4.3.7.* Let $u_1, u_2$ be two solutions with $u_1|_{\partial\mathbb{D}} = u_2|_{\partial\mathbb{D}} = \varphi$. By Proposition 4.3.8, there exists a constant $C$ such that

$$-C\sqrt{1-|z|^2} \leq u_1(z) - u_2(z) \leq C\sqrt{1-|z|^2}\,.$$

Hence the function $u_1 - u_2$ extends continuously to zero at the boundary $\partial\mathbb{D}$. Therefore $u_1 - u_2$ has a minimum on $\overline{\mathbb{D}}$. By Theorem 3.3.2, the minimum cannot be achieved at an interior point. Therefore the minimum is achieved on $\partial\mathbb{D}$, which means that $u_1 \geq u_2$. By exchanging the roles of $u_1$ and $u_2$, one can conclude that $u_1 \equiv u_2$. □

### 4.3.3 The solution is an entire graph

In this subsection we prove that the solutions constructed in Theorem 4.3.1 are the support functions of entire graphs in $\mathbb{R}^{2,1}$. We will make use of barriers which are constant curvature surfaces invariant under a parabolic group, as constructed in Subsection 4.2.1.

In order to use the surface $S_{C,v_0}(K)$ (described in Proposition 4.2.2) as a barrier, we need to prove a technical lemma.

**Lemma 4.3.9.** *Let $\varphi_1, \varphi_2 : \partial\mathbb{D} \to \mathbb{R}$ be two bounded lower semicontinuous functions and let $\psi_1, \psi_2 : \mathbb{H}^2 \to [a, b]$ be two smooth functions, for some $0 < a < b$. If $\varphi_1 \leq \varphi_2$ and $\psi_1 \leq \psi_2$, then the smooth solutions $u_i : \mathbb{D} \to \mathbb{R}$ (for $i = 1, 2$) to the equation*

$$\det D^2 u_i(z) = \frac{1}{\psi_i(z)}(1-|z|^2)^{-2}$$

*with*

$$u_i|_{\partial\mathbb{D}} = \varphi_i$$

*satisfy $u_1 \leq u_2$ on $\mathbb{D}$.*

*Proof.* Suppose first $\varphi_1$ is continuous. Therefore also the solution $u_1$ is continuous on $\overline{\mathbb{D}}$, since it satisfies the condition

$$h_1(z) - C\sqrt{1-|z|^2} \leq u_1(z) \leq h_1(z)\,,$$

where $h_1$ is the convex envelope of $\varphi_1$. Then the function $u_2 - u_1$ is lower semicontinuous and is positive on the boundary, therefore it achieves a minimum. By Theorem 3.3.2, the minimum has to be on the boundary, hence $u_2 \geq u_1$ on $\overline{\mathbb{D}}$.

Now for the general case, let $\varphi_1$ be lower semicontinuous and let $\varphi_n$, $n \geq 3$, be a sequence of continuous functions which converge to $\varphi_1$ monotonically from below, namely $\varphi_n \leq \varphi_{n+1}$ and $\varphi_n \leq \varphi_1$ for every $n \geq 3$. Let $u_n$ be the solution of the equation

$$\det D^2 u_n(z) = \frac{1}{\psi_1(z)}(1-|z|^2)^{-2}$$

with

$$u_n|_{\partial\mathbb{D}} = \varphi_n\,.$$

By the previous case, we know (if $n \geq 3$) that $u_n \leq u_{n+1}$ and $u_n \leq u_1$. Hence the $u_n$ are uniformly bounded and convex, thus by convexity the sequence $u_n$ converges uniformly on compact sets (up to a subsequence) to a generalized solution $u_\infty$ of the same equation:

$$\det D^2 u_\infty(z) = \frac{1}{\psi_1(z)}(1-|z|^2)^{-2}\,.$$

It is clear that $u_\infty \leq u_1$. Let $z \in \partial\mathbb{D}$. Recall the value of $u_\infty$ on $z$ coincides with the limit on radial geodesics. Hence we have $u_\infty(z) = \lim_{r\to 1} u_\infty(rz) \geq \lim_{r\to 1} u_n(rz) = \varphi_n(z)$ for every $n$. Therefore $u_\infty|_{\partial\mathbb{D}} \equiv \varphi_1$. By the uniqueness proved in Proposition 4.3.7, $u_\infty \equiv u_1$. Since $u_n(z) \leq u_2(z)$ for $n \geq 3$ and for every $z \in \mathbb{D}$, we conclude that $u_1 \leq u_2$. □

We are finally ready to conclude the proof of Theorem 4.A.

*Proof of Theorem 4.A.* We have showed in Theorem 4.3.1 and Proposition 4.3.7 that there exists a unique solution $u$ to Equation (MA), hence having the required curvature function. Moreover, the solution satisfies Equation (CT), which ensures that the surface $S$ with support function $u$ is a Cauchy surface and satisfies the estimate on the cosmological time.

It only remains to show that $S$ is a spacelike entire graph. Suppose it is not. Therefore $S$ is tangent to the boundary of the domain of dependence and develops a lightlike ray $R$ at the tangency point. Suppose the lightlike ray is parallel to the null vector $v_0$ of $\mathbb{R}^{2,1}$. Let $\alpha$ be such that $\varphi(z) \leq \alpha$ for every $z \in \partial\mathbb{D}$.

We consider the function

$$\varphi_0(z) = \begin{cases} \varphi(z) & [z] = [v_0] \\ \alpha & [z] \neq [v_0] \end{cases}.$$

From Proposition 4.2.2, there exists a $K_0$-surface $S_0$ with support function at infinity $\varphi_0$, for $K_0 = \inf K = -\sup|K|$, which is obtained by translating vertically in $\mathbb{R}^{2,1}$ a suitably chosen surface $S_{C,v_0}(K_0)$. Geometrically, this is equivalent to choosing the domain of dependence whose boundary is the future of a parabola (see Figure 4.1). The parabola is obtained by intersecting the plane containing the lightlike ray $R$ with a cone $\mathrm{I}^+(p)$ over a point $p$ on the $z$-axis, sufficiently in the past, so as to contain the original surface $S$.

Applying Lemma 4.3.9, we see that $S$ is in the future of $S_0$. However, $S_0$ is an entire graph (see Proposition 4.2.2) and both $S$ and $S_0$ have the same lightlike support plane with normal vector $v_0$. This gives a contradiction and concludes the claim that $S$ is an entire graph. □

*Remark* 4.3.10. In the above proof of Theorem 4.A, we have actually showed that every smooth convex bounded solution $u : \mathbb{D} \to \mathbb{R}$ of

$$\det D^2 u(z) = \frac{1}{\psi(z)}(1-|z|^2)^{-2} \qquad \text{(MA)}$$

with $\psi : \mathbb{D} \to (0,\infty)$, $\sup\psi < \infty$, corresponds to a spacelike entire graph in $\mathbb{R}^{2,1}$ with curvature $K(G^{-1}(z)) = -\psi(z)$. On the other hand, the hypothesis that $\sup\psi <$

$\infty$ is essential. We give an explicit counterexample. Let us consider the function $u : \mathbb{D} \to \mathbb{R}$ defined by $u(z) = |z|^2/2$. It is easily checked that the dual surface $S$ defined by $u$ is the graph of $u$ itself, and is only defined over $\mathbb{D}$. One can see that $S$ is tangent to the lightcone centered at $(0, 0, -1/2)$ and that its curvature function is $K(G^{-1}(z)) = -\psi(z) = -\frac{1}{(1-|z|^2)^2}$, hence is unbounded. See also Figure 4.11.

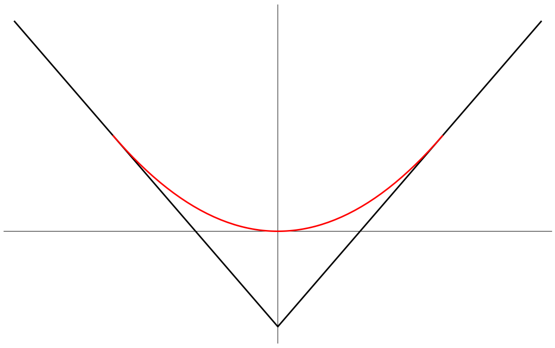


**Figure 4.11:** A Cauchy surface in the cone over a point, of unbounded negative curvature, which is not a spacelike entire graph. The surface (red) is obtained by revolution around the vertical axis.

## 4.4 Foliations by constant curvature surfaces

In this section we prove that every domain of dependence defined by a bounded support function at infinity is foliated by $K$-surfaces, as $K$ varies in $(-\infty, 0)$. The main statement is the following.

**Theorem 4.D.** *Every domain of dependence $D$, with bounded support function at infinity $\varphi : \partial\mathbb{D} \to \mathbb{R}$, is foliated by smooth spacelike entire graphs of constant curvature $K \in (-\infty, 0)$.*

The existence of such $K$-surfaces follows from Theorem 4.A, by choosing the constant function $\psi \equiv |K|$. We now show that the $K$-surfaces foliate the domain of dependence $D$.

**Lemma 4.4.1.** *Let $S_n = \mathrm{graph}(f_n)$ and $S_\infty = \mathrm{graph}(f_\infty)$ be spacelike entire graphs in $\mathbb{R}^{2,1}$ with $C^1$ support functions $u_n : \mathbb{D} \to \mathbb{R}$ and $u_\infty : \mathbb{D} \to \mathbb{R}$. If $u_n$ converges to $u_\infty$ uniformly on compact sets, then $f_n$ converges uniformly on compact sets to $f_\infty$.*

*Proof.* By a slightly abuse of notation, we consider here the Gauss map $G_n : S_n \to \mathbb{D}$ using the canonical identification $\pi : \mathbb{H}^2 \to \mathbb{D}$. As by convexity we have that $Du_n(z) \to Du(z)$ for any $z \in \mathbb{D}$, Formula (4.1) implies that

$$G_n^{-1}(z) \to G_\infty^{-1}(z)$$

for all $z \in \mathbb{D}$. Let us set $G_n^{-1}(z) = (p_n(z), f_n(p_n(z))$, where $p_n(z)$ is the vertical projection to $\mathbb{R}^2$. We have that

$$p_n(z) \to p_\infty(z) \quad \text{and} \quad f_n(p_n(z)) \to f_\infty(p_\infty(z)) \,. \tag{4.28}$$

Using that $f_n$'s are 1-Lipschitz, by a standard use of Ascoli-Arzelà Theorem, we get that up to a subsequence, $f_n$ converges uniformly on compact subset of $\mathbb{R}^2$ to some function $g$.

In order to prove that $g = f_\infty$, let us use again (4.28). We get that $f_\infty(p_\infty(z)) = \lim f_n(p_n(z)) = g(p_\infty(z))$. So $f$ and $g$ coincide on the image of $p_\infty$. As we are assuming that $S_\infty$ is a spacelike entire graph we conclude that they coincide everywhere. □

Recall that a $K_0$-surface $S(K_0)$ in $D$ is constructed as limit of $K_0$-surfaces $S_n(K_0)$ invariant under the action of a surface group. By the work of [BBZ11], the $K$-surfaces $S_n(K)$ foliate the domain of dependence $D_n$ of $S_n(K_0)$, as $K$ varies in $(-\infty, 0)$.

**Theorem 4.4.2** ([BBZ11])**.** *Let $G_0$ be a Fuchsian cocompact group and let $\Gamma_0$ be a discrete subgroup of* $\mathrm{Isom}(\mathbb{R}^{2,1})$ *whose linear part is $G_0$. Then the Cauchy surfaces $S_0(K)$ of constant curvature $K \in (-\infty, 0)$ foliate the maximal domain of dependence $D_0$ invariant under the action of the group $\Gamma_0$, in such a way that if $K_1 < K_2$, then $S_0(K_2)$ is contained in the future of $S_0(K_1)$.*

*Proof of Theorem 4.D.* The proof is split in several steps. First we prove the constant curvature surfaces are pairwise disjoint, then that the portion contained between two constant curvature surfaces is filled by other constant curvature surfaces, and finally that one can find a constant curvature surface arbitrarily close to the boundary of the domain of dependence and to infinity.

*Step 1.* Let us show that, if $K_1 < K_2$, then the constant curvature surfaces $S(K_1)$ and $S(K_2)$ are disjoint, and $S(K_2)$ is in the future of $S(K_1)$. Let $S_n(K_1)$ and $S_n(K_2)$ be approximating sequences as in the proof of Theorem 4.3.1, and let $u_n(K_1)$ and $u_n(K_2)$ be the corresponding support functions. From Theorem 4.4.2 of [BBZ11], we know that $u_n(K_2) < u_n(K_1)$. Hence in the limit $u(K_2) \leq u(K_1)$, where $u(K_i)$ is the support function of $S(K_i)$. Hence $S(K_1)$ and $S(K_2)$ do not intersect transversely. Moreover $S(K_1)$ is in the closure of the past of $S(K_2)$. Finally $S(K_1)$ and $S(K_2)$ cannot be tangent at a point, since $|K_1| > |K_2|$ and thus at least one of the eigenvalues of the shape operator of $S(K_1)$ is larger than the largest eigenvalue of $S(K_2)$.

*Step 2.* We show that, given two Cauchy surfaces $S(K_1)$, $S(K_2)$ in $D$ of constant curvature $K_1 < K_2$, every point between $S(K_1)$ and $S(K_2)$ lies on a Cauchy surface of constant curvature. Let $x$ be a point in $\mathbb{R}^{2,1}$ contained in the past of $S(K_2)$ and in the future of $S(K_1)$. For $n$ large, $x$ is in the past of $S_n(K_2)$ and in the future of $S_n(K_1)$. Therefore there exists a surface $S_n(K_n)$ through $x$, with $K_1 < K_n < K_2$. Up to a subsequence, let us assume $K_n \to K_\infty$. Using the same argument we gave in the proof of Theorem 4.3.1, the support functions $u_n(K_n)$ converge (up to a subsequence) uniformly of compact sets to $u_\infty(K_\infty)$, which is the support function of the $K_\infty$-surface $S(K_\infty)$ in $D$. Since $x \in S_n(K_n)$ for every $n$, Lemma 4.4.1 implies that $x \in S(K_\infty)$.

*Step 3.* We show that for every point $x$ nearby the boundary of the domain of dependence there is a constant curvature Cauchy surface $S(K_0)$ such that $x \in S(K_0)$. This follows from Equation (CT), which states that the $K$-surface $S(K)$ in $D$ is contained in the past of the $(1/\sqrt{|K|})$-level surface of the cosmological time. Since the level surfaces $L_C = \{T = C\}$ of the cosmological time get arbitrarily close to the boundary of the domain of dependence as $C$ gets close to 0, it is clear that $x$ is in the future of $S(K_0)$ if $|K_0|$ is large enough. The claim follows by Step 2.

*Step 4.* Finally, we show that every point far off at infinity lies on some constant curvature Cauchy surface. By contradiction, suppose there is a point $x$ which is in the future of every $K$-surface $S(K)$. Let $S(K)$ be the graph of $f_K : \mathbb{R}^2 \to \mathbb{R}$. Then the $f_K$ are uniformly bounded (they are all smaller than the function which defines $\mathrm{I}^+(x)$) and convex. Up to a subsequence $f_K \to f_\infty$ uniformly on compact sets. The function $f_\infty$ defines a surface $S_\infty$ contained in the domain of dependence $D$.

Now, $f_K$ satisfies (see for instance [Li95])

$$\det D^2 f_K = |K|(1 - ||Df||^2)^2 \,.$$

Therefore, taking the limit as $K \to 0$, $\det D^2 f_\infty = 0$ in the generalized sense. The following Lemma states that $f_\infty$ is affine along a whole line of $\mathbb{R}^2$, and this gives a contradiction, since $S_\infty$ would contain an entire line and thus could not be contained in the domain of dependence $D$. □

**Lemma 4.4.3.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex function which satisfies the equation $\det D^2 f = 0$ in the generalized sense. Then there exist a point $x_0 \in \mathbb{R}^2$, a vector $v \in \mathbb{R}^2$, and $\alpha \in \mathbb{R}$ such that $f(x_0 + tv) = f(x_0) + \alpha t$ for every $t \in \mathbb{R}$.*

*Proof.* By Theorem 3.3.7 for any bounded convex domain $\Omega \subset \mathbb{R}^2$, $f|_\Omega$ coincides with the convex envelope of $f|_{\partial\Omega}$. It follows that for any $x \in \mathbb{R}^2$ there is $v = v(x) \in \mathbb{R}^2$ and $\alpha = \alpha(x) \in \mathbb{R}$ such that $f(x+tv) = f(x) + \alpha t$ for $t \in (-\epsilon, \epsilon)$, for some $\epsilon = \epsilon(x) > 0$.

Fix a point $x_1$, and set $v_1 = v(x_1)$. Up to adding an affine function we may assume that $\alpha(x_1) = 0$ and that $f(x) \geq f(x_1)$ for any $x \in \mathbb{R}^2$. If $f$ is affine along the whole line $x_1 + \mathbb{R}v_1$, we have done. Otherwise take the maximal $t_1$ such that $f(x_1 + t_1 v_1) = f(x_1)$ and put $x_0 = x_1 + t_1 v_1$. Let $v_0 = v(x_0)$, clearly $v_0 \neq v_1$. As $f(x) \geq f(x_0) = f(x_1)$ for every $x \in \mathbb{R}^2$, necessarily $\alpha(x_0) = 0$. See Figure 4.12.

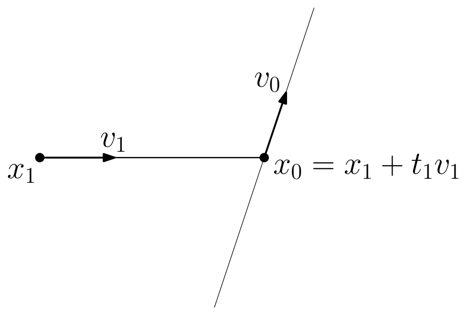



**Figure 4.12:** The setting of the proof of Lemma 4.4.3. Composing with an affine map, we can assume $f$ is constant on the drawn segments.

We claim that $f(x) > f(x_0)$ on the half-plane $P_0$ bounded by $x_0 + \mathbb{R}v_0$ which does not contain $x_1$. Otherwise we should have that $f \equiv f(x_0)$ on the triangle with vertices $x, x_0 + \epsilon v_0, x_0 - \epsilon v_0$, but then $f$ would be constant equal to $f(x_1)$ on some segment $[x_1, x_0 + \eta v_1]$, violating he maximality of $t_1$. See Figure 4.13.

Now suppose that $f(x_0 + tv_0) > f(x_0)$ for some $t$ and take the maximal $t_0$ for which $f(x_0 + t_0v_0) = f(x_0)$. Define $x_2 = x_0 + t_0v_0$. As before we have that $v_2$ is different form $v_0$ and that $f \equiv f(x_0)$ on some segment of the form $[x_2 - \epsilon v_2, x_2 + \epsilon v_2]$ (see Figure 4.14). As this segment contains points of $P_0$ we get a contradiction. $\square$

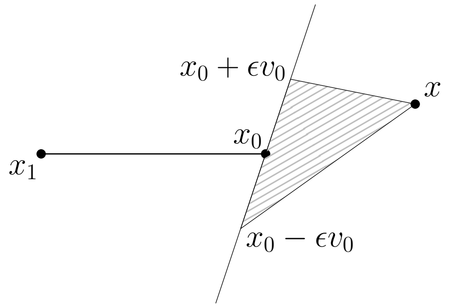



**Figure 4.13:** We have $f(x) > f(x_1)$ on the right halfplane bounded by the line $x_0 + \mathbb{R}v_0$, for otherwise $f$ would be constant on the ruled triangle, contradicting the maximality of $t_1$.

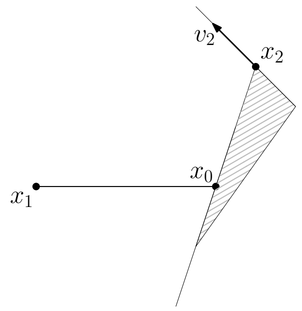



**Figure 4.14:** By a similar argument, $f$ has to be constant on the entire line $x_0 + \mathbb{R}v_0$.

## 4.5 Constant curvature surfaces and boundedness of principal curvatures

In this section we give a characterization of $K$-surfaces with bounded principal curvatures. The following statement contains a series of equivalences which basically include the statements of Theorem 4.B and Theorem 4.C.

**Theorem 4.5.1.** *Let $D$ be a the domain of dependence in $\mathbb{R}^{2,1}$. The following are equivalent:*

*i) The measured geodesic lamination $\mu$ dual to $\partial_s D$ is bounded, i.e. $||\mu||_{Th} < +\infty$.*

*ii) The support function at infinity $h = H|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \mathbb{R}$ of $D$ is in the Zygmund class.*

*iii) The domain of dependence $D$ contains a convex Cauchy surface with principal curvatures bounded below by some constant $d > 0$.*

*iv) The domain of dependence $D$ is foliated by complete convex Cauchy surfaces of constant curvature $K$ with principal curvatures bounded below by some constant $d = d(K) > 0$, where $K \in (-\infty, 0)$.*

We will give the proof in several steps. It is obvious that $iv) \Rightarrow iii)$. In Subsection 4.5.1 we prove that $i) \Leftrightarrow ii)$. The existence part of $ii) \Rightarrow iv)$ follows by Theorem 4.D; in Subsection 4.5.2 we complete the proof by showing that, if the dual lamination has finite Thurston norm, then the principal curvatures are bounded and the surface is complete. Finally, in Subsection 4.5.3 we give a proof of the last step. Indeed, we will show directly *iii*) $\Rightarrow$ *i*), by giving an explicit estimate of Thurston norm of the dual lamination in terms of the supremum of the principal curvatures, which holds for any convex Cauchy surface.

### 4.5.1 Zygmund fields and bounded measured geodesic laminations

In this part, we discuss the equivalence between *i*) and *ii*). We prove here the key fact for this equivalence. Given a function $\varphi : S^1 \to \mathbb{R}$, we denote by $\hat{\varphi}$ the vector field on $S^1$ associated to $\varphi$ by means of the standard trivialization.

**Proposition 4.5.2.** *Given an infinitesimal earthquake $\hat{\varphi} = \frac{d}{dt}\big|_{t=0} E^{t\mu}$, the function $\varphi : S^1 \to \mathbb{R}$ is the support function at infinity on $\partial\mathbb{D} = S^1$ of a domain of dependence $D$ with dual lamination $\mu$.*

*Proof.* By composing $\hat{\varphi}$ with infinitesimal Möbius tranformation, (compare Section 2.3.3) we can suppose the point $x_0 \in \mathbb{H}^2$ lies in a stratum of $\mu$ which is fixed by the earthquakes $E^{t\mu}$, for $t \in \mathbb{R}$. By Proposition 4.1.10, the support function at infinity of the domain of dependence $D = D(\mu, x_0, 0)$ which has dual lamination $\mu$ and $x_0^\perp$ as a support plane is

$$H(\eta) = \int_{\mathcal{G}[x_0,\eta)} \langle \eta, \boldsymbol{\sigma} \rangle d\mu \,, \tag{4.29}$$

for every $\eta$ in $\overline{\partial \mathrm{I}^+(0)}$. Here $[x_0, \eta)$ denotes the geodesic ray obtained by projecting to $\mathbb{H}^2$ the line segment connecting $x_0$ and $\eta$ (recall the convention introduced before Proposition 4.1.10).

By Lemma 4.1.6, the vector field $\hat{\varphi}$ on $S^1$ defines a 1-homogeneous function $\Phi$ on $\partial \mathrm{I}^+(0)$. Since $H$ is 1-homogeneous, it suffices to check that $H$ and $\Phi$ agree on $\partial\mathbb{D} = \partial \mathrm{I}^+(0) \cap \{x_3 = 1\}$. Let $\eta \in \partial\mathbb{D}$ and let $v$ be the unit vector tangent to $\partial\mathbb{D}$ in the counterclockwise orientation. By Lemma 4.1.6, under the standard identification of $\partial\mathbb{D}$ with $S^1$, we have

$$\varphi(\eta) = \langle \hat{\varphi}(\eta), v \rangle = \Phi(\eta) \,.$$

We now compute the infinitesimal earthquake $\varphi$ at a point $\eta$. If $l$ is a leaf of $\mu$, the infinitesimal earthquake along the lamination composed of the only leaf $l$ (as in Example 4.1.7) is

$$\dot{E}_l(\eta) = \langle \dot{A}^l(\eta), v \rangle v = \langle \eta \boxtimes \boldsymbol{\sigma}(l), v \rangle v \,,$$

where $\dot{A}^l = \frac{d}{dt}\big|_{t=0} A^l(t) \in \mathfrak{so}(2,1)$ is the infinitesimal generator of the 1-parameter subgroup of hyperbolic isometries $A^l(t)$ which translate on the left (as seen from $x_0$) along the geodesic $l$ by lenght $t$. In the second equality we have used the fact that $\dot{A}^l(\eta) = \eta \boxtimes \boldsymbol{\sigma}(l)$ (see for instance [BS12, Appendix B]).

Using Equation (2.6) in Theorem 2.3.11, we obtain

$$\langle \hat{\varphi}(\eta), v \rangle = \int_{\mathcal{G}[x_0,\eta)} \dot{E}_l(\eta) d\mu = \int_{\mathcal{G}[x_0,\eta)} \langle \eta \boxtimes \boldsymbol{\sigma}, v \rangle d\mu \,. \tag{4.30}$$

We will show that $\langle \eta, \boldsymbol{\sigma}(l) \rangle = \langle \eta \boxtimes \boldsymbol{\sigma}(l), v \rangle$, from which the claim follows, by comparing Equations (4.29) and (4.30). For this purpose, let $p = (0,0,1)$ and $\eta = p + w$, so that $(p, w, v)$ gives an orthonormal oriented triple. Suppose $\boldsymbol{\sigma}(l) = ap + bw + cv$. Then

$$\langle \eta, \boldsymbol{\sigma}(l) \rangle = a\langle p, p\rangle + b\langle w, w\rangle = b - a$$

whereas

$$\eta \boxtimes \boldsymbol{\sigma}(l) = b(p \boxtimes w) + a(w \boxtimes p) + c(p \boxtimes v + w \boxtimes v)$$

and in conclusion $\langle \eta \boxtimes \boldsymbol{\sigma}(l), v\rangle = b - a = \langle \eta, \boldsymbol{\sigma}(l)\rangle$. □

*Remark* 4.5.3. We observe that if a different base point $x_0'$ (out of any weighted leaf of $\mu$) is chosen, we obtain the 1-homogeneous function $H'$ such that

$$H'(\eta) = \int_{\mathcal{G}[x_0', \eta)} \langle \eta, \boldsymbol{\sigma}\rangle d\mu = \int_{\mathcal{G}[x_0', x_0]} \langle \eta, \boldsymbol{\sigma}\rangle d\mu + \int_{\mathcal{G}[x_0', \eta)} \langle \eta, \boldsymbol{\sigma}\rangle d\mu$$

where this equality follows from the fact that $\mu$ is a measured geodesic lamination, and hence in Equation (4.29) the interval $[x_0, x]$ can be replaced by any path from $x_0$ to $x$ transverse to the support of the lamination (see also [Mes07]). Clearly, if a suitable normalization is chosen, also the infinitesimal earthquake $\hat{\varphi}'$ changes in the same way. Therefore $H'$ agrees with the function $\Phi'$ obtained from $\hat{\varphi}'$.

Finally, recall that a tangent element to Universal Teichmüller space is defined as a vector field on $S^1$ satisfying the Zygmund condition (2.4) up to a first-order deformation by Möbius transformations. Therefore a different representative $\hat{\varphi}'$ in the same equivalence class of $\hat{\varphi}$ differs by

$$\hat{\varphi}'(\eta) = \hat{\varphi}(\eta) + \left.\frac{d}{dt}\right|_{t=0} A(t)(\eta) = \hat{\varphi}(\eta) + \dot{A}(\eta) = \hat{\varphi}(\eta) + \eta \boxtimes y_0$$

where $A(t) \in \mathrm{Isom}(\mathbb{R}^{2,1})$, $A(0) = \mathrm{I}$ and $\dot{A} = \frac{d}{dt}\big|_{t=0} A(t) \in \mathfrak{so}(2,1)$. By the same computation as above, $\langle \hat{\varphi}'(\eta), v\rangle = H'(\eta)$ where $H'$ is the support function of the domain of dependence $D(\mu, x_0, y_0)$.

By applying Proposition 4.5.2 and results presented in [GHL02] or [MŠ12] on the convergence of the integral in Equation (2.6), one deduces that, if $D$ is a domain of dependence whose dual lamination has finite Thurston norm, then the support function at infinity of $D$ is finite. We will give a quantitative version of this fact in Proposition 4.5.6 - which will be useful because it gives a uniform bound on the support function in terms of the Thurston norm. Here we draw another consequence of Proposition 4.5.2, namely the equivalence of conditions *i*) and *ii*).

**Corollary 4.5.4.** *Given a domain of dependence $D$, the dual lamination $\mu$ has finite Thurston norm if and only if the support function $h$ of $D$ extends to a Zygmund field on $\partial\mathbb{D}$.*

*Proof.* If $||\mu||_{Th} < +\infty$, from Proposition 4.5.2 we know that $h|_{\partial\mathbb{D}}$ coincides with the infinitesimal earthquake along $\mu$, hence is a Zygmund field. Viceversa, if $h|_{\partial\mathbb{D}}$ is a Zygmund field, by Theorem 2.3.11 there exists a bounded lamination $\mu$ such that $h|_{\partial\mathbb{D}}$ is the infinitesimal earthquake along $\mu$, and we conclude again by Proposition 4.5.2. □

### 4.5.2 Boundedness of curvature

To prove the implication $i) \Rightarrow iv)$, we have showed in Theorem 4.A the existence of constant curvature Cauchy surfaces $S(K)$, while in Theorem 4.D we proved that the surfaces $S(K)$ foliate the domain of dependence as $K \in (-\infty, 0)$. It remains to show that the principal curvatures of the Cauchy $K$-surfaces $S(K)$ we constructed are bounded provided the dual measured geodesic lamination has finite Thurston norm. This will also imply that $S(K)$ is complete, since (by boundedness of the curvature and of the principal curvatures) the Gauss map is bi-Lipschitz with respect to the induced metric on $S$ and the hyperbolic metric of $\mathbb{H}^2$.

**Proposition 4.5.5.** *Given a $K$-surface $S$, if the lamination $\mu$ dual to the domain of dependence $D(S)$ has finite Thurston norm, then the principal curvatures of $S$ are uniformly bounded.*

We will prove the proposition by contradiction. If the statement did not hold, there would exist a sequence of points $x_n \in S$ such that the principal curvatures diverge (since the product of the principal curvatures is constant, necessarily one principal curvature will tend to zero and the other to infinity). Roughly speaking, we will choose isometries $A_n$ so that the points $x_n$ are sent to a compact region of $\mathbb{R}^{2,1}$, and consider the surfaces $S_n = A_n(S)$. Essentially, a contradiction will be obtained by showing that the sequence $S_n$ contains a subsequence converging to a constant curvature smooth surface $S_\infty$ - using the boundedness of the dual lamination - and that this gives bounds on the principal curvatures at $x_n$. Hence it is not possible that principal curvatures diverge.

In order to apply the above argument, we need to prove a uniform bound on the support functions, depending only on the Thurston norm of the dual lamination.

**Proposition 4.5.6.** *Let $D_0 = D(\mu_0, x_0, y_0)$ be a domain of dependence whose dual lamination $\mu_0$ has Thurston norm $||\mu_0||_{Th} < M$, such that $P = y_0 + x_0^\perp$ is a support plane tangent to $\partial_s D_0$ at $y_0$. Let $h_0$ be the support function of $D_0$. Then $h_0 \leq C$ on $\overline{\mathbb{D}}$ for a constant $C$ which only depends on $M, x_0, y_0$.*

*Proof.* The support function of $D_0$ restricted to $\mathbb{H}^2$, under the hypothesis, is given by (see Proposition 4.1.10):

$$\bar{h}_0(x) = \langle x, y_0 \rangle + \int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma} \rangle d\mu_0 \,.$$

It is harmless to assume that $x_0 = (0, 0, 1)$ and $y_0 = 0$; indeed, composing with an isometry of $\mathbb{R}^{2,1}$, the support function $h$ changes by an affine map on $\overline{\mathbb{D}}$. Hence we give an estimate of the integral term in $\bar{h}_0$. Let $\gamma$ be a unit speed parametrization of the geodesic segment $[x_0, x]$. Note that if $\gamma(s)$ is on a geodesic $l$, for every $x \in \mathbb{H}^2$,

$$|\langle x, \boldsymbol{\sigma}(l) \rangle| = \sinh d_{\mathbb{H}^2}(x, l) \leq \sinh d_{\mathbb{H}^2}(x, \gamma(s)).$$

Hence, consider the partition $\gamma(0) = x_0, \gamma(1), \ldots, \gamma(N), \gamma(d_{\mathbb{H}^2}(x_0, x)) = x$, for $N$ the integer part of $d_{\mathbb{H}^2}(x_0, x)$. We have

$$\begin{aligned}\left|\int_{\mathcal{G}[x_0,x]} \langle x, \boldsymbol{\sigma}\rangle d\mu_0\right| &\leq \sum_{i=1}^{N+1} \sinh(i)\mu_0([\gamma(i-1),\gamma(i)]) \\ &\leq \sum_{i=1}^{N+1} \frac{M}{2}e^i = \frac{M}{2}e\frac{e^{N+1}-1}{e-1} \leq \frac{M}{2}\frac{e^2}{e-1}e^{d_{\mathbb{H}^2}(x_0,x)}\,.\end{aligned}$$

We can finally give a bound for the support function $h_0$ on $\mathbb{D}$. If $\pi(x) = z \in \mathbb{D}$,

$$h_0(z) = \frac{\bar{h}_0(x)}{\cosh d_{\mathbb{H}^2}(x_0, x)} \leq \frac{M}{2}\frac{e^2}{e-1}\frac{e^{d_{\mathbb{H}^2}(x_0,x)}}{\cosh d_{\mathbb{H}^2}(x_0,x)} \leq M\frac{e^2}{e-1}\,.$$

This shows that the function $h_0$ is bounded by a constant which only depends on $M$ and on $x_0, y_0$. By Lemma 4.1.3, the bound also holds on $\partial\mathbb{D}$. □

*Proof of Proposition 4.5.5.* Let $A_n \in SO_0(2,1)$ be a linear isometry of $\mathbb{R}^{2,1}$ such that $A_n(G(x_n)) = (0,0,1)$, where $G : S \to \mathbb{H}^2$ is the Gauss map of $S$. Let $D'_n$ be the domain of dependence of the surface $A_n(S)$. Now let $t_n \in \mathbb{R}^{2,1}$ be such that the $t_n$-translate of $\partial_s D'_n$ has a support plane with normal vector $(0,0,1)$ with tangency point the origin.

Let $S_n = A_n(S) + t_n$. Let $u_n$ be the support function on $\mathbb{D}$ of $S_n$ and $h_n$ be the support function of its domain of dependence $D_n = D'_n + t_n$.

The support functions $u_n$ are uniformly bounded on any $\Omega$ with compact closure in $\mathbb{D}$, since we have $h_n - (1/\sqrt{|K|})\sqrt{1-|z|^2} \leq u_n \leq h_n$ and by Proposition 4.5.6 the support functions $h_n$ of $D_n$ are uniformly bounded on $\Omega$. Hence $||u_n||_{C^0(\mathbb{D})} < C$ for some constant $C$. Let $B_n$ be the shape operator of $S_n$. Equation (4.2) in Lemma 4.1.4, for $z = 0$, gives $B_n^{-1} = \mathrm{Hess}(u_n)$. Applying Lemma 4.1.12, the inverse of the principal curvatures of $S_n$, which are the eigenvalues of $B_n^{-1}$, cannot go to infinity at the origin. This concludes the proof that principal curvatures of $S_n$ cannot become arbitrarily small. □

### 4.5.3 Cauchy surfaces with bounded principal curvatures and the Thurston norm of the dual lamination

In this part, we will show that a Cauchy surface $S$ with principal curvatures bounded below, $\lambda_i \geq d$ for some $d > 0$, are such that the measured geodesic lamination $\mu$ dual to $D(S)$ has finite Thurston norm. This shows the implication *iii)* $\Rightarrow$ *i)*. More precisely, we prove:

**Proposition 4.5.7.** *Let $B$ be the shape operator of a convex spacelike surface $S$ in $\mathbb{R}^{2,1}$ such that the Gauss map is a homeomorphism. Let $\mu$ be the measured geodesic lamination dual to $D(S)$, where $D(S)$ is the domain of dependence of $S$. Then*

$$||\mu||_{Th} \leq 2\sqrt{2(1+\cosh(1))}||B^{-1}||_{op} \tag{4.31}$$

*where $||B^{-1}||_{op} = \sup\{||B^{-1}(v)||/||v|| : v \in TS\}$ is the operator norm.*

Note that $||B^{-1}||_{op}$ is the supremum of the inverse of the principal curvatures of $S$; alternatively, it is the inverse of the infimum of the principal curvatures of $S$. To prove Proposition 4.5.7, we consider $x_1, x_2 \in \mathbb{H}^2$ with $d_{\mathbb{H}^2}(x_1, x_2) \leq 1$ and take points $y_1$, $y_2$ on $\partial_s D(S)$ such that $P_1 = y_1 + x_1^\perp$ and $P_2 = y_2 + x_2^\perp$ are support planes of $D$. Recall $\mu(\mathcal{G}[x_1, x_2])$ denotes the value taken by the dual lamination $\mu$ on the geodesic segment $[x_1, x_2]$ which joins $x_1$ and $x_2$. We will also denote $||v||_- = \sqrt{\langle v, v\rangle}$ if $v \in \mathbb{R}^{2,1}$ is spacelike.

**Lemma 4.5.8.** *Let $y_1, y_2 \in \partial_s D(S)$ and $P_1 = y_1 + x_1^\perp$ and $P_2 = y_2 + x_2^\perp$ be support planes for $\partial_s D(S)$ tangent to $\partial_s D(S)$ at $y_1$ and $y_2$. If $x_1$ and $x_2$ do not lie on any weighted leaf of the dual lamination $\mu$ of $D(S)$, then*

$$\mu(\mathcal{G}[x_1, x_2]) \leq ||y_1 - y_2||_- \,.$$

*Proof.* Assume first supp$\mu \cap \mathcal{G}[x_1, x_2]$ determines a finite lamination, i.e. $\mu$ restricted to $\mathcal{G}[x_1, x_2]$ is composed of a finite number of weighted leaves $g_1, \ldots, g_p$ with weights $a_1, \ldots, a_p$. Then we have (compare Proposition 4.1.10)

$$y_2 - y_1 = \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu = \sum_{i=1}^{p} a_i \boldsymbol{\sigma}(g_i).$$

Since the geodesics $g_1, \ldots, g_p$ are pairwise disjoint, the unit oriented normal vectors $\boldsymbol{\sigma}(g_1), \ldots, \boldsymbol{\sigma}(g_p)$ are such that $\langle \boldsymbol{\sigma}(g_i), \boldsymbol{\sigma}(g_j)\rangle \geq 1$. Hence

$$\langle y_2 - y_1, y_2 - y_1\rangle = \sum_{i=1}^{p} a_i^2 + 2\sum_{i<j} a_i a_j \langle \boldsymbol{\sigma}(g_i), \boldsymbol{\sigma}(g_j)\rangle \geq \left(\sum_{i=1}^{p} a_i\right)^2 = \mu(\mathcal{G}[x_1, x_2])^2.$$

This shows that the following inequality holds for a finite lamination $\mu$:

$$\left(\int_{\mathcal{G}[x_1,x_2]} d\mu\right)^2 \leq \langle \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu, \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu\rangle \,. \tag{4.32}$$

In general, if $\mu$ restricted to $\mathcal{G}[x_1, x_2]$ is not a finite lamination, we can approximate in the weak* topology the lamination $\mu$ by finite laminations $\mu_n$ (compare Lemma 4.3.3). As in Lemma 4.3.4, one can show

$$\int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu_n \xrightarrow{n\to\infty} \int_{\mathcal{G}[x_1,x_2]} \boldsymbol{\sigma} d\mu = y_2 - y_1 \tag{4.33}$$

and

$$\int_{\mathcal{G}[x_1,x_2]} d\mu_n \xrightarrow{n\to\infty} \int_{\mathcal{G}[x_1,x_2]} d\mu = \mu(\mathcal{G}[x_1, x_2]) \,. \tag{4.34}$$

Since (4.32) holds for the finite laminations in the LHS of Equations (4.33) and (4.34), the proof is complete. □

**Lemma 4.5.9.** *Let $S$ be a convex surface in $\mathbb{R}^{2,1}$ such that the principal curvatures $\lambda_i$ of $S$ are bounded below, $\lambda_i \geq d > 0$. For every point $p \in S$, the surface $S$ is contained in the future of the hyperboloid through $p$, tangent to $T_pS$ and with curvature $-d^2$.*

*Proof.* Let $u : \mathbb{D} \to \mathbb{R}$ and $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$, as usual, denote the support function of $S$ restricted to $\mathbb{D}$ and to $\mathbb{H}^2$. Analogously, let $v : \mathbb{D} \to \mathbb{R}$ and $\bar{v} : \mathbb{H}^2 \to \mathbb{R}$ be the support function of the hyperboloid $p + (1/d)\mathbb{H}^2$, as in the hypothesis. Composing with an isometry, we can assume $p = (0, 0, 1/d)$ and $T_pS$ is the horizontal plane $x_3 = 1/d$. Hence $\bar{v} \equiv -1/d$, while from Equation (4.1) in Lemma 4.1.4 for the inverse of the Gauss map of $S$

$$G^{-1}(x) = \operatorname{grad} \bar{u}(x) - \bar{u}(x)x\,,$$

one can deduce $\bar{u}((0, 0, 1)) = -1/d$ and $\operatorname{grad} \bar{u}((0, 0, 1)) = 0$. From the hypothesis, the eigenvalues of the shape operator of $S$ are larger than $d$ at every point. On the other hand the shape operator of the hyperboloid of curvature $-d^2$ is $d\mathrm{I}$. Therefore $\mathrm{Hess}\bar{u} - \bar{u}\mathrm{I} < \mathrm{Hess}\bar{v} - \bar{v}\mathrm{I}$ and it follows that $v - u$ is a convex function on $\mathbb{D}$ with a minimum at $0 \in \mathbb{D}$, where $(v - u)(0) = 0$. Therefore $v - u$ is positive on $\mathbb{D}$, which shows that $v \geq u$ and thus proves the statement. $\square$

The following Lemma is a direct consequence.

**Lemma 4.5.10.** *Let $S$ be a convex spacelike surface in $\mathbb{R}^{2,1}$ such that the principal curvatures of $S$ are bounded below, $\lambda_i \geq d > 0$, and that the Gauss map $G$ is a homeomorphism. Then for every $p \in S$, $D(S) \subset \mathrm{I}^+(r_d(p))$, where*

$$r_d(p) = p - \frac{1}{d}G(p).$$

*Proof.* By Lemma 4.5.9, $S \subset \mathrm{I}^+(r_d(p))$ for every $p \in S$. It follows that the entire domain of dependence of $S$ is contained in $\mathrm{I}^+(r_d(p))$. $\square$

*Proof of Proposition 4.5.7.* Assume the principal curvatures of $S$ are bounded below by $d > 0$, and $d$ is the infimum of the principal curvatures of $S$. Hence $||B^{-1}||_{op} = 1/d$.

Let us take $x_1, x_2 \in \mathbb{H}^2$, which do not lie on any weighted leaf of the dual lamination $\mu$ of $D(S)$, with $d_{\mathbb{H}^2}(x_1, x_2) \leq 1$. Suppose $y_1, y_2 \in \partial_s D(S)$ are such that $P_1 = y_1 + x_1^\perp$ and $P_2 = y_2 + x_2^\perp$ are tangent planes for $\partial_s D(S)$. We will show that $||y_1 - y_2||_- \leq 2\sqrt{2(1 + \cosh(1))}/d$ and thus the estimate (4.31) will follow by Lemma 4.5.8.

Suppose moreover $p_1, p_2 \in S$ are such that $p_1 + x_1^\perp$ and $p_2 + x_2^\perp$ are tangent planes for $S$. Let us denote $U_i = \mathrm{I}^+(r_d(p_i)) \cap \overline{\mathrm{I}^-(T_{p_i}S)}$ and $V_i = \mathrm{I}^+(r_d(p_i)) \cap \overline{\mathrm{I}^-(P_i)}$, for $i = 1, 2$. See Figure 4.15 and 4.16. Note that $y_1, y_2 \in \mathrm{I}^+(r_d(p_i))$, for $i = 1, 2$, by Lemma 4.5.10.

By construction, $y_1 \in V_1 \subset U_1$ and $y_2 \in V_2 \subset U_2$. Let us consider separately three cases:

*Case 1*: $y_2 \in V_1$. Then both $y_1$ and $y_2$ are contained in $U_1$. Since $U_1$ can be mapped isometrically to the region $\left\{(x_1, x_2, x_3) : x_3^2 \geq x_1^2 + x_2^2,\ x_3 \leq 1/d\right\}$ (see Figure 4.17), it is easy to see that a spacelike segment contained in $U_1$ can have lenght at most $2/d$, which gives the statement in this particular case.

*Case 2*: $y_1 \in V_2$. The estimate $||y_1 - y_2||_- \leq 2/d$ is obtained in a completely analogous way.

*Case 3*: $y_1 \notin V_2$ and $y_2 \notin V_1$. We claim that in this case $P_1 \cap P_2$ contains a point $y_3$ in $\mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$. Indeed, if $P_1 \cap P_2$ did not contain such a point,

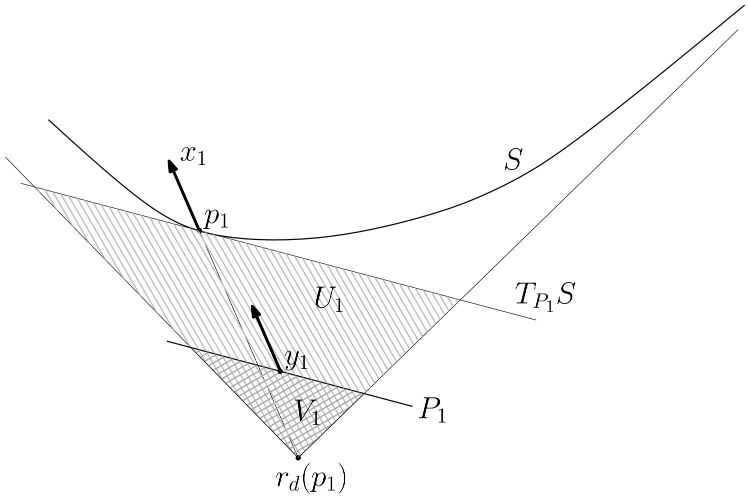



**Figure 4.15:** The setting of the proof and the definitions of the sets $U_1$ and $V_1$.

then the line $P_1 \cap P_2$ would be disjoint from $\mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$ and there would be two possibilities. Either $P_2 \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$ is contained in $\mathrm{I}^+(P_1) \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$, which is not possible since by assumption $y_1 \in P_1$ is not in $V_2$, or $P_1 \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$ is contained in $\mathrm{I}^+(P_2) \cap \mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$, which contradicts $y_2 \notin V_1$.

Consider now the geodesic segments $y_3 - y_1$ and $y_2 - y_3$. Since $y_3$ and $y_1$ are both contained in $V_1$, we have $||y_3 - y_1||_- \leq 2/d$. Analogously $||y_2 - y_3||_- \leq 2/d$. If the plane $Q$ containing $y_1, y_2, y_3$ is spacelike or lightlike, then

$$||y_2 - y_1||_- \leq ||y_2 - y_3||_- + ||y_3 - y_1||_- \leq \frac{4}{d}\,.$$

If $Q$ is timelike (meaning that the induced metric on $Q$ is a Lorentzian metric), then

$$\langle y_2 - y_1, y_2 - y_1 \rangle = ||y_2 - y_3||_-^2 + ||y_3 - y_1||_-^2 + 2\langle y_2 - y_3, y_3 - y_1 \rangle\,.$$

Let $v_1$ and $v_2$ be the future unit vectors in $Q$ orthogonal to $y_3 - y_1$ and $y_2 - y_3$. It is easy to check that

$$\begin{aligned}|\langle y_2 - y_3, y_3 - y_1 \rangle| &= ||y_2 - y_3||_- ||y_3 - y_1||_- |\langle v_1, v_2 \rangle| \\ &= ||y_2 - y_3||_- ||y_3 - y_1||_- \cosh d_{\mathbb{H}^2}(v_1, v_2)\,.\end{aligned}$$

We claim that $v_i$ is the orthogonal projection in $\mathbb{H}^2$ of $x_i$ to the geodesic determined by $Q$ (namely, the geodesic through $v_1$ and $v_2$). Composing with an isometry, we can assume $y_3 = 0$, the direction spanned by $y_3 - y_1$ is the line $x_2 = x_3 = 0$ and $Q = \{x_2 = 0\}$. Then $v_1 = (0, 0, 1)$ and $x_1 = (0, \sinh t, \cosh t)$, where $t = d_{\mathbb{H}^2}(v_1, x_1)$, and thus the claim holds. Of course the proof for $x_2$ and $v_2$ is analogous. This concludes the proof by Lemma 4.5.8, since

$$||y_2 - y_1||_-^2 \leq \frac{4}{d^2}(2 + 2\cosh(1))\,,$$

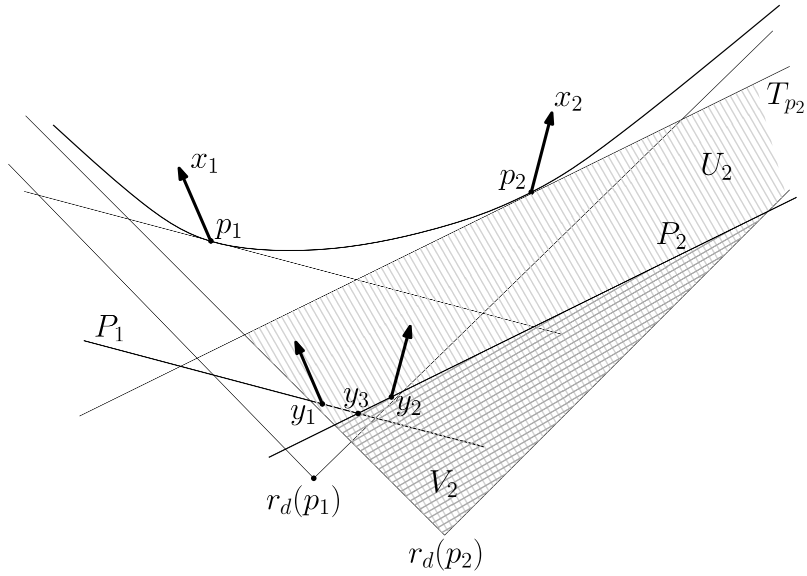



**Figure 4.16:** Analogously, the definitions of $U_2$ and $V_2$. The case pictured is for $y_1 \notin V_2$ and $y_2 \notin V_1$, hence there is a point $y_3$ in $P_1 \cap P_2$, which lies in $\mathrm{I}^+(r_d(p_1)) \cap \mathrm{I}^+(r_d(p_2))$.

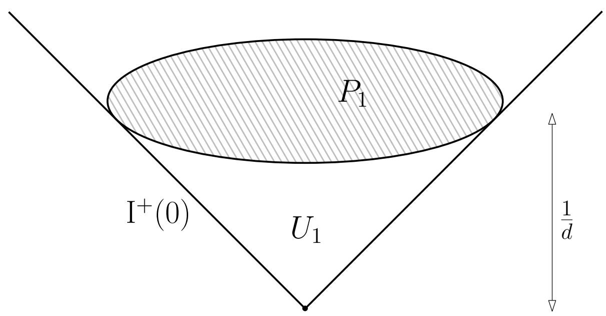



**Figure 4.17:** An isometric image of $U_1$.

where we have used that $y_3$ and $y_2$ are contained in $P_2 \subset U_2$, $y_1$ and $y_3$ are contained in $P_1 \subset U_1$ and so $||y_2 - y_3||_-, ||y_3 - y_1||_- \leq 2/d$ as above, and (since the projection to a line in $\mathbb{H}^2$ is distance-contracting) $d_{\mathbb{H}^2}(v_1, v_2) \leq d_{\mathbb{H}^2}(x_1, x_2) \leq 1$. $\square$

# Chapter 5

# Maximal globally hyperbolic flat spacetimes

The purpose of this chapter is to study convex surfaces in maximal globally hyperbolic flat spacetimes. We have already explained (see Subsection 2.2.1) that, by the pioneering work of Mess, such three-manifolds are parametrized by the holonomy representation, which consists of a Fuchsian representation and a cocycle representing the translation part. Hence this provides a parametrization of the moduli space of MGHF spacetimes whose topology is $S \times \mathbb{R}$ (where $S$ has genus at least 2) by means of the tangent bundle $T\mathcal{T}(S)$ of Teichmüller space. The first result of the chapter is an explicit description of the holonomy of a maximal globally hyperbolic flat spacetime in terms of the embedding data of a strictly convex Cauchy surface (recall the description of embedding data in Subsection 1.1.1).

The reason why we assume the strict convexity of $S$ is that it permits a convenient *change of variables*. Instead of the pair $(I, B)$, one can in fact consider the pair $(h, b)$, where $h$ is the third fundamental form $h = I(B\cdot, B\cdot)$ and $b = B^{-1}$. The fact that $(I, B)$ solves Gauss-Codazzi equations corresponds to the conditions that $h$ is a hyperbolic metric and $b$ is a self-adjoint solution of Codazzi equation for $h$.

It is simple to check that the holonomy of the hyperbolic surface $(S, h)$ is the linear part of the holonomy of $M$, so the isotopy class of $h$ does not depend on the choice of a Cauchy surface in $M$ and corresponds to the basepoint in $\mathcal{T}(S)$ for Mess' parameterization. Our first theorem gives a recipe to recover the translation part of the holonomy of $M$ in terms of $(h, b)$, by using the fact that ([OS83]) every $h$-self-adjoint operator on the hyperbolic surface $(S, h)$ which solves the Codazzi equation can be locally expressed as $\operatorname{Hess} u - u\, E$ for some smooth function $u$. Using this result we construct a short sequence of sheaves

$$0 \to \mathcal{F} \to \mathrm{C}^\infty \to \mathcal{C} \to 0\,, \tag{5.1}$$

where $\mathcal{C}$ is the sheaf of self-adjoint Codazzi operators on $S$ and $\mathcal{F}$ is the sheaf of flat sections of the $\mathbb{R}^{2,1}$-valued flat bundle associated to the holonomy of $h$. Passing to cohomology, this gives a connecting homomorphism

$$\delta : \mathcal{C}(S, h) \to H^1(S, \mathcal{F}) \cong H^1_{\mathrm{hol}}(\pi_1(S), \mathbb{R}^{2,1})\,,$$

where $\mathrm{hol} : \pi_1(S) \to \mathrm{SO}_0(2, 1)$ is the holonomy of $h$.

**Theorem 5.A.** *Let $M$ be a globally hyperbolic spacetime and $S$ be a uniformly convex Cauchy surface with embedding data $(I, B)$. Let $h$ be the third fundamental form of $S$ and $b = B^{-1}$. Then*

- *the linear holonomy of $M$ coincides with the holonomy of $h$;*
- *the translation part of the holonomy of $M$ coincides with $\delta b$.*

In the case where $S$ is a closed surface, we also provide a 2-dimensional geometric interpretation of $\delta b$. This is based on the simple remark that $b$ can also be regarded as a first variation of the metric $h$. As any Riemannian metric determines a complex structure over $S$, $b$ determines an infinitesimal variation of the complex structure $X$ underlying the metric $h$, giving in this way an element $\Psi(b) \in T_{[X]}\mathcal{T}(S)$.

**Theorem 5.B.** *Let $h$ be a hyperbolic metric on a closed surface $S$, $X$ denote the complex structure underlying $h$ and $\mathcal{C}(S, h)$ be the space of $h$-self-adjoint Codazzi tensors. Then the following diagram is commutative*

$$
\begin{array}{ccc}
\mathcal{C}(S,h) & \xrightarrow{\Lambda\circ\delta} & H^1_{Ad\circ hol}(\pi_1(S), \mathfrak{so}(2,1)) \\
\Psi\big\downarrow & & d\mathbf{hol}\big\uparrow \\
T_{[X]}\mathcal{T}(S) & \xrightarrow[\mathcal{J}]{} & T_{[X]}\mathcal{T}(S)
\end{array}
\tag{5.2}
$$

*where $\Lambda : H^1_{hol}(\pi_1(S), \mathbb{R}^{2,1}) \to H^1_{Ad\circ hol}(\pi_1(S), \mathfrak{so}(2,1))$ is the natural isomorphism, and $\mathcal{J}$ is the complex structure on $\mathcal{T}(S)$.*

Here **hol** is the map which associates to a point in Teichmüller space $\mathcal{T}(S)$ its (conjugacy class of) holonomy representation in $\mathcal{R}(\pi_1(S), \mathrm{SO}_0(2,1))/\!\!/\mathrm{SO}_0(2,1)$. A consequence of Theorem 5.B is the following corollary.

**Corollary 5.C.** *Two embedding data $(I, B)$ and $(I', B')$ correspond to Cauchy surfaces contained in the same spacetime if and only if*

- *the third fundamental forms $h$ and $h'$ are isotopic;*
- *the infinitesimal variation of $h$ induced by $b = B^{-1}$ is Teichmüller equivalent to the infinitesimal variation of $h'$ induced by $b' = (B')^{-1}$.*

Finally, we give an application of the commutativity of the diagram (5.2) to hyperbolic geometry. Goldman proved in [Gol84] that the Weil-Petersson symplectic form on $\mathcal{T}(S)$ coincides up to a factor with the Goldman pairing on the character variety $\mathcal{R}(\pi_1(S), \mathrm{SO}_0(2,1))/\!\!/\mathrm{SO}_0(2,1)$ through the map **hol**. We give a new *Lorentzian proof* of this fact. It directly follows by the commutativity of (5.2): we show by an explicit computation that the pull-back of those forms through the maps $\Lambda \circ \delta$ and $\Psi$ coincide (up to a factor) on $\mathcal{C}(S, h)$. While Goldman's proof highly relies on the complex analytical theory of Teichmüller space, our proof is basically only differential geometric.

## 5.1 Cauchy surfaces and Codazzi operators

### 5.1.1 Codazzi operators on a hyperbolic surface

Let $(S, h)$ be any hyperbolic (possibly open and non-complete) surface. Denote by

$$\text{hol} : \pi_1(S) \to \mathrm{SO}_0(2,1)$$

the corresponding holonomy. The only assumption we will make on $h$ is that hol is not elementary.

We will consider the Codazzi operator on the space of linear maps on $TS$

$$d^\nabla : \Gamma(T^*S \otimes TS) \to \Gamma(\Lambda^2 T^*S \otimes TS)$$

defined in the following way. Given a linear map $b : TS \to TS$, and given $v_1, v_2 \in T_xM$ we have

$$d^\nabla b(v_1, v_2) = (\nabla_{v_1} b)(v_2) - (\nabla_{v_2} b)(v_1) = \nabla_{v_1}(b(\hat{v}_2)) - \nabla_{v_2}(b(\hat{v}_1)) - b([\hat{v}_1, \hat{v}_2]) ,$$

where $\nabla$ is the Levi Civita connection of $h$ and $\hat{v}_1$ and $\hat{v}_2$ are local extensions of $v_1$ and $v_2$ in a neighborhood of $x$.

Given a representation $\rho : \pi_1(S) \to \mathrm{SO}_0(2,1)$, the first cohomology group $H^1_\rho(\pi_1(S), \mathbb{R}^{2,1})$ is the vector space obtained as a quotient of cocycles over coboundaries. A cocycle is a map $t : \pi_1(S) \to \mathbb{R}^{2,1}$ satisfying $t_{\alpha\beta} = \rho(\alpha)t_\beta + t_\alpha$. Such a cocycle $t$ is a coboundary if $t_\alpha = \rho(\alpha)t - t$ for some $t \in \mathbb{R}^{2,1}$. In this section we will show that self-adjoint operators $b$ satisfying the Codazzi equation $d^\nabla b = 0$ naturally describe the elements of the cohomology group $H^1_{\text{hol}}(\pi_1(S), \mathbb{R}^{2,1})$.

Let us recall that associated with $h$ there is a natural flat $\mathbb{R}^{2,1}$-bundle $F \to S$, whose holonomy is hol. Basically, $F$ is the quotient of $\tilde{S} \times \mathbb{R}^{2,1}$ by the product action of $\pi_1(S)$ as deck transformation on the first component and through the representation hol on the second one. Since $\tilde{S}$ is contractible, the group $H^1_{\text{hol}}(\pi_1(S), \mathbb{R}^{2,1})$ can be canonically identified with the first cohomology group $H^1(S, \mathcal{F})$ of the sheaf $\mathcal{F}$ of flat sections of $F$.

The relation between Codazzi tensors and the cohomology group $H^1_{\text{hol}}(\pi_1(S), \mathbb{R}^{2,1}) = H^1(S, \mathcal{F})$ relies on the construction of a short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathrm{C}^\infty \to \mathcal{C} \to 0 ,$$

where $\mathcal{C}$ is the sheaf of self-adjoint Codazzi tensors on $S$.

First we construct a map $H : \mathrm{C}^\infty \to \mathcal{C}$. This is is based on the following simple remark. In this chapter, we will usually denote by $u$ functions defined on $\mathbb{H}^2$, or more generally on a hyperbolic surface, although those were previously denoted by $\bar{u}$. Indeed, in the following it will be not important to consider the restriction of such functions to the Klein model, as in the previous chapters.

**Lemma 5.1.1.** *Let $U$ be any hyperbolic surface. For every $u \in \mathrm{C}^\infty(S)$, $b = \mathrm{Hess}\, u - u\, E$ is a self-adjoint Codazzi operator with respect to $h$. Here $E$ denotes the identity operator and* $\mathrm{Hess} u = \nabla \operatorname{grad} u$ *is considered as an operator on $TS$.*

*Proof.* The fact that $b$ is self-adjoint is clear. Let us prove that $b$ satisfies the Codazzi equation. By a simple computation $d^{\nabla}(u\,E) = du \wedge E$. On the other hand, since by definition $\mathrm{Hess}u = \nabla(\mathrm{grad}\,u)$ we get $d^{\nabla}\mathrm{Hess}u = R(\cdot,\cdot)\,\mathrm{grad}\,u$. As for a hyperbolic surface $R(v_1, v_2)v_3 = h(v_3, v_1)v_2 - h(v_3, v_2)v_1$, we get $d^{\nabla}\mathrm{Hess}u = du \wedge E$, so the result follows. □

Hence we define
$$H(u) = \mathrm{Hess}\,u - u\,E\,.$$
The second step is to construct a map $V : \mathcal{F} \to \mathrm{C}^\infty$ whose image is the kernel of $H$. Notice that the developing map $\mathrm{dev} : \tilde{S} \to \mathbb{H}^2 \subset \mathbb{R}^{2,1}$ induces to the quotient a section $\boldsymbol{\iota} : S \to F$ called the developing section. Using the natural Minkowski product on $F$, for any section $\sigma$ of $F$ the smooth function $V(\sigma)$ is defined by taking the product of $\sigma$ with $\boldsymbol{\iota}$:
$$V(\sigma) = \langle \sigma, \boldsymbol{\iota} \rangle\ .$$

**Theorem 5.1.2.** *The short sequence of sheaves*
$$0 \longrightarrow \mathcal{F} \xrightarrow{\ V\ } \mathrm{C}^\infty \xrightarrow{\ H\ } \mathcal{C} \longrightarrow 0 \tag{5.3}$$
*is exact.*

Since the statement is of local nature, it suffices to check exactness on an open convex subset $U$ of $\mathbb{H}^2$. The surjectivity of the map $H : \mathrm{C}^\infty(U) \to \mathcal{C}(U, h)$ follows by the general results in [OS83].

Notice that $\mathcal{F}(U)$ is naturally identified with $\mathbb{R}^{2,1}$. On the other hand, by identifying $\mathbb{H}^2$ with a subset of $\mathbb{R}^{2,1}$, the developing section is the standard inclusion. So the exactness of the first part of the sequence (5.3) is proved by the following Proposition.

**Proposition 5.1.3.** *Let $U$ be a convex neighborhood of $\mathbb{H}^2$. For any vector $t_0 \in \mathbb{R}^{2,1}$ the corresponding function $v = V(t_0)$*
$$v(x) = \langle t_0, x \rangle$$
*satisfies the equation $H(v) = 0$. Conversely, if $u$ is a smooth function on $U$ such that $H(u) = 0$, there exists a unique vector $t \in \mathbb{R}^{2,1}$ such that $u(x) = \langle t, x \rangle$ for any $x \in U$.*

*Proof.* We start by showing that for any fixed $t_0 \in \mathbb{R}^{2,1}$ the function $v(x) = \langle t_0, x \rangle$ satisfies $H(v) = 0$. Note that, for $w \in T_x\mathbb{H}^2$,
$$dv_x(w) = \langle t_0, w \rangle = \langle t_0^{T_x}, w \rangle \tag{5.4}$$
where $t_0^{T_x}$ is the projection of $t_0$ to $T_x\mathbb{H}^2$. Thus $\mathrm{grad}\,v(x) = t_0^{T_x}$. Note that $T_x\mathbb{H}^2$ coincides with the orthogonal plane to $x$. So $t_0 = t_0^{T_x} - \langle t_0, x \rangle x = \mathrm{grad}\,v(x) - v(x)x$. We will denote by $\overline{\nabla}$ the covariant derivative in the ambient $\mathbb{R}^{2,1}$ and by $\nabla$ that of $\mathbb{H}^2$ and use the fact that the second fundamental form of $\mathbb{H}^2$ coincides with the metric. Upon covariant differentiation in $\mathbb{R}^{2,1}$,
$$0 = \nabla_w\,\mathrm{grad}\,v(x) + \langle w, \mathrm{grad}\,v(x) \rangle x - dv_x(w)x - v(x)\,w = (\mathrm{Hess}\,v - v\,E)(w)\,.$$

Since elements of $U$, regarded as vectors of $\mathbb{R}^{2,1}$, generate the whole Minkowski space, the map

$$V : \mathbb{R}^{2,1} \to \mathrm{C}^\infty(U)$$

is injective and the image is a subspace of dimension 3 contained in the kernel of $H$. In order to conclude it is sufficient to prove that the dimension of $\ker H$ is 3.

To this aim it will suffice to show that any $u$ satisfying $\mathrm{Hess} u - u\mathrm{I} = 0$ such that $u(x_i) = 0$ on three non-collinear points $x_1, x_2, x_3$ vanishes everywhere.

Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a unit-speed geodesic in $U$ connecting two points $\gamma(s_1), \gamma(s_2)$ where $u(\gamma(s_1)) = u(\gamma(s_2)) = 0$. We claim that $u \circ \gamma \equiv 0$. Using that $\mathrm{Hess}\, u - u\, E = 0$, one gets that $y = u \circ \gamma$ satisfies the linear differential equation $y'' = y$. Since $y(s_1) = y(s_2) = 0$, for a standard maximum argument, $y \equiv 0$ on the interval $[s_1, s_2]$ and, by uniqueness, $y \equiv 0$ on $\mathbb{R}$.

Then $u \equiv 0$ on any geodesic connecting two points where $u$ takes the value 0. By hypothesis, $u$ takes the value 0 on three non-collinear points of $U$. By convexity of $U$, it is easy to see that the geodesics on which $u \equiv 0$ exaust the whole $U$, and this concludes the proof. □

Let us stress that in general the sequence (5.3) is not globally exact. The following example shows a family of Codazzi tensors which cannot be expressed as $\mathrm{Hess}\, u - u\, E$.

*Example* 5.1.4. Suppose $S$ is a closed surface. As observed by Hopf, see also [KS07], a self-adjoint Codazzi operator $b : TS \to TS$ is traceless if and only if the symmetric form $g(v, w) = h(b(v), w)$ on $S$ is the real part of a holomorphic quadratic differential $q$ on $S$. This gives an isomorphism of real vector spaces between the space of holomorphic quadratic differentials on $S$ and the space of traceless Codazzi tensors. We denote the image of $q$ under this isomorphism by $b_q$. So traceless Codazzi tensors form a vector space of finite dimension $6g - 6$ where $g$ is the genus of $S$.

On the other hand, if $\mathrm{Hess}\, u - u\, E$ is traceless, then $u$ satisfies the equation $\Delta u - 2u = 0$. A simple application of the maximum principle shows that the only solution of that equation is $u \equiv 0$. It follows that non-trivial traceless Codazzi tensors on $S$ cannot be expressed as $\mathrm{Hess}\, u - u\, E$.

The next Proposition shows that however the examples above are in a sense the most general possible. Although the proof is contained in [OS83], we give a short argument.

**Proposition 5.1.5.** *Let $S$ be a closed surface. Given $b \in \mathcal{C}(S, h)$ self-adjoint tensor satisfying Codazzi equation with respect to the hyperbolic metric $h$, a holomorphic quadratic differential $q$ and a smooth function $u \in \mathrm{C}^\infty(S)$ are uniquely determined so that $b = b_q + \mathrm{Hess}\, u - u\, E$.*

*Proof.* The subspaces $\{b \in \mathcal{C}(S, h) : b \text{ is traceless}\}$ and $\{b \in \mathcal{C}(S, h) : b = \mathrm{Hess} u - uE\}$ have trivial intersection by Example 5.1.4. Now, let $b \in \mathcal{C}(S, h)$ and $f = \mathrm{tr}(b) \in \mathrm{C}^\infty(S)$. Again, since $\Delta - 2\mathrm{id}$ is invertible, there exists some function $u$ such that $f = \Delta u - 2u = \mathrm{tr}(\mathrm{Hess}\, u - u\, E)$. Therefore $b - (\mathrm{Hess}\, u - u\, E)$ is traceless. This concludes the proof of the direct sum decomposition. □

From the exact sequence (5.3) we have a long exact sequence in cohomology

$$0 \to H^0(S,\mathcal{F}) \to H^0(S,\mathrm{C}^\infty) \to H^0(S,\mathcal{C}) \to H^1(S,\mathcal{F}) \to H^1(S,\mathrm{C}^\infty)\ . \qquad (5.5)$$

Since the holonomy representation hol is irreducible, then $H^0(S,\mathcal{F})$ is trivial. Moreover, $H^1(S,\mathrm{C}^\infty)$ vanishes since $\mathrm{C}^\infty$ is a fine sheaf. So we have a short exact sequence

$$0 \longrightarrow \mathrm{C}^\infty(S) \xrightarrow{H} \mathcal{C}(S,h) \xrightarrow{\delta} H^1(S,\mathcal{F}) \longrightarrow 0\ . \qquad (5.6)$$

Since $\tilde{S}$ is contractible, the cohomology group $H^1(S,\mathcal{F})$ is naturally identified with the group $H^1_{\mathrm{hol}}(\pi_1(S),\mathbb{R}^{2,1})$. The identification goes as follows. Take a good cover $\mathcal{U}$ of $S$ and let $\tilde{\mathcal{U}}$ be its lifting on $\tilde{S}$. By Leray Theorem $H^1(S,\mathcal{F}) = H^1(\check{C}^\bullet(\mathcal{U},\mathcal{F}))$.

The pull-back $\pi^*\mathcal{F}$ of the sheaf $\mathcal{F}$ on the universal cover is isomorphic to the sheaf $\underline{\mathbb{R}}^{2,1}$ of $\mathbb{R}^{2,1}$-valued locally constant function. Moreover there is a natural left action of $\pi_1(S)$ on $\check{C}^k(\tilde{\mathcal{U}},\underline{\mathbb{R}}^{2,1})$ given by

$$(\alpha\star s)(i_0,\ldots,i_k) = \mathrm{hol}(\alpha)s(\alpha^{-1}i_0,\ldots,\alpha^{-1}i_k)\,,$$

where we are using the fact that $\pi_1(S)$ permutes the open subsets in $\tilde{\mathcal{U}}$.

Now, the complex $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ is identified by pull-back with the sub-complex of $\check{C}^\bullet(\tilde{\mathcal{U}},\mathbb{R}^{2,1})$, say $\check{C}^\bullet(\tilde{\mathcal{U}},\mathbb{R}^{2,1})^{\pi_1(S)}$, made of elements invariant by the action of $\pi_1(S)$.

Since $H^1(\tilde{S},\mathbb{R}^{2,1}) = 0$, given a $\pi_1(S)$-invariant cocycle $s \in \check{Z}^\bullet(\tilde{\mathcal{U}},\mathbb{R}^{2,1})^{\pi_1(S)}$, there is a 0-cochain $r \in \check{C}^0(\tilde{\mathcal{U}},\mathbb{R}^{2,1})$ such that $\check{d}(r) = s$, that is $s(i_0,i_1) = r(i_1) - r(i_0)$ for any pair of open sets in $\tilde{\mathcal{U}}$ which have nonempty intersection.

Although in general $r$ is not $\pi_1(S)$-invariant, for any $\alpha \in \pi_1(S)$ we have that $\check{d}(\alpha\star r - r) = 0$, so there is an element $t_\alpha \in \mathbb{R}^{2,1}$ such that $\mathrm{hol}(\alpha)r(\alpha^{-1}i_0) - r(i_0) = t_\alpha$ for every $i_0$.

It turns out that the collection $(t_\alpha)$ verifies the cocycle condition, so it determines an element of $H^1_{\mathrm{hol}}(\pi_1(S),\mathbb{R}^{2,1})$. This construction provides the required isomorphism

$$H^1(\check{C}^\bullet(\tilde{\mathcal{U}},\mathbb{R}^{2,1})^{\pi_1(S)}) \to H^1_{\mathrm{hol}}(\pi_1(S),\mathbb{R}^{2,1})\ .$$

Using this natural identification we will explicitly describe the connecting homomorphism $\delta : \mathcal{C}(S,h) \to H^1_{\mathrm{hol}}(\pi_1(S),\mathbb{R}^{2,1})$. Let $b \in \mathcal{C}(S,h)$ and let $\tilde{b} : T\mathbb{H}^2 \to T\mathbb{H}^2$ be the lifting of $b$ to the universal cover. By Proposition 5.1.3, there exists $\hat{u} \in \mathrm{C}^\infty(\tilde{S})$ such that $\tilde{b} = \mathrm{Hess}\hat{u} - \hat{u}\mathrm{I}$. By the equivariance of $b$, for every $\alpha \in \pi_1(S)$, $\hat{u}\circ\alpha^{-1}$ is again such that $\tilde{b} = \mathrm{Hess}(\hat{u}\circ\alpha^{-1}) - (\hat{u}\circ\alpha^{-1})\mathrm{I}$. By the exactness of (5.3) there is a vector $t_\alpha \in \mathbb{R}^{2,1}$ such that

$$(\hat{u} - \hat{u}\circ\alpha^{-1})(x) = \langle t_\alpha, \mathrm{dev}(x)\rangle$$

where $\mathrm{dev} : \tilde{S} \to \mathbb{H}^2$ is a developing map for the hyperbolic structure on $\tilde{S}$. The map $\alpha \to t_\alpha$ gives a cocyle. Since the definition depends on the choice of $\hat{u}$, it is easy to check that $t_\bullet : \pi_1(S) \to \mathbb{R}^{2,1}$ is well-defined up to a coboundary. The cohomology class of $t_\bullet$ in $H^1_{\mathrm{hol}}(\pi_1(S),\mathbb{R}^{2,1})$ coincides with $\delta(b)$.

### 5.1.2 Geometric interpretation

**Definition 5.1.6.** Let $S$ be a $C^2$ Cauchy surface in a flat spacetime $M$ and denote by $B$ its shape operator computed with respect to the future-pointing normal vector. We say that $S$ is strictly future convex if $B$ is positive.

In this section we fix a topological surface $S$ of genus $g \geq 2$. We consider pairs $(M, \sigma)$ where $M$ is a maximal globally hyperbolic flat spacetime (we will use the acronym MGHF hereafter) and $\sigma : S \to M$ is an embedding onto a strictly future-convex Cauchy surface. Recall $M$ is homeomorphic to $S \times \mathbb{R}$; we will always implicitly consider embeddings $\sigma : S \to M$ which are isotopic to the standard embedding $S \hookrightarrow S \times \{0\}$.

By a classical result of [CB68], those pairs are parameterized by the embedding data of $\sigma$ which are the Riemannian metric $I$ induced by $\sigma$ on $S$ and the shape operator $B$ of the immersion (computed with respect to the future-pointing normal vector). Pairs $(I, B)$ are precisely the solutions of the so Gauss-Codazzi equations, as explained for Theorem 1.1.6:

$$\begin{cases} \det B = -K_I \\ d^{\nabla^I} B = 0 \end{cases} \tag{GC-$\mathbb{R}^{2,1}$}$$

We consider the set of embedding data of strictly convex Cauchy surfaces in a MGHF spacetime, namely:

$$\mathbb{D} = \left\{ (I, B) : \begin{array}{c} I \text{ Riemannian metric on } S \\ B : TS \to TS \text{ positive, self-adjoint for } I \\ (I, B) \text{ solves equations (GC-}\mathbb{R}^{2,1}) \end{array} \right\}.$$

Observe that the Riemannian metric $I$ in a pair $(I, B) \in \mathbb{D}$ is necessarily of negative curvature. First of all we want to show that the space $\mathbb{D}$ can be naturally identified with the space

$$\mathbb{E} = \left\{ (h, b) : \begin{array}{l} h \text{ hyperbolic metric on } S \\ b : TS \to TS \text{ self-adjoint for } h, d_h^{\nabla} b = 0, b > 0 \end{array} \right\}.$$

**Proposition 5.1.7.** *Let $(I, B)$ be an element of $\mathbb{D}$. Then the metric $h(v, w) = I(Bv, Bw)$ is hyperbolic and the operator $b = B^{-1}$ satisfies the Codazzi equation for $h$. Conversely if $(h, b) \in \mathbb{E}$ then $I(v, w) = h(bv, bw)$ and $B = b^{-1}$ are solutions of* (GC-$\mathbb{R}^{2,1}$).

*Proof.* Using the formula $\nabla^h = B^{-1} \nabla^I B$ which relates the Levi-Civita connection of $h$ and $I$ (see [Lab92] or [KS07]), it is easy to check that $b$ is an $h$-Codazzi tensor, and the same implication with the roles of $h$ and $I$ switched. We also have $K_h = K_I / \det B$ when $h = I(B, B)$, where $K_h$ and $K_I$ are the curvatures of $h$ and $I$. Therefore $K_h = -1$. Vice versa, starting from $(h, b)$, one obtains the Gauss equation $K_I = -1/\det b = -\det B$. $\square$

*Remark* 5.1.8. The group Diffeo($S$) naturally acts both on $\mathbb{E}$ and $\mathbb{D}$. It is important to remark here that the identification given by Proposition 5.1.7 commutes with those actions.

The following theorem shows the relation between the embedding data $(I, B)$ of a convex Cauchy embedding $S$ into $M$ and the holonomy of $M$.

**Theorem 5.A.** *Let $M$ be a globally hyperbolic spacetime and $S$ be a uniformly convex Cauchy surface with embedding data $(I, B)$. Let $h$ be the third fundamental form of $S$ and $b = B^{-1}$. Then*

- *the linear holonomy of $M$ coincides with the holonomy of $h$;*
- *the translation part of the holonomy of $M$ coincides with $\delta b$.*

The rest of this section is devoted to the proof of this theorem. We first give a more geometric meaning to the correspondence between $\mathbb{D}$ and $\mathbb{E}$. The key ingredient is the Gauss map. Given a Cauchy immersion $\sigma : S \to M$ we can consider the equivariant immersion of the universal cover of $S$, $\tilde{\sigma} : \tilde{S} \to \mathbb{R}^{2,1}$ obtained by composing the inclusion of $\tilde{S}$ into $\tilde{M}$ with the developing map of $M$. Notice that the holonomy of $\tilde{\sigma}$ coincides with the holonomy of $M$.

The Gauss map of the immersion is then the map $G : \tilde{S} \to \mathbb{H}^2$ sending a point $x$ to the future normal of the immersion $\tilde{\sigma}$. Notice that $d\sigma(T_x\tilde{S}) = \langle G(x)\rangle^\perp = T_{G(x)}\mathbb{H}^2$. Denote by $\tilde{B}$ and $\tilde{I}$ the lifting of $B$ and $I$ to the universal cover. Using that $\tilde{B}$ is the covariant derivative of the future normal field by the flat $\mathbb{R}^{2,1}$-connection, it is immediate to see that

$$dG_x(v) = d\sigma(\tilde{B}(v)). \tag{5.7}$$

This identity shows that the pull-back of the hyperbolic metric through $G$ is the metric $h(v, w) = I(Bv, Bw)$ so $G$ is a local isometry between $(\tilde{S}, \tilde{h})$ and $\mathbb{H}^2$. This implies that $G$ is the developing map of $h$.

**Proposition 5.1.9.** *Let $(I, B)$ be the embedding data of a strictly convex spacelike Cauchy surface in some MGHF spacetime $M$, and denote by $(h, b)$ the pair in $\mathbb{E}$ corresponding to $(I, B)$. If $\tilde{\sigma} : \tilde{S} \to \mathbb{R}^{2,1}$ is the space-like immersion corresponding to the data $(I, B)$, then the corresponding Gauss map $G : \tilde{S} \to \mathbb{H}^2$ is a developing map for $h$.*

We now want to compute the translation part of the holonomy of $M$, once the embedding data $(I, B)$ are known. In particular we want to show that the translation part of the holonomy equals $\delta b$ where $(h, b)$ is the pair corresponding to $(I, B)$.

To this aim we need to construct a function $\hat{u} : \tilde{S} \to \mathbb{R}$ such that $\tilde{b} = \mathrm{Hess}_{\tilde{h}}\hat{u} - \hat{u}\, E$.

**Proposition 5.1.10.** *Let $(I, s) \in \mathbb{D}$ and $(h, b) \in \mathbb{E}$ be the corresponding pair. Let $\tilde{\sigma} : \tilde{S} \to \mathbb{R}^{2,1}$ be an embedding whose first fundamental form is $\tilde{I}$ and whose shape operator is $\tilde{B}$, and denote by $G : \tilde{S} \to \mathbb{H}^2$ its Gauss map. Let us define $\hat{u} : \tilde{S} \to \mathbb{R}$ as $\hat{u}(x) = \langle \tilde{\sigma}(x), G(x)\rangle$. Then $\tilde{B}^{-1} = \tilde{b} = \mathrm{Hess}_{\tilde{h}}\hat{u} - \hat{u}\, E$.*

*Proof.* Notice that the statement is local so we may suppose that $G$ is an isometry between $\tilde{S}$ and an open subset $U$ of $\mathbb{H}^2$. So we can identify $(\tilde{S}, \tilde{h})$ to $U$ through the map $G$. Under this identification we have $\hat{u}(x) = \langle \tilde{\sigma}(x), x \rangle$ and (5.7) becomes $d\tilde{\sigma}_x(v) = b_x(v)$. By a computation we have

$$d\hat{u}_x(v) = \langle d\tilde{\sigma}_x(v), x \rangle + \langle \tilde{\sigma}(x), v \rangle = \langle \tilde{\sigma}(x)^{T_x}, v \rangle \,,$$

where we are using that $d\tilde{\sigma}_x(v) = b_x(v) \in T_x\mathbb{H}^2 = x^\perp$. This shows that $\operatorname{grad} \hat{u}(x) = \tilde{\sigma}(x)^{T_x}$ and concludes that $\tilde{\sigma}(x) = \operatorname{grad} \hat{u}(x) - \hat{u}(x)x$. Furthermore,

$$\begin{aligned}\tilde{b}_x(v) =& d\tilde{\sigma}_x(v) = \nabla_v \operatorname{grad} \hat{u} + \langle \operatorname{grad} \hat{u}, v \rangle x - d\hat{u}_x(v)x - \hat{u}(x)v \\ =& (\mathrm{Hess}\hat{u}(x))(v) - \hat{u}(x)v \,.\end{aligned}$$

□

The argument of the proof shows that in general the map $\tilde{\sigma}$ can be reconstructed using $G$ and $\hat{u}$ by the formula

$$\tilde{\sigma}(x) = dG(\operatorname{grad} \hat{u}(x)) - \hat{u}(x)G(x) \;. \tag{5.8}$$

*Remark* 5.1.11. By a result of Mess, if the metric $\tilde{I}$ on $\tilde{S}$ is complete (that is the case if it comes from a metric $I$ on the closed surface $S$), then the immersion $\tilde{\sigma}$ is in fact an embedding on a space-like surface of $\mathbb{R}^{2,1}$.

It turns out that $S$ is future strictly convex iff the future $F$ of $\tilde{\sigma}(\tilde{S})$ in $\mathbb{R}^{2,1}$ is convex. The function $\hat{u} \circ G^{-1} : \mathbb{H}^2 \to \mathbb{R}$ coincides with the support function of $F$ (see [FV13] for details on the support function of convex subsets in Minkowski space).

We can now conclude the proof of Theorem 5.A.

**Lemma 5.1.12.** *Let $(I, B) \in \mathbb{D}$ and $(b, h)$ the corresponding pair in $\mathbb{E}$. Denote by $\tilde{\sigma} : \tilde{S} \to \mathbb{R}^{2,1}$ the space-like immersion corresponding to $(I, B)$ and by $G : \tilde{S} \to \mathbb{H}^2$ the corresponding Gauss map. Consider the function $\hat{u} : \tilde{S} \to \mathbb{R}$ defined by $\hat{u}(x) = \langle \tilde{\sigma}(x), G(x) \rangle$. Then*

$$(\hat{u} - \hat{u} \circ \alpha^{-1})(x) = \langle t_\alpha, G(x) \rangle \,,$$

*where $t_\alpha$ is the translation part of the holonomy of the immersion $\tilde{\sigma}$.*

*Proof.* We have

$$\tilde{\sigma}(\alpha^{-1}x) = \mathrm{hol}(\alpha^{-1})\tilde{\sigma}(x) - \mathrm{hol}(\alpha)^{-1}t_\alpha \,,$$

and

$$G(\alpha^{-1}x) = \mathrm{hol}(\alpha^{-1})(G(x)) \;.$$

So

$$\hat{u}(\alpha^{-1}x) = \langle G(\alpha^{-1}x), \tilde{\sigma}(\alpha^{-1}x) \rangle = \hat{u}(x) - \langle t_\alpha, G(x) \rangle \;.$$

and this concludes the proof. □

*Remark* 5.1.13. If $S$ is a closed surface of genus $g \geq 2$, the holonomy distinguishes MGHF structures on $S \times \mathbb{R}$ containing a convex Cauchy surface. So Theorem 5.A implies that that two elements of $\mathbb{E}$, say $(h, b)$, and $(h', b')$, correspond to isotopic Cauchy immersions into the same spacetime if and only if

- there is an isometry $F : (S, h) \to (S, h')$ isotopic to the identity.

- $\delta(b) = \delta(b')$ (this makes sense as the holonomies of $h$ and $h'$ coincide for the previous point).

Let us denote by $\mathcal{S}_+(S)$ the set of isotopy classes of MGHF structures on $S \times \mathbb{R}$ containing a future convex surface. Then, $\mathcal{S}_+(S)$ can be realized as the quotient of $\mathbb{E}$ up to the identify $(h, b)$ and $(h', b')$ if the previous conditions are satisfied.

### 5.1.3 A de Rham approach to Codazzi tensors

We want to give a different description of the connection homomorphism

$$\delta : \mathcal{C}(S, h) \to H^1(S, \mathcal{F}) \ .$$

Let us denote by $D$ the flat connection over $F \to S$, and let $\Omega^k(S, F)$ be the space of $F$-valued $k$-forms over $S$, namely the space of smooth sections of the bundle $\Lambda^k T^*S \otimes F$. The exterior differential is the operator

$$d^D : \Omega^k(S, F) \to \Omega^{k+1}(S, F)$$

defined on simple elements by $d^D(\omega \otimes t) = d(\omega) \otimes t + (-1)^k \omega \wedge Dt$.

As $F$ is flat, $d^D \circ d^D = 0$ and the de Rham cohomology of the bundle $F$ is defined as the cohomology of the complex $(\Omega^\bullet(S, F), d^D)$. As

$$0 \to \mathcal{F} \to \Omega^0(-, F) \to \Omega^1(-, F) \to \dots$$

is a fine resolution of $\mathcal{F}$, by de Rham theorem $H^1(\mathcal{F})$ is naturally identified with $H^1_{\mathrm{dR}}(S, F)$.

The first aim of this section is to give a characterization of Codazzi operators over $S$ in terms of de Rham complex. Recall that we have a developing section $\boldsymbol{\iota} : S \to F$ obtained as the projection of the developing map. The covariant derivative of $\boldsymbol{\iota}$ provides a natural monomorphism $\boldsymbol{\iota}_* : TS \to F$, where $\boldsymbol{\iota}_*(v) = D_v \boldsymbol{\iota}$. If $S = \mathbb{H}^2$, then $\boldsymbol{\iota}$ corresponds to the natural inclusion $\mathbb{H}^2 \to \mathbb{R}^{2,1}$, and $\boldsymbol{\iota}_*$ corresponds to the inclusion of tangent spaces of $\mathbb{H}^2$ in $\mathbb{R}^{2,1}$.

Now, given any operator $b : TS \to TS$ we can consider the composition $\boldsymbol{\iota}_* b$ as an $F$-valued 1-form on $S$. The following simple computation gives a characterization of self-adjoint Codazzi tensors:

**Proposition 5.1.14.** *Let $b : TS \to TS$ be any operator. Then $b$ is self-adjoint and Codazzi if and only if $\boldsymbol{\iota}_* b$ is closed.*

*Proof.* The usual splitting of the flat connection of $\mathbb{R}^{2,1}$ into the Levi-Civita connection of $\mathbb{H}^2$ and second fundamental form gives in this setting the following formula

$$D_v(\boldsymbol{\iota}_* X) = \boldsymbol{\iota}_*(\nabla_v X) + h(X, v)\boldsymbol{\iota}(x)$$

for any vector field $X$ over $S$ and any tangent vector $v$ at $x$. Given two vector fields $X, Y$ on $S$ we get

$$\begin{aligned} d^D(\boldsymbol{\iota}_* b)(X, Y) =& D_X(\iota_*(bY)) - D_Y(\iota_*(bX)) - \boldsymbol{\iota}_* b[X, Y] \\ =& \iota_*(\nabla_X(bY) - \nabla_Y(bX) - b[X, Y]) + (h(X, bY) - h(Y, bX))\boldsymbol{\iota}(x) \\ =& \iota_*(d^\nabla b(X, Y)) + (h(X, bY) - h(Y, bX))\boldsymbol{\iota}(x) \end{aligned}$$

As the image of $\iota_*$ at $T_xS$ is the orthogonal complement of $\boldsymbol{\iota}(x)$ in $F_x$, the previous computation proves the statement. □

We can now give a description of the connection homomorphism $\delta$ under the usual identification between $H^1(S, \mathcal{F})$ and $H^1_{\mathrm{dR}}(S, F)$.

**Proposition 5.1.15.** *The connecting homomorphism $\delta : \mathcal{C}(S, h) \to H^1_{\mathrm{dR}}(S, F)$ of the short exact sequence* (5.6) *is expressed by the formula*

$$\delta(b) = [\boldsymbol{\iota}_* b] \,.$$

Before proving the Proposition, we give a preliminary Lemma.

**Lemma 5.1.16.** *Given a function $u \in \mathrm{C}^\infty(S)$ and $b \in \mathcal{C}(S, h)$, we have $b = \mathrm{Hess}u - u\mathrm{I}$ if and only if $\boldsymbol{\iota}_* b = d^D(\boldsymbol{\iota}_* \operatorname{grad} u - u\boldsymbol{\iota})$.*

*Proof.* By an explicit computation,

$$\begin{aligned} d^D(\boldsymbol{\iota}_* \operatorname{grad} u - u\boldsymbol{\iota}) =& D(\boldsymbol{\iota}_* \operatorname{grad} u) - (du)\boldsymbol{\iota} - uD\boldsymbol{\iota} \\ =& \boldsymbol{\iota}_*(\mathrm{Hess}\, u) + h(\operatorname{grad} u, \cdot)\boldsymbol{\iota} - (du)\boldsymbol{\iota} - u\boldsymbol{\iota}_* \\ =& \boldsymbol{\iota}_*(\mathrm{Hess}\, u - u\, E) \,. \end{aligned}$$

□

*Remark* 5.1.17. If $b$ is positive and $(I, B) \in \mathbb{D}$ are the embedding data associated with $(h, b)$, the corresponding map $\tilde{\sigma} : \tilde{S} \to \mathbb{R}^{2,1}$, considered as a section of the trivial flat $\mathbb{R}^{2,1}$-bundle on $\tilde{S}$, solves the equation

$$d^D \tilde{\sigma} = \boldsymbol{\iota}_* b \,,$$

so Lemma 5.1.16 is a generalization of Formula (5.8).

*Proof of Proposition 5.1.15.* The construction of the operator $\delta$ works as follows. Take a good cover $\{U_i\}$ of $S$ (namely, such that all the $U_i$ and their finite intersections are contractible). On each $U_i$ there is a function $u_i$ such that $b|_{U_i} = \mathrm{Hess}u_i - u_i E$. Now $u_{i_1} - u_{i_0} = V(t_{i_0 i_1})$ for some flat sections $t_{i_0 i_1}$ of $F$ on $U_{i_0} \cap U_{i_1}$. The family $\{t_{i_0 i_1}\}$ forms an $F$-valued 1-cocycle. Since $\Omega^0(-, F)$ is fine, there are smooth (but in general non flat) sections $\eta_i$ over $U_i$ such that

$$t_{i_0 i_1} = \eta_{i_1} - \eta_{i_0} \,. \tag{5.9}$$

The differentials of $\eta_i$ glue to a global $F$-valued closed form which represents $\delta b$ in the de Rham cohomology. We claim that $\eta_i = \boldsymbol{\iota}_*(\operatorname{grad} u_i) - u_i \boldsymbol{\iota}$ satisfy the condition (5.9). From the claim and Lemma 5.1.16 we easily get that $\delta b = [\boldsymbol{\iota}_* b]$.

To prove the claim it is sufficient to check the following formula

$$u_{i_1}(x) - u_{i_0}(x) = \langle (\eta_{i_1} - \eta_{i_0})(x), \boldsymbol{\iota}(x) \rangle \,,$$

which is immediate once one recalls that the image of $\boldsymbol{\iota}_*$ is orthogonal to $\boldsymbol{\iota}$. □

## 5.2 Relations with Teichmüller theory

In this section we will consider symmetric Codazzi tensors as infinitesimal deformations of a metric, inducing in this way an infinitesimal deformation of the conformal structure.

We will see that in the closed case the first order variation of the holonomy map

$$\mathbf{hol} : \mathcal{T}(S) \to \mathcal{R}(\pi_1(S), \mathrm{SO}(2,1))/\!\!/\mathrm{SO}(2,1)$$

which associates to a conformal structure the holonomy of the hyperbolic structure in its conformal class, can be explicitly computed in terms of the coboundary operators $\delta$ we already considered.

As a by-product we will see that the corresponding map $\mathbb{E} \to T\mathcal{T}(S)$, induces to the quotient a bijective map between the space $\mathcal{S}_+(S)$ of MGHF structures on $S \times \mathbb{R}$ and $T\mathcal{T}(S)$.

### 5.2.1 Killing vector fields and Minkowski space

The Lie algebra $\mathfrak{so}(2,1)$ is naturally identified to the set of Killing vector fields on $\mathbb{H}^2$, so it is realized as a subalgebra of the space of smooth vector fields $\mathfrak{X}(\mathbb{H}^2)$ on $\mathbb{H}^2$. There is a natural action of $\mathrm{SO}(2,1)$ on $\mathfrak{X}(\mathbb{H}^2)$ that is simply defined by

$$(A_*X)(x) = dA(X(A^{-1}x))\,.$$

This action restricts to the adjoint action on $\mathfrak{so}(2,1)$.

The Minkowski cross product is defined by $v \boxtimes w = *(v \wedge w)$, where $* : \Lambda^2(\mathbb{R}^{2,1}) \to \mathbb{R}^{2,1}$ is the Hodge operator associated to the Minkowski product. As in the Euclidean $3D$ case, it leads to a natural identification between $\mathbb{R}^{2,1}$ and $\mathfrak{so}(2,1)$ which commutes with the action of $\mathrm{SO}(2,1)$. Basically any vector $t \in \mathbb{R}^{2,1}$ is associated with the Killing vector field on $\mathbb{H}^2$ defined by $X_t(x) = t \boxtimes x$. This identification will be denoted by $\Lambda : \mathbb{R}^{2,1} \to \mathfrak{so}(2,1)$.

Notice that any hyperbolic surface $S$ is equipped with a $\mathrm{SO}(2,1)$-flat bundle $F_{\mathfrak{so}(2,1)}$ whose flat sections correspond to Killing vector fields on $S$. Elements of $F_{\mathfrak{so}(2,1)}$ are germs of Killing vector fields on $S$, so an evaluation map $\mathrm{ev} : F_{\mathfrak{so}(2,1)} \to TS$ is defined.

On the other hand, the isomorphism $\Lambda$ provides a flat isomorphism, that with a standard abuse we still denote by $\Lambda$, between the $\mathbb{R}^{2,1}$-flat bundle $F$ and $F_{\mathfrak{so}(2,1)}$. Under this identification the developing section $\boldsymbol{\iota} : S \to F$ corresponds to the section sending each point to the infinitesimal generator of the rotation about the point.

We recall that the Zariski tangent space at a point $\rho : \pi_1(S) \to \mathrm{SO}(2,1)$ to the character variety $\mathcal{R}(\pi_1(S), \mathrm{SO}(2,1))/\!\!/\mathrm{SO}(2,1)$ is canonically identified with the cohomology group $H^1_{\mathrm{Ad}\rho}(\pi_1(S), \mathfrak{so}(2,1))$ ([Gol84]). The identification goes in the following way. Given a differentiable path of representations $\rho_t$ such that $\rho_0 = \rho$, we put

$$\dot\rho(\alpha)(x) = \left.\frac{d}{dt}\right|_{t=0} \rho_t(\alpha) \circ \rho_0(\alpha)^{-1}(x).$$

### 5.2.2 Deformations of hyperbolic metrics

We fix a hyperbolic surface $(S, h)$ – that in this subsection we will not necessarily assume complete – with holonomy $\mathrm{hol} : \pi_1(S) \to \mathrm{SO}(2,1)$. We still make the assumption that hol is not elementary.

It is well known that if $h_t$ is a family of hyperbolic metrics on $S$ which smoothly depend on $t$ and such that $h_0 = h$, then the $h$-self-adjoint operator $b = h^{-1}\dot{h}$ satisfies the so called Lichnerowicz equation (see [FT84a])

$$L(b) = -(\Delta - 1/2)\mathrm{tr}b + \delta_h\delta_h b = 0\,,$$

where $\delta_h b$ is the 1-form obtained by contracting $\nabla b$ namely, $\delta_h(b)(v) = \mathrm{tr}(\nabla_\bullet b)(v)$, whereas the second $\delta_h$ is the divergence on 1-forms, $\delta(\omega) = \mathrm{tr}h^{-1}\nabla\omega$.

If $X$ is a vector field on $S$ with compact support, and $f_t : S \to S$ is the flow generated by $X$, then putting $h_t = f_t^*(h)$, we have that $h^{-1}\dot{h} = 2\mathbf{S}\nabla X$, where $\mathbf{S}\nabla X$ denotes the symmetric part of the operator $\nabla X$. It follows that $\mathbf{S}\nabla X$ is a solution of Lichnerowicz equation. Being $L$ a local operator, we deduce that $\mathbf{S}\nabla X$ is a solution of Lichnerowicz equation for any vector field $X$.

Conversely, as any deformation of a hyperbolic metric is locally trivial, it can be readily shown that any solution of Lichnerowicz equation can be locally written as $\mathbf{S}\nabla X$ for some vector field $X$.

Denoting by $\mathfrak{X}$ the sheaf of smooth vector fields and by $\mathcal{L}$ the sheaf of solutions of Lichnerowicz equation, the sheaf morphism $\mathbf{S}\nabla : \mathfrak{X} \to \mathcal{L}$, defined by $X \mapsto \mathbf{S}\nabla X$, is surjective. On the other hand the kernel of this morphism is the subsheaf of $\mathfrak{X}$ of Killing vector fields. This is simply the image of the sheaf $\mathcal{F}_{\mathfrak{so}(2,1)}$ of flat sections of $F_{\mathfrak{so}(2,1)}$ through the evaluation map $\mathrm{ev} : \mathcal{F}_{\mathfrak{so}(2,1)} \to \mathfrak{X}$.

So we have a short exact sequence of sheaves

$$0 \to \mathcal{F}_{\mathfrak{so}(2,1)} \to \mathfrak{X} \to \mathcal{L} \to 0$$

that in cohomology gives a sequence

$$0 \longrightarrow \mathfrak{X}(S) \longrightarrow \mathcal{L}(S) \xrightarrow{\ \mathfrak{d}\ } H^1_{\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1)) \longrightarrow 0\,, \tag{5.10}$$

where hol is the holonomy of $h$ and again we are using the canonical identification

$$H^1(S, \mathcal{F}_{\mathfrak{so}(2,1)}) = H^1_{\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1)).$$

We claim that if $b$ is the first order deformation of a family of hyperbolic metrics, namely $b = h^{-1}\dot{h}$, then $\mathfrak{d}$ coincides with the derivative of the holonomy map (which we still denote by **hol**)

$$\mathbf{hol} : \mathcal{M}_{-1} \to \mathcal{R}(\pi(S), \mathrm{SO}(2,1)) /\!\!/ \mathrm{SO}(2,1)$$

along the family of metrics.

More precisely we will prove the following result.

**Proposition 5.2.1.** *Let $h_t$ be a family of hyperbolic metrics on a surface $S$, not necessarily complete, and suppose that $h$ depends smoothly on $t$. Then we can find a family of representations $hol_t$ so that*

- *$hol_t$ is a representative of the holonomy of $h_t$ in its conjugacy class.*
- *$hol_t$ smoothly depends on $t$.*

*Moreover we have*

$$2\dot{hol} = \mathfrak{d}(h^{-1}\dot{h}) \ . \tag{5.11}$$

*Remark* 5.2.2. A simple way to understand the operator $\mathfrak{d}$ is the following. Given $b \in \mathcal{L}(S)$, on the universal covering there is a field $T$ such that $\tilde{b} = \mathbf{S}\nabla T$. As $\tilde{b}$ is $\pi_1(S)$-invariant, it turns out that $T - \alpha_* T$ is a Killing vector field on $\tilde{S}$ for any $\alpha \in \pi_1(S)$. Thus there is an element $\tau_\alpha \in \mathfrak{so}(2,1)$ such that $d\mathrm{dev}(T - \alpha_* T)(x) = \tau_\alpha(\mathrm{dev}(x))$ and $\mathfrak{d}(b)$ coincides with the cocycle $\tau_\bullet$.

*Proof.* Notice that the exponential maps $\exp_t$ of $h_t$ define a differentiable map on an open subset $\Omega$ of $(-\epsilon, \epsilon) \times T\tilde{S}$

$$\exp : \Omega \to \tilde{S}$$

such that $\exp(t, x, v) = \exp_t(x, v)$.

Now, let $\mathrm{dev}_t$ be the developing map of $h_t$ normalized so that $\mathrm{dev}_t(p_0) = \mathrm{dev}_0(p_0)$ and $d(\mathrm{dev}_t)(p_0) = d(\mathrm{dev}_0)(p_0)$, and consider the map

$$\mathrm{dev} : (-\epsilon, \epsilon) \times \tilde{S} \to \mathbb{H}^2$$

defined by $\mathrm{dev}(t, p) = \mathrm{dev}_t(p)$. As $\mathrm{dev}_t$ commutes with the exponential map

$$\mathrm{dev}_t(\exp_t(x, v)) = \exp_{\mathbb{H}^2}((\mathrm{dev}_t)_*(x, v)) \,,$$

one readily sees that dev is differentiable.

As a representative for the holonomy representation $\mathrm{hol}_t$ of $h_t$ can be chosen so that

$$\mathrm{dev}_t \circ \alpha = \mathrm{hol}_t(\alpha) \circ \mathrm{dev}_t \,,$$

it turns out that the map $\mathrm{hol} : (-\epsilon, \epsilon) \to \mathcal{R}(\pi_1(S), \mathrm{SO}(2,1)$ is differentiable as well.

Now differentiating the identity

$$h_t(v, w) = \langle d\mathrm{dev}_t(v), d\mathrm{dev}_t(w) \rangle$$

one sees that on the universal covering

$$h^{-1}\dot{h} = 2\mathbf{S}\nabla \dot{\mathrm{dev}} \,,$$

where we have put $\dot{\mathrm{dev}} = d(\mathrm{dev}_0)^{-1} \frac{d\mathrm{dev}_t}{dt}|_{t=0}$. On the other hand, for any $\alpha \in \pi_1(S)$, differentiating the identity

$$\mathrm{dev}_t(\alpha x) = \mathrm{hol}_t(\alpha)\mathrm{dev}_t(x)$$

one gets that

$$(d\mathrm{dev}_0)\dot{\mathrm{dev}}(\alpha x) = (d\mathrm{dev}_0)(d\alpha)\dot{\mathrm{dev}}(x) + \dot{\mathrm{hol}}(\alpha)\mathrm{dev}_0(\alpha x) \ .$$

It can be checked (compare Remark 5.2.2) that $\mathfrak{d}(h^{-1}\dot{h})_\alpha = 2d\mathrm{dev}_0(\dot{\mathrm{dev}} - \alpha_* \dot{\mathrm{dev}})$, hence the conclusion follows. □

### 5.2.3 Codazzi tensors as deformations of conformal structures on a closed surface

In this section we restrict to the case $S$ closed. We fix a hyperbolic metric $h$ with holonomy hol and denote by $J : TS \to TS$ the almost-complex structure induced by $h$.

We remark that in general a Codazzi tensor $b$ for $h$ is not an infinitesimal deformation of a family of hyperbolic metrics, unless $b$ is traceless. Indeed the following computation holds.

**Lemma 5.2.3.** *If $b$ is a self-adjoint Codazzi tensor of a hyperbolic surface $(S, h)$ then $L(b) = \mathrm{tr}(b)/2$ .*

*Proof.* Notice that $(\delta_h b)(v) = \mathrm{tr}(\nabla_{\bullet} b)(v) = \mathrm{tr}\nabla_v b$, where the last equality holds as $b$ is Codazzi. As the trace commutes with $\nabla$, $(\delta_h b)(v) = (d\mathrm{tr}b)(v)$ so $\delta_h b = d\mathrm{tr}b$, and $\delta_h\delta_h b = \Delta(\mathrm{tr}b)$. The conclusion follows immediately. □

We may consider $b$ as an infinitesimal deformation of the conformal structure. Indeed for small $t$ the bilinear form

$$\hat{h}_t(v, w) = h((E + tb)v, (E + tb)w)$$

defines a path of Riemannian metrics which smoothly depends on $t$.

So for each $t$, there is a uniformization function $\psi_t : S \to \mathbb{R}$ such that $h_t = e^{2\psi_t}\hat{h}_t$ is the unique hyperbolic metric conformal to $h_t$. It is well known that the conformal factor $\psi_t$ smoothly depends on $t$ (compare [FT84a]), so the path of holonomies $[\mathrm{hol}_t]$ defines a smooth path in the character variety.

The following proposition computes the first order variation of $\mathrm{hol}_t$ (that is an element of $H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2, 1))$ in terms of $b$.

**Proposition 5.2.4.** *Let $h$ be a hyperbolic metric on $S$, $X_h \in \mathcal{T}(S)$ be its complex structure and let $b \in \mathcal{C}(S, h)$. Let $b = b_q + \mathrm{Hess}u - u\,E$ the decomposition of $b$ given in Proposition 5.1.5. Then*

$$d\mathbf{hol}_{X_h}([b_0]) = -\Lambda(\delta(Jb_q))$$

*where $\Lambda : H^1_{hol}(\pi_1(S), \mathbb{R}^{2,1}) \to H^1_{Ad\circ hol}(\pi_1(S), \mathfrak{so}(2, 1))$ is the isomorphism induced by the $\mathrm{SO}(2, 1)$-equivariant isomorphism $\Lambda : \mathbb{R}^{2,1} \to \mathfrak{so}(2, 1)$.*

*Remark* 5.2.5. As $b_q$ is traceless and self-adjoint, $Jb_q$ is traceless and self-adjoint as well, and it turns out that $Jb_q = b_{iq}$.

*Proof.* Let $\mathrm{dev}_t$ be a family of developing maps of $h_t$ depending smoothly on $t$ and denote by $\mathrm{hol}_t$ the holonomy representative for which $\mathrm{dev}_t$ is equivariant.

By Proposition 5.2.1, $2\dot{\mathrm{hol}} = \mathfrak{d}(h^{-1}\dot{h})$, where $\mathfrak{d} : \mathcal{L}(S) \to H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2, 1))$ is the connecting homomorphism defined in (5.10).

By differentiating the identity $h_t(v, w) = e^{2\psi_t}h((E + tb)v, (E + tb)w)$ one gets

$$\dot{h}(v, w) = 2h((\dot{\psi}E + b)v, w)$$

that is

$$\frac{1}{2}h^{-1}\dot{h} = (\dot{\psi} - u)E + \mathrm{Hess}u + b_q \ .$$

Now $h^{-1}\dot{h}$, $b_q$ and $\mathrm{Hess}u = \mathbf{S}\nabla(\mathrm{grad}\, u)$ are solutions of Lichnerowicz equation, so by linearity $L((\dot{\psi} - u)E) = 0$. An explicit computation shows that this precisely means that the function $\phi = \dot{\psi} - u$ must satisfy the equation $\Delta\phi - \phi = 0$. As $\phi$ is a regular function on $S$ and we are assuming $S$ is closed, we deduce that $\phi \equiv 0$.

Thus $h^{-1}\dot{h} = 2\mathrm{Hess}u + 2b_q$. Notice that $\mathfrak{d}(\mathrm{Hess}u) = 0$ as $\mathrm{Hess}u = \mathbf{S}\nabla(\mathrm{grad}\, u)$ on $S$. So we get

$$2\dot{\mathrm{hol}} = \mathfrak{d}(h^{-1}\dot{h}) = 2\mathfrak{d}(b_q) \ .$$

To compute $\mathfrak{d}(b_q)$ we have to find a vector field $T$ on the universal covering, such that $b_q = \mathbf{S}\nabla T$, and $\mathfrak{d}(b_q)$ is determined by the equivariance of $T$.

Now as $Jb_q$ is still a symmetric Codazzi tensor, on the universal cover we can find a function $v$ such that $Jb_q = \mathrm{Hess}v - vE$. This implies that $b_q = -J\mathrm{Hess}v + Jv = -\nabla(J\,\mathrm{grad}\, v) + Jv$. That is, the field $T = -J\,\mathrm{grad}\, v$ satisfies the property we need. It follows that

$$(d\mathrm{dev}_0)^{-1}(\dot{\mathrm{hol}}(\alpha)) = (-J\,\mathrm{grad}\, v) - \alpha_*(-J\,\mathrm{grad}\, v) \ , \tag{5.12}$$

where $\mathrm{dev} : \tilde{S} \to \mathbb{H}^2$ is the developing map for $h$.

The conclusion then follows by comparing Equation (5.12) with the formula proved in the following Lemma, that we prove separately as it does not depend on the fact that $S$ is closed. □

**Lemma 5.2.6.** *Let $b$ a Codazzi tensor on any hyperbolic surface $S$ and let $v$ be a function on the universal cover such that $b = \mathrm{Hess}f - fE$. Then for any $\alpha \in \pi_1(S)$ and $x \in \tilde{S}$ we have that*

$$J\,\mathrm{grad}\, f - \alpha_*(J\,\mathrm{grad}\, f) = -(d\mathit{dev})^{-1}\Lambda(\delta b)_\alpha \,,$$

*where $\mathit{dev} : \tilde{S} \to \mathbb{H}^2$ is the developing map.*

*Proof.* Let us put $t_\alpha = (\delta b)_\alpha$. As $\Lambda(t)(\cdot) = t \boxtimes \cdot$, we have to prove that for any $x$ in $\tilde{S}$

$$d\mathrm{dev}_x(J\,\mathrm{grad}\, f(x) - \alpha_*(J\,\mathrm{grad}\, f))(x) = -t_\alpha \boxtimes \mathrm{dev}(x) \,.$$

The point is that $(f - f \circ \alpha^{-1})(x) = \langle t_\alpha, \mathrm{dev}(x)\rangle$ for any $x \in \tilde{S}$. Thus

$$d\mathrm{dev}_x(\mathrm{grad}\, f - \alpha_*\,\mathrm{grad}\, f) = t_\alpha + \langle t_\alpha, \mathrm{dev}(x)\rangle \mathrm{dev}(x) \ .$$

As for any $v \in T_x\tilde{S}$ we have that $d\mathrm{dev}_x(Jv) = J_{\mathbb{H}^2} d\mathrm{dev}_x(v) = \mathrm{dev}(x) \boxtimes d\mathrm{dev}_x(v)$ we get

$$d\mathrm{dev}_x(J(\mathrm{grad}\, f - \alpha_*\,\mathrm{grad}\, f)) = \mathrm{dev}(x) \boxtimes d\mathrm{dev}_x(\mathrm{grad}\, f - \alpha_*\,\mathrm{grad}\, f) = \mathrm{dev}(x) \boxtimes t_\alpha \,,$$

and the conclusion follows. □

*Remark* 5.2.7. We want to emphasize the reasons we restricted Proposition 5.2.4 to the case of closed surfaces.

There are some technical issues. For instance the metric $\hat{h}_t$ is not well defined if the eigenvalues of $b$ are not bounded and one should use $E + \chi_t b$, where $\chi_t$ is a function going sufficiently fast to 0 at infinity for each $t$ and such that $\partial_t \chi_t(0) = 1$. Moreover the uniformization factor $\psi_t$ on $t$ is well defined only if we restrict on some classes of hyperbolic metrics (e.g. complete metrics, metrics with cone singularities) and its smooth dependence on the factor is more complicated.

More substantial problems are related to the splitting of $b$ as traceless part and trivial part. The splitting is related to the solvability of the equation $\Delta u - 2u = \mathrm{tr}(b)$. Now in the non closed case to get existence and uniqueness of the solution, some asymptotic behavior of $\mathrm{tr}(b)$ must be required.

A related problem is that on an open surface there are smooth non trivial solutions of the equation $\Delta\phi - \phi = 0$. So in order to prove that $\phi \equiv 0$, $\phi$ is needed to have some good behavior at infinity (which can be obtained only requiring some extra hypothesis on the behavior of $b$ in the ends).

We will discuss the case of hyperbolic metrics with cone singularity in Chapter 6, where these problems will become evident.

### 5.2.4 A global parameterization of MGHF spacetimes with closed Cauchy surfaces

We consider the space $\mathbb{E}$ introduced in Section 5.1. We know that this space parameterizes embedding data of uniformly convex surfaces in some MGHF spacetime, and we have already remarked that the space $\mathcal{S}_+(S)$ of MGHF structures on $S \times \mathbb{R}$ containing a closed convex Cauchy surface is the quotient of $\mathbb{E}$ by identifying $(h, b)$ and $(h', b')$ if $h$ and $h'$ are isotopic and $\delta b = \delta b'$ (see Remark 5.1.13)

We want to use results of the previous section to construct a natural bijection between $\mathcal{S}_+(S)$ and the tangent bundle of the Teichmüller space of $S$. Let us briefly recall some basic facts of Teichmüller theory that we will use. See [Gar87] for more details.

Elements of $\mathcal{T}(S)$ are complex structures on $S$, say $X = (S, \mathcal{A})$, up to isotopy. In the classical Ahlfors-Bers theory, the tangent space of $\mathcal{T}(S)$ at a point $[X] \in \mathcal{T}(S)$ is identified with a quotient of the space of Beltrami differentials $\mathcal{B}(X)$. Recall that a Beltrami differential is a $L^\infty$ section of the bundle $K^{-1} \otimes \bar{K}$, where $K$ is the canonical bundle of $X$, that simply means that a Beltrami differential is a $(0,1)$-form with value in the holomorphic tangent bundle of $X$. There is a natural pairing between quadratic differentials and Beltrami differentials, given by the integration of the $(1,1)$ form obtained by contraction

$$\langle q, \mu \rangle = \int_S q \bullet \mu \,,$$

where in complex chart $q \bullet \mu := q(z)\mu(z) dz \wedge d\bar{z}$, if $\mu = \mu(z) d\bar{z}/dz$ and $q = q(z)dz^2$.

We say that a Beltrami differential $\mu$ is trivial if $\langle q, \mu \rangle = 0$ for any holomorphic quadratic differential. We will denote by $\mathcal{B}(X)^\perp$ the subspace of trivial Beltrami differentials.

The tangent space $T_{[X]}\mathcal{T}(S)$ is naturally identified with $\mathcal{B}(X)/\mathcal{B}(X)^{\perp}$ as a complex vector space. The identification goes as follows: suppose to have a $C^1$-path $\gamma : [0,1] \to \mathcal{T}(S)$ such that $\gamma(0) = [X]$. Then it is possible to choose a family of representatives $\gamma(t) = [X_t]$ such that the Beltrami differential $\mu_t$ of the identity map $\mathrm{I} : X \to X_t$ is a $C^1$ map of $[0,1]$ in $\mathcal{B}(X)$. It is a classical fact that $\dot{\mu}(0)$ does not depend on the choice of the representatives $X_t$ up to a trivial differential, and thus one can identify $\dot{\gamma}(0)$ with the class of $[\dot{\mu}]$ in $\mathcal{B}(X)/\mathcal{B}(X)^{\perp}$. Smooth trivial Beltrami operators can be expressed as $\bar{\partial}\sigma$, where $\sigma$ is a section of $K^{-1}$.

In order to link this theory with our construction it seems convenient to identify the holomorphic tangent bundle $K^{-1}$ of a Riemann surface $X = (S, \mathcal{A})$ with its real tangent bundle $TS$. Basically if $z = x + iy$ is a complex coordinate one identifies the tangent vector $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ with the holomorphic tangent vector $(a + ib)\frac{\partial}{\partial z}$. Notice that under this identification the multiplication by $i$ on $K^{-1}$ corresponds to the multiplication by the almost-complex structure $J$ associated with $X$. Moreover Beltrami differentials correspond to operators $m$ on $TS$ which are anti-linear for $J$: $mJ = -Jm$. By some simple linear algebra this is equivalent to $\mathrm{tr}(m) = 0$ and $\mathrm{tr}(Jm) = 0$, or analogously Beltrami differentials correspond to traceless operators which are symmetric for some conformal metric on $(TS, J)$.

More explicitly, if in local complex coordinate $\mu = \mu(z)d\bar{z}/dz$, the corresponding operator in the real coordinates is

$$m = \begin{pmatrix} \Re(\mu) & \Im(\mu) \\ \Im(\mu) & -\Re(\mu) \end{pmatrix}. \tag{5.13}$$

We recall that $\mathcal{T}(S)$ is a complex manifold. Its almost-complex structure $\mathcal{J}$ corresponds in the complex notation to the multiplication by $i$ of the Beltrami differential. In the real notation, this is the same as $\mathcal{J}([m]) = [Jm]$.

Now we denote by $X = X_h$ the complex structure determined by a hyperbolic metric $h$. Given a Codazzi operator for the metric $h$, we have considered a smooth path of metrics $\hat{h}_t = h(E + tb, E + tb)$, which determines a smooth path in the Teichmuller space $[X_t]$, where $X_t$ is the complex structure on $S$ determined by $\hat{h}_t$.

It turns out that the tangent vector of this path is simply the class of the Beltrami differential $b_0$, where $b_0 = b - (\mathrm{tr}b/2)E$ is the traceless part of $b$.

The main theorem we prove in this section is the following:

**Theorem 5.B.** *Let $h$ be a hyperbolic metric on a closed surface $S$, $X$ denote the complex structure underlying $h$, and $\mathcal{C}(S,h)$ be the space of self-adjoint $h$-Codazzi tensors. Then the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{C}(S,h) & \xrightarrow{\Lambda\circ\delta} & H^1_{Ad\circ hol}(\pi_1(S), \mathfrak{so}(2,1)) \\ {\scriptstyle\Psi}\big\downarrow & & \big\uparrow{\scriptstyle d\mathbf{hol}} \\ T_{[X]}\mathcal{T}(S) & \xrightarrow[\mathcal{J}]{} & T_{[X]}\mathcal{T}(S) \end{array} \tag{5.14}$$

*where $\Lambda : H^1_{hol}(\pi_1(S), \mathbb{R}^{2,1}) \to H^1_{Ad\circ hol}(\pi_1(S), \mathfrak{so}(2,1))$ is the natural isomorphism, and $\mathcal{J}$ is the complex structure on $\mathcal{T}(S)$.*

*Remark* 5.2.8. In order to prove this Theorem, we need to link the Levi-Civita connection on $S$ with the complex structure $X$.

Let $X = (S, \mathcal{A})$ be a Riemann surface with underlying space $S$, and let $h$ be a Riemannian metric on $S$ which is conformal in the complex charts of $\mathcal{A}$. Then through the canonical identification between $TS$ and $K^{-1}$, $h$ corresponds to a Hermitian product over $K^{-1}$.

Being $K^{-1}$ a holomorphic bundle over $S$, there is a Chern connection $D$ on $K^{-1}$ associated with $h$. As in complex dimension 1 any Hermitian form is Kähler, the connection $D$ corresponds to the Levi Civita connection of $h$ (regarded as a real Riemannian structure on $S$), through the identification $K^{-1} \cong TS$ (see Proposition 4.A.7 of [Huy05]).

Now if in a conformal coordinate $z$ the metric is of the form $h = e^{2\eta}|dz|^2$, then the connection form of $D$ is simply

$$\omega = 2\partial\eta \ , \tag{5.15}$$

where $\partial\eta = \frac{\partial\eta}{\partial z}dz$. Thus a simple computation shows that the connection form of $\nabla$ with respect to the real conformal frame $\partial_x, \partial_y$ is

$$A = d\eta \otimes \mathrm{I} - (d\eta \circ J) \otimes J \, . \tag{5.16}$$

Finally, as the $\bar{\partial}$-operator on $K^{-1}$ corresponds to the $(0,1)$-part of $D$, holomorphic sections of $K^{-1}$ correspond to vector fields $Y$ such that $(\nabla Y)$ commutes with $J$. This means that $\mathbf{S}\nabla Y$ must be a multiple of the identity, i.e. $\nabla Y = \lambda \mathrm{I} + \mu J$ for some functions $\lambda$ and $\mu$.

*Proof of Theorem 5.B.* The proof is based on the computation in Proposition 5.1.5. Using the decomposition $b = b_q + \mathrm{Hess}\, u - u\, E$, we see that $Jb_0 = Jb_q + \mathbf{S}\nabla(J\mathrm{Hess}\, u) = Jb_q + \bar{\partial}(J \,\mathrm{grad}\, u)$, where we are using that the $\bar{\partial}$-operator on $K^{-1}$ coincides with the anti linear part of $\nabla X$ (under the identification $K^{-1} = TS$). In particular $[Jb_0] = [Jb_q]$ as elements of $T_X\mathcal{T}(S)$. As $d\mathbf{hol}([Jb_0]) = \Lambda\delta(b_q) = \Lambda\delta(b)$ by Proposition 5.2.4 we conclude that the diagram is commutative. □

**Corollary 5.C.** *Two embedding data $(I, B)$ and $(I', B')$ correspond to Cauchy surfaces contained in the same flat globally hyperbolic spacetime if and only if*

- *the third fundamental forms $h$ and $h'$ are isotopic;*
- *the infinitesimal variation of $h$ induced by $b$ is Teichmüller equivalent to the infinitesimal variation of $h'$ induced by $b'$.*

*In particular the map induces to the quotient a bijective map*

$$\bar{\Psi} : \mathcal{S}_+(S) \to T\mathcal{T}(S) \, .$$

*Proof.* As it is known ([Gol84]) that

$$d\mathbf{hol} : T_X\mathcal{T}(S) \to T_{[\mathrm{hol}]}\left(\mathcal{R}(\pi_1(S), \mathrm{SO}(2,1)) /\!\!/ \mathrm{SO}(2,1)\right)$$

is an isomorphism, and that the holonomy distinguishes maximal globally hyperbolic flat spacetimes with compact surface ([Mes07]), the result follows by the commutativity of (5.14). The only point to check is that the restriction of the map $\Lambda \circ \delta : \mathcal{C}(S,h) \to H^1_{\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1))$ on the subset of positive Codazzi tensors $\mathcal{C}^+(S,h) = \{b \in \mathcal{C}(S,h) : b > 0\}$ is surjective. This follows from the fact that $\delta : \mathcal{C}(S,h) \to H^1_{\mathrm{hol}}(\pi_1(S), \mathbb{R}^{2,1})$ is surjective and that for any smooth Codazzi tensor we can find a constant $M$ such that $b + ME$ is positive. □

## 5.3 Symplectic forms

We fix a hyperbolic metric $h$ on a closed surface $S$ and use the same notation as in the previous section. We consider the Goldman symplectic form $\omega^B$ on $H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1))$, which depends on the choice of a non-degenerate Ad-invariant symmetric form $B$ on $\mathfrak{so}(2,1)$, and the Weil-Petersson symplectic form $\omega_{WP}$ on $T\mathcal{T}(S)$. In [Gol84], Goldman proved that

$$d\mathbf{hol} : (T_{[X_h]}\mathcal{T}(S), \omega_{WP}) \to (H^1_{\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1)), \omega^B)$$

is symplectic up to a multiplicative factor (which depends on the choice of $B$).

In this section we give a different proof of this fact. We will compute in a simple way the pull-back of the forms $\omega^B$ and $\omega_{WP}$ respectively through the maps $\Lambda \circ \delta : \mathcal{C}(S,h) \to H^1_{\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1))$ and $\Psi : \mathcal{C}(S,h) \to T_{X_h}\mathcal{T}(S)$ introduced in the previous section and show that they coincide up to a factor. The thesis will directly follow by the commutativity of (5.14).

In the definition of the Goldman form $\omega^{\mathbf{B}}$ given in [Gol84], the model $\mathfrak{sl}(2,\mathbb{R})$ of the algebra $\mathfrak{so}(2,1)$ is considered, and in that model the following Ad-invariant form is taken:

$$\mathbf{B}(X,Y) = \mathrm{tr}(XY)\,.$$

With this choice we can compute $B$ in terms of the Minkowski product on the bundle $F$.

**Lemma 5.3.1.** *Let* $\mathbf{B}$ *be the form on* $\mathfrak{so}(2,1)$ *obtained by identifying* $\mathfrak{so}(2,1)$ *with* $\mathfrak{sl}(2,\mathbb{R})$*. Then* $\mathbf{B}(\Lambda(t), \Lambda(s)) = (1/2)\langle t, s\rangle$.

*Proof.* As the space of Ad-invariant symmetric forms on $\mathfrak{so}(2,1)$ is 1-dimensional, there exists $\lambda_0$ such that $\mathbf{B}(\Lambda(t), \Lambda(s)) = \lambda_0\langle t, s\rangle$.

In order to compute $\lambda_0$, let us consider an isometry $\Gamma$ between $\mathbb{H}^2$ and the upper half-space $H^+$, sending $t_0 = (1,0,0)$ to $i$. As $\Lambda(t_0)$ is a generator of the elliptic group around $t_0$, we have that $\Gamma\Lambda(t_0)\Gamma^{-1}$ is a multiple of the matrix

$$X_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Using that $\exp(t\Lambda(t_0))$ is $2\pi$ periodic, whereas $\exp(tX)$ is $\pi$-periodic in $PSL(2,\mathbb{R})$ we deduce that $\Gamma\Lambda(t_0)\Gamma^{-1} = \pm(1/2)X_0$, so $\mathbf{B}(\Lambda(t_0), \Lambda(t_0)) = -1/2 = 1/2\langle t_0, t_0\rangle$ and $\lambda_0 = 1/2$. □

To define the symplectic form $\omega^{\mathbf{B}}$, the cohomology group $H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1))$ is identified with $H^1_{\mathrm{dR}}(S, F_{\mathfrak{so}(2,1)})$. Then we set

$$\omega^{\mathbf{B}}(\sigma, \sigma') = \int_S \mathbf{B}(\sigma \wedge \sigma')\,,$$

where $\mathbf{B}(\sigma \wedge \sigma')$ is obtained by alternating the real 2-form $\mathbf{B}(\sigma(\cdot), \sigma'(\cdot))$. Analogously a symplectic form $\omega^F$ is defined on $H^1_{\mathrm{dR}}(S, F)$ by setting

$$\omega^F(s, s') = \int_S \langle s \wedge s' \rangle\,,$$

where $s, s'$ are $F$-valued closed 1-forms.

By Lemma 5.3.1 one gets that $\omega^{\mathbf{B}}(\Lambda(s), \Lambda(s')) = (1/2)\omega^F(s, s')$.

**Proposition 5.3.2.** *Let $\delta : \mathcal{C}(S, h) \to H^1_{dR}(S, F)$ be the connecting homomorphism. Then*

$$\omega^F(\delta(b), \delta(b')) = \frac{1}{2}\int_S \mathrm{tr}(Jbb')\omega_h\,, \tag{5.17}$$

*or analogously*

$$\omega^B(\Lambda(\delta(b)), \Lambda(\delta(b'))) = \frac{1}{4}\int_S \mathrm{tr}(Jbb')\omega_h\,. \tag{5.18}$$

*Proof.* By Proposition 5.1.15, $\delta(b) = [\boldsymbol{\iota}_* b]$ and $\delta(b') = [\boldsymbol{\iota}_* b']$, where $\boldsymbol{\iota}_* : TS \to F$ is the inclusion induced by the developing section. In particular if $\{e_1, e_2\}$ is an orthonormal frame on $S$ we have

$$\begin{aligned}
\langle (\delta b) \wedge (\delta b') \rangle =& \frac{1}{2}\big(\langle \boldsymbol{\iota}_* b(e_1), \boldsymbol{\iota}_* b'(e_2)\rangle - \langle \boldsymbol{\iota}_* b(e_2), \boldsymbol{\iota}_* b'(e_1)\rangle\big)\\
=& \frac{1}{2}\big(h(be_1, b'e_2) - h(be_2, b'e_1)\big)\\
=& \frac{1}{2}\big(h(be_1, b'Je_1) + h(be_2, b'Je_2)\big)\\
=& \frac{1}{2}\mathrm{tr}(Jbb')\,.
\end{aligned}$$

Formula (5.17) immediately follows. $\square$

*Remark* 5.3.3. A consequence of the previous proposition is that if $b$ and $b'$ are Codazzi operators, then

$$\int_S \mathrm{tr}(Jbb')\omega_h = 0$$

whenever one of the two factors is of the form $\mathrm{Hess}\, u - u\,E$. This could also be deduced by a direct computation.

We now consider the computation of the Weil-Petersson symplectic form $\omega_{WP}$. In conformal coordinates, if $q(z) = f(z)dz^2$, $q'(z) = g(z)dz^2$ and $h(z) = e^{2\eta}|dz|^2$, then the 2-form

$$\frac{f\bar{g}}{e^{2\eta}}dx \wedge dy$$

is independent of the coordinates. Recall that the Codazzi tensor $b_q$ is defined as the $h^{-1}\Re(q)$. A simple computation shows that in the conformal basis $\{\partial_x, \partial_y\}$ the operator $b_q$ is represented by the matrix

$$e^{-2\eta}\begin{pmatrix} \Re(f) & -\Im(f) \\ -\Im(f) & -\Re(f) \end{pmatrix} . \tag{5.19}$$

The Weil-Petersson product is defined as

$$g_{WP}(q, q') = \int_S \frac{f\bar{g}}{e^{2\eta}} dx \wedge dy \ .$$

A local computation, using expression (5.19) of the matrices $b_q$ and $b_{q'}$ shows that

$$g_{WP}(q, q') = \frac{1}{2}\int_S \operatorname{tr}(b_q b_{q'})\omega_h \ + \ \frac{i}{2}\int_S \operatorname{tr}(Jb_q b_{q'})\omega_h \, . \tag{5.20}$$

This expression is at the heart of the following computation.

**Proposition 5.3.4.** *Given $b, b' \in \mathcal{C}(S, h)$, the following formula holds:*

$$\omega_{WP}(\Psi(b), \Psi(b')) = 2\int_S \operatorname{tr}(Jbb')\omega_h \, . \tag{5.21}$$

*Proof.* Let $q$ be a holomoprhic quadratic differential. By a local computation, using Equations (5.13) and (5.19) which relate the expression in complex charts of the Beltrami differential $\Psi(b) = [b_0]$ and the holomorphic quadratic $q$ differential to their expression as operators on the real tangent space $TS$, the contraction form of $\Psi(b)$ and $q$ equals

$$q \bullet \Psi(b) = -(\operatorname{tr}(Jb_0 b_q) + i\operatorname{tr}(b_0 b_q))\omega_h \, .$$

It follows that

$$\langle q, \Psi(b)\rangle = -\int_S (\operatorname{tr}(Jb_0 b_q) + i\operatorname{tr}(b_0 b_q))\omega_h \, . \tag{5.22}$$

Comparing this equation with (5.20) we see that the antilinear map $\mathcal{K}^2(S) \to T_{X_h}\mathcal{T}(S)$ defined by the Weil-Petersson product is

$$q \to \Psi\left(\frac{Jb_q}{2}\right) .$$

So we have dually that

$$g_{WP}(\Psi(b_q), \Psi(b'_q)) = 4g_{WP}(\Psi(Jb_q/2), \Psi(Jb'_q/2)) = 4g_{WP}(q, q')$$

and using (5.20) we see that

$$\omega_{WP}(\Psi(b_q), \Psi(b'_q)) = 2\int_S \operatorname{tr}(Jb_q b'_q)\omega_h \, .$$

To get the formula in general, notice that, if $b = b_q + \operatorname{Hess} u - u\,E$ and $b' = b_{q'} + \operatorname{Hess} u' - u'E$,

$$\begin{aligned}\omega_{WP}(\Psi(b),\Psi(b')) =& \omega_{WP}(\Psi(b_q),\Psi(b_{q'}))\\ =& 2\int_S \mathrm{tr}(Jb_q b_{q'})\omega_h\\ =& 2\int_S \mathrm{tr}(Jbb')\omega_h\,.\end{aligned}$$

where the last equality holds by Remark 5.3.3. □

**Corollary 5.3.5.** *The Weil-Petersson symplectic form $\omega_{WP}$ and the Goldman symplectic form $\omega^{\mathbf{B}}$ are related by:*

$$\mathbf{hol}^*(\omega^{\mathbf{B}}) = \frac{1}{8}\omega_{WP}\,.$$

The proof, which is a new proof of Goldman's Theorem presented in [Gol84], follows directly by the commutativity of diagram (5.14) and formulae (5.18) and (5.21).

# Chapter 6

# Flat spacetimes with massive particles

In this chapter we apply a machinery similar to Chapter 5 to study globally hyperbolic spacetimes containing particles, that is, cone singularities along timelike lines. To develop the study of Cauchy surfaces in a spacetime with particles, we will need to make the assumption that the shape operator of the surface is bounded and uniformly positive (meaning that the principal curvatures are uniformly far from 0 and $+\infty$). We will briefly say that the Cauchy surface is *bounded and uniformly convex*. Moreover, we will assume that the cone singularity of every particle is in $(0, 2\pi)$.

Under this assumption we prove that the surface is necessarily orthogonal to the singular locus and intrinsically carries a Riemannian metric with cone angles equal to the cone singularities of the particle (here we use the definition given by Troyanov [Tro91] of metrics with cone angles on a surface with variable curvature). We will show that the third fundamental form of such a surface is a hyperbolic surface with the same cone angles and $b = B^{-1}$ is a bounded and uniformly positive Codazzi operator for $(S, h)$. Hence the first result of this chapter is the following:

**Theorem 6.A.** *Let us fix a* divisor $\boldsymbol{\beta} = \sum \beta_i p_i$ *on a surface with* $\beta_i \in (-1, 0)$ *and consider the following sets:*

- $\mathbb{E}_{\boldsymbol{\beta}}$ *is the set of embedding data* $(I, B)$ *of bounded and uniformly convex Cauchy surfaces on flat spacetimes with particles such that for every* $i = 1, \ldots, k$ *a particle of angle* $2\pi(1 + \beta_i)$ *passes through* $p_i$*.*

- $\mathbb{D}_{\boldsymbol{\beta}}$ *is the set of pairs* $(h, b)$*, where* $h$ *is a hyperbolic metric on* $S$ *with a cone singularity of angle* $2\pi(1 + \beta_i)$ *at each* $p_i$ *and* $b$ *is a self-adjoint solution of Codazzi equation for* $h$*, bounded and uniformly positive.*

*Then the correspondence* $(I, B) \to (h = I(B, B), b = B^{-1})$ *induces a bijection between* $\mathbb{E}_{\boldsymbol{\beta}}$ *and* $\mathbb{D}_{\boldsymbol{\beta}}$*.*

By Gauss-Bonnet formula, in order to have $\mathbb{D}_{\boldsymbol{\beta}}$ (and consequently $\mathbb{E}_{\boldsymbol{\beta}}$) non empty one has to require that $\chi(S, \boldsymbol{\beta}) := \chi(S) + \sum \beta_i$ is negative. We will always make this assumption.

A consequence of the construction which will be used in the proof of is the following, which might have an independent interest.

**Theorem 6.B.** *Let $h$ be a hyperbolic metric with cone singularities and let $b$ be a Codazzi, self-adjoint operator for $h$, bounded and uniformly positive. Then $I = h(b\cdot, b\cdot)$ defines a singular metric with the same cone angles as $h$. Moreover if $I = e^{2\xi}|w|^{2\beta}|dw|^2$ in a conformal coordinate $w$ around a singular point $p$, the factor $\xi$ extends to a Hölder continuous function at $p$.*

The second main theorem is the analogue of Theorem 5.B in the context of cone singularities, proving that the moduli space of maximal globally hyperbolic flat spacetimes with particles is the tangent bundle of the Teichmüller space of the punctured surface. In particular it does not depend on the cone angles.

To give a precise statement we use the Troyanov uniformization result [Tro91] which ensures that, given a conformal structure on $S$, there is a unique conformal hyperbolic metric with prescribed cone angles at the points $p_i$ (notice we are assuming $\chi(S, \boldsymbol{\beta}) < 0$). So once the divisor $\boldsymbol{\beta}$ is chosen we have a holonomy map

$$\mathbf{hol} : \mathcal{T}(S, \mathfrak{p}) \to \mathcal{R}(\pi_1(S \setminus \mathfrak{p}), \mathrm{SO}_0(2,1))/\!\!/\mathrm{SO}_0(2,1)\,,$$

where $\mathfrak{p} = \{p_1, \ldots, p_k\}$ is the support of $\boldsymbol{\beta}$, and $\mathcal{T}(S, \mathfrak{p})$ is the Teichmüller space of the punctured surface.

As in the closed case fix a hyperbolic metric $h$ on $S$ with cone angles $2\pi(1 + \beta_i)$ at $p_i$. Let $X$ denote the complex structure underlying $h$. Any Codazzi operator $b$ on $(S, h)$ can be regarded as an infinitesimal deformation of the metric on the regular part of $S$. If $b$ is bounded this deformation is quasiconformal so it extends to an infinitesimal deformation of the underlying conformal structure at the punctures, providing an element $\Psi(b)$ in $T_{[X]}\mathcal{T}(S, \mathfrak{p})$.

**Theorem 6.C.** *Let $\mathcal{C}_\infty(S, h)$ be the space of bounded Codazzi tensors on $(S, h)$. The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{C}_\infty(S,h) & \xrightarrow{\Lambda\circ\delta} & H^1_{Ad\circ hol}(\pi_1(S \setminus \mathfrak{p}), \mathfrak{so}(2,1)) \\ {\scriptstyle\Psi}\Big\downarrow & & d\mathbf{hol}\Big\uparrow \\ T_{[X]}\mathcal{T}(S,\mathfrak{p}) & \xrightarrow[\mathcal{J}]{} & T_{[X]}\mathcal{T}(S,\mathfrak{p}) \end{array}\,, \tag{6.1}$$

*where $\Lambda : H^1_{hol}(\pi_1(S \setminus \mathfrak{p}), \mathbb{R}^{2,1}) \to H^1_{Ad\circ hol}(\pi_1(S \setminus \mathfrak{p}), \mathfrak{so}(2,1))$ is the natural isomorphism, and $\mathcal{J}$ is the complex structure on $\mathcal{T}(S, \mathfrak{p})$.*

A consequence is the following theorem on the classification of maximal globally hyperbolic flat spacetimes, in terms of the embedding data of a Cauchy surface:

**Theorem 6.D.** *Two embedding data $(I, B)$ and $(I', B')$ in $\mathbb{E}_{\boldsymbol{\beta}}$ correspond to Cauchy surfaces contained in the same spacetime with particles if and only if*

- *the third fundamental forms $h$ and $h'$ are isotopic;*
- *the infinitesimal variation of $h$ induced by $b = B^{-1}$ is Teichmüller equivalent to the infinitesimal variation of $h'$ induced by $b' = (B')^{-1}$.*

It should be remarked that in this context, at least if the cone angles are in $[\pi, 2\pi)$, the holonomy does not distinguish the structures, so Theorem 6.D is not a direct consequence of Theorem 6.C. Indeed we believe that for the same reason, the direct application of Mess' arguments to this context is not immediate. In Anti-de Sitter case in [BS09] a generalization of Mess' techniques has been achieved, at least if cone angles are in $(0, \pi)$.

In Section 6.3 we address the question of the coincidence of the Weil-Petersson metric and the Goldman pairing in this context of structures with cone singularities. Once a divisor $\boldsymbol{\beta}$ is fixed, the hyperbolic metrics with prescribed cone angles allow to determine a Weil-Petersson product on $\mathcal{T}(S, \mathfrak{p})$, as it has been studied in [ST11]. In [Mon10], Mondello showed that also in this singular case the Weil-Petersson product corresponds to an intersection form on the subspace of $H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S \setminus \mathfrak{p}), \mathfrak{so}(2,1))$ corresponding to cocycles which are trivial around the punctures. Actually Mondello's proof is based on a careful generalization of Goldman argument in the case with singularity. Like in the closed case, we give a substantially different proof of this coincidence by using the commutativity of Equation (6.1).

In the last section of the chapter we discuss to what extent the condition of containing a uniformly convex surface is restrictive. Indeed, a natural question is whether there are globally hyperbolic spacetimes with particles with negative characteristic which do not contain uniformly convex surfaces. Some simple counterexamples can be obtained by doubling a cylinder in Minkowski space based on some polygon on $\mathbb{R}^2$. However the spacetimes obtained in this way have the property that Euler characteristic $\chi(S, \boldsymbol{\beta})$ of its Cauchy surfaces is 0. Hence, in Section 6.4 we construct some counterexamples in this direction, based on simple surgery ideas, showing some spacetimes which do not contain any uniformly convex Cauchy surface. Similar problems regarding the existence of spacetimes with certain properties on the Cauchy surfaces have been tackled in [BG00]. In all those exotic examples at least one particle must have cone angle in $[\pi, 2\pi)$.

## 6.1 Metrics with cone singularities

We now consider more deeply the case of surfaces with cone singularities. Let us fix a closed surface $S$ of genus $g$ and a finite set of points $\mathfrak{p} = \{p_1, \ldots, p_k\}$ on $S$. Finally fix $\theta_1, \ldots \theta_k \in (0, 2\pi)$. Recall by [Tro91] that a singular metric on $S$ with cone angles $\theta_i$ at $p_i$ is a smooth metric $h$ on $S \setminus \mathfrak{p}$ such that for any $i = 1 \ldots, k$. there is a conformal coordinate $z$ in a neighborhood $U_i$ of $p_i$ such that $h|_{U_i \setminus \{p_i\}} = |z|^{2\beta_i} e^{2\xi_i(z)} |dz|^2$, where $\beta_i = \frac{\theta_i}{2\pi} - 1 \in (-1, 0)$ and $\xi_i$ is a continuous function on $U_i$. We will denote by $\boldsymbol{\beta} = \sum \beta_i p_i$ the divisor associated with the metric $h$. We will always assume $\chi(S, \boldsymbol{\beta}) := \chi(S) + \sum \beta_i < 0$.

*Example* 6.1.1. The local model of a hyperbolic metric with cone singularity of angle $\theta_0$ is obtained by taking a wedge in $\mathbb{H}^2$ of angle $\theta_0$ and glueing its edges by a rotation. See Figure 6.1 below, in the Poincaré disc model.

More formally one can consider the universal cover $H$ of $\mathbb{H}^2 \setminus \{p\}$. Its isometry group is the universal cover of the stabilizer of $p$ in $\mathrm{Isom}(\mathbb{H}^2)$. Indeed we can consider on $H \cong (0, +\infty) \times \mathbb{R}$ the coordinates $(r, \theta)$ obtained by pulling back the polar

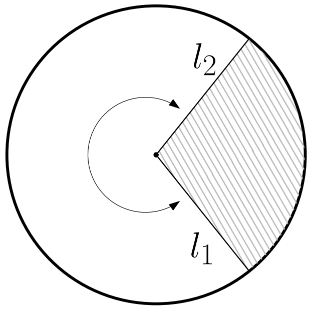



**Figure 6.1:** The model of a hyperbolic surface with a cone point. The wedge in $\mathbb{H}^2$ is the intersection of the half-planes bounded by two geodesic $l_1, l_2$. The edges are glued by a rotation fixing $l_1 \cap l_2$.

coordinates of $\mathbb{H}^2$ centered at $p$. If we take $p = (0, 0, 1)$ the projection map is simply

$$d(r, \theta) = (\sinh r \cos \theta, \sinh r \sin \theta, \cosh r) \ .$$

We have $d^*(h_{\mathbb{H}^2}) = dr^2 + \sinh^2(r)d\theta^2$. The isometry group of $H$ coincides with the group of horizontal translations $\tau_{\theta_0}(r, \theta) = (r, \theta_0 + \theta)$. For a fixed $\theta_0$, the completion $\mathbb{H}_{\theta_0}$ of the quotient of $H$ by the group generated by $\tau_{\theta_0}$ is the model of a hyperbolic surface with cone singularity $\theta_0$.

According to the definition given in [JMR11], *polar coordinates* on $\mathbb{H}_{\theta_0}$ are obtained by taking $r$ and $\phi = (2\pi/\theta_0)\theta \in [0, 2\pi]$ and the metric takes the form

$$dr^2 + \left(\frac{\theta_0}{2\pi}\right)^2 (\sinh r)^2 d\phi^2.$$

To construct a conformal coordinate on $\mathbb{H}_{\theta_0}$ it is convenient to consider the holomorphic covering of $\pi : H \to \mathbb{H}^2 \setminus \{p\}$. Taking the Poincaré model of the hyperbolic plane centered at $p$, we can realize $H$ as the upper half plane with projection $\pi(w) = \exp(iw)$. In this model the pull-back metric is simply

$$\pi^* \left( \frac{4|dz|^2}{(1 - |z|^2)} \right) = \frac{4e^{-2y}}{(1 - e^{-2y})^2} |dw|^2 \,,$$

and isometries are horizontal translations. It can be readily shown that the dependence of coordinates $(r, \theta)$ on the conformal coordinate $w = x + iy$ is of the form $r = \log \frac{1+e^{-y}}{1-e^{-y}}$, $\theta = x$. In particular the map $\tau_{\theta_0}$ in this model is still of the form $\tau_{\theta_0}(w) = w + \theta_0$.

Now we have a natural holomorphic projection $\pi_{\theta_0} : H \to \mathbb{D} \setminus \{0\}$ given by $\pi_{\theta_0}(w) = \exp(ikw)$ where $k = 2\pi/\theta_0$. The automorphism group of $\pi_{\theta_0}$ is generated by the translation $\tau_{\theta_0}$. So a conformal metric $h_{\theta_0}$ is induced on $\mathbb{D} \setminus \{0\}$ that makes $\mathbb{D}$ a model of $\mathbb{H}_{\theta_0}$. Putting $z = \exp(ikw)$ and $w = x + iy$ it turns out that $|z| = e^{-ky}$ so $e^{-y} = |z|^{1/k}$ and as $dz = ik \exp(ikw)dw$ one gets that

$$h_{\theta_0} = \frac{4}{k^2} \frac{|z|^{2(1/k-1)}}{(1 - |z|^{2/k})^2} |dz|^2 = 4(1 + \beta)^2 \frac{|z|^{2\beta}}{(1 - |z|^{2(1+\beta)})^2} |dz|^2 \,, \qquad (6.2)$$

where we have put $\beta = \theta_0/2\pi - 1$.

Analogously, one can construct the models of Euclidean and spherical singular points. In polar coordinates the flat cone metric takes the form

$$d\rho^2 + \left(\frac{\theta_0}{2\pi}\right)^2 \rho^2 d\phi^2 \ .$$

In the conformal coordinate the flat metric is simply $|z|^{2\beta}|dz|^2$.

We mainly consider the case where $h$ is a hyperbolic metric with cone singularities. In particular we denote by $\mathcal{H}(S, \boldsymbol{\beta})$ the set of singular hyperbolic metrics on $S$ with divisor $\boldsymbol{\beta}$. We will endow a neighborhood of a cone point of a hyperbolic surface with a Euclidean metric with the same cone singularity, that we call the *Klein Euclidean metric* associated with $h$. The construction goes as follows.

Take a point $p \in \mathbb{H}^2$ and consider the radial projection of $\mathbb{H}^2$ to the affine plane $P$ tangent to $\mathbb{H}^2$ at $p$,

$$\pi : \mathbb{H}^2 \to P \, .$$

The map $\pi$ is frequently used to construct the Klein model of $\mathbb{H}^2$. The pull-back of the Euclidean metric $g_P$ of $P$ to $\mathbb{H}^2$ is invariant by the whole stabilizer of $p$. Hence the pull-back of this metric on the universal cover $H$ of $\mathbb{H}^2 \setminus \{p\}$ is a Euclidean metric invariant by the isometry group of $H$. The latter metric thus projects to a Euclidean metric $g_K$ on $\mathbb{H}_{\theta_0}$, still having a cone singularity of angle $\theta_0$ at the cone point. We observe that $h$ and $g_K$ are bi-Lipschitz metrics in a neighborhood of the singular point.

*Remark* 6.1.2. On a surface with cone singularity, the metric $g_K$ is defined only in a regular neighborhood of a cone point.

One of the reasons we are interested in this metric is that there is a useful relation between the Hessian computed with respect to the metric $h$ and the Hessian computed with respect to the metric $g_K$.

**Lemma 6.1.3** (Lemma 2.8 of [BF14]). *Let $S$ be a hyperbolic surface with a cone point at $p$. Let $u$ be a function defined in a neighborhood of $p$ and $r$ the the hyperbolic distance from $p$. Consider the function $\bar{u} = (\cosh r)^{-1}u$, then*

$$D^2\bar{u}(\cdot, \cdot) = (\cosh r)^{-1} h((\operatorname{Hess} u - u\, E)\cdot, \cdot) \, , \tag{6.3}$$

*where $D^2\bar{u}$ is the Euclidean Hessian of $\bar{u}$ for the metric $g_K$, considered as a bilinear form.*

The computation in [BF14] was done only locally in $\mathbb{H}^2$, but as the result is of local nature it works also in the neighborhood of a cone point.

### 6.1.1 Codazzi tensors on a hyperbolic surface with cone singularities

A Codazzi operator on the singular hyperbolic surface $(S, h)$ is a smooth self-adjoint operator $b$ on $S \setminus \mathfrak{p}$ which solves the Codazzi equation $d_h^\nabla b = 0$. It is often convenient to require some regularity of $b$ around the singularity.

We basically consider two classes of regularity. We say $b$ is bounded if the eigenvalues of $b$ are uniformly bounded on $S$, and denote by $\mathcal{C}_\infty(S,h)$ the space of bounded Codazzi operators. We say that $b$ is of class $L^2$ (with respect to $h$) if

$$\int_S \mathrm{tr}(b^2)\omega_h < +\infty\,,$$

where $\omega_h$ is the area form of the singular metric. We denote by $\mathcal{C}_2(S,h)$ the space of $L^2$-Codazzi tensors. Since $\mathrm{tr}(b^2)=||b||^2$, $b$ is in $\mathcal{C}_2(S,h)$ if and only if $||b||\in L^2(S,h)$. Notice that if $u\in \mathrm{C}^\infty(S\backslash\mathfrak{p})$, then the operator $b=\mathrm{Hess}\,u-u\,E$ is a Codazzi operator of the singular surface. We have that $b\in\mathcal{C}_2(S,h)$ if $u$ and $||\mathrm{Hess}u||$ are in $L^2(S,h)$.

As in the general case, given a holomorphic quadratic differential $q$ on $S\setminus\mathfrak{p}$, the self-adjoint operator $b_q$ defined by $\Re q(u,v)=h(b_q(v),w)$ is a traceless Codazzi operator of the singular surface $(S,h)$. The regularity of $b_q$ close to the singular points can be easily understood in terms of the singularity of $q$. In particular we have

**Proposition 6.1.4.** *The Codazzi operator $b_q$ is in $\mathcal{C}_2(S,h)$ if and only if $q$ has at worst simple poles at the singularities. Moreover if $\theta_i\leq\pi$ then $b_q$ is bounded around $p_i$, and if $\theta_i<\pi$ then $b_q$ continuously extends at $p_i$.*

For the sake of completeness we prove an elementary Lemma, that will be used in the proof of Proposition 6.1.4.

**Lemma 6.1.5.** *Let $D$ be a disc in $\mathbb{C}$ centered at $0$ and $f$ be a holomorphic function on $D\setminus\{0\}$. If $|z|^a|f|^p$ is integrable for some $a\in(0,2)$ and $p\in[1,2]$, then $f$ has at worst a pole of order $3$ at $0$. If moreover $2p-a\geq 2$, then the pole at $0$ is at most simple.*

*Proof.* First we notice that with our assumption $\hat f(z)=z^2f(z)\in L^1(D,|dz|^2)$. As the function $|\hat f(z)|$ is subharmonic, the value at a point $z$ of $|\hat f|$ is estimated from above by the mean value of $|\hat f|$ on the ball centered at $z$ with radius $|z|$

$$|\hat f(z)|\leq \frac{1}{2\pi i|z|^2}\int_{B(z,|z|)}|\hat f(\zeta)|d\zeta\wedge d\bar\zeta\;.$$

As the integral is estimated by the norm $L^1$ of $\hat f$ we get that $|z|^2|\hat f(z)|\leq C$. Thus $\hat f$ has a pole of order at worst 2. As $1/z\in L^1(D,|dz|^2)$, whereas $1/z^2\notin L^1(D,|dz|^2)$, the pole in 0 cannot be of order 2.

This implies that $f$ has at most a pole of order 3. Suppose $f$ has a pole of order 2 or 3, then close to 0 we have $|f(z)|>C|z|^{-2}$ where $C$ is some positive constant. Thus $|z|^a|f(z)|^p\geq C^p|z|^{a-2p}$, that implies $2p-a<2$. □

*Proof of Proposition 6.1.4.* The problem is local around the punctures. Let us fix a conformal coordinate $z$ in a neighborhood of a puncture $p_i$, so that the metric takes the form $h=e^{2\xi}|z|^{2\beta}|dz|^2$ where $\xi$ is a bounded function, whereas $q=f(z)dz^2$ where $f$ is a holomorphic function on the punctured disc $\{z\,|\,0<|z|<\epsilon\}$. Using the expression (5.19) for the operator $b_q$ in real coordinates, we get

$$||b_q||^2=\mathrm{tr}(b_q^2)=2e^{-4\xi}|z|^{-4\beta}|f|^2\,.$$

On the other hand the area form is $\omega_h = e^{2\xi}|z|^{2\beta}dx\wedge dy$, so the problem is reduced to the integrability of the function $|z|^{-2\beta}|f(z)|^2$. As $-2\beta\in(0,2)$, Lemma 6.1.5 implies that this happens if and only if $f$ has at worst a simple pole at 0.

The same computation shows that $||b_q||^2(z) < C|z|^{-4\beta-2}$. In particular if $\beta\in(-1,-1/2]$ (that is if the cone angle is $\theta\leq\pi$) the operator $b_q$ is bounded around $p_i$, whereas if $\beta<-1/2$ the operators continuously extends at $p_i$ with $b_q(p_i)=0$. □

We want to prove now that $\delta b$ is a continuous function of $b$ with respect to the $L^2$-distance.

**Proposition 6.1.6.** *The map $\delta : \mathcal{C}_2(S,h)\to H^1_{hol}(\pi_1(S),\mathbb{R}^{2,1})$ is continuous for the $L^2$-distance on $\mathcal{C}_2(S,h)$.*

We premise to the proof an elementary lemma.

**Lemma 6.1.7.** *Let $f_n$ be a sequence of smooth functions defined on a planar disc $U$ of radius $R$ such that*

- *$f_n(0)\to 0$ and $df_n(0)\to 0$ as $n\to+\infty$;*
- *$||D^2 f_n||_{L^2(U)}\to 0$.*

*then $df_n\to 0$ and $f_n\to 0$ in $L^2(U)$.*

*Proof.* By Taylor expansion along the path $\gamma(t)=ty$, we write

$$df_n(y) = df_n(0) + \int_0^1 D^2(f_n)_{ty}(y,\cdot)dt\,,$$

so by applying Cauchy-Schwartz inequality we deduce that

$$||df_n(y)-df_n(0)||^2 < C\int_0^1 ||D^2(f_n)_{ty}||^2dt\,,$$

where $C$ is a constant depending on $R$. By integrating, using Fubini-Tonelli Theorem we get

$$\int_U ||df_n(y)-df_n(0)||^2 < C||D^2 f_n||^2_{L^2(U)}\,.$$

Using that $df_n(0)\to 0$ we conclude that $df_n\to 0$ in $L^2(U)$. A similar computation shows that $f_n\to 0$ in $L^2(U)$. □

*Proof of Proposition 6.1.6.* Let $b_n$ be a sequence of Codazzi operators on $(S,h)$ such that $||b_n||_{L^2(h)}\to 0$. We have to prove that $\delta b_n\to 0$.

Lifting $b_n$ on the universal covering we have a family of functions $u_n:\tilde{S}\to\mathbb{R}$ such that

$$\tilde{b}_n = \operatorname{Hess} u_n - u_n\,E\,,$$

and $u_n(x)-u_n(\alpha^{-1}x) = \langle \operatorname{dev}(x), \delta(b_n)_\alpha\rangle$ for every $\alpha\in\pi_1(S)$. We claim that we can choose $u_n$ such that $u_n(x)\to 0$ almost everywhere on $\tilde{S}$. The claim immediately implies that $\delta(b_n)_\alpha\to 0$ for every $\alpha\in\pi_1(S)$.

In order to prove the claim we fix a point $x_0 \in \tilde{S}$ and normalize $u_n$ so that $u_n(x_0) = 0$ and $du_n(x_0) = 0$ (this is always possible by adding some linear function to $u_n$).

Consider now the Klein Euclidean metric $g_K$ on $\tilde{S}$ obtained by composing the developing map with the projection of $\mathbb{H}^2$ to the tangent space $T_{\mathrm{dev}(x_0)}\mathbb{H}^2$. (Notice that covering transformations are not isometries for $g_K$.) Take the function $\bar{u}_n = (\cosh r)^{-1}u_n$ where $r(x)$ is the hyperbolic distance between $\mathrm{dev}(x)$ and $\mathrm{dev}(x_0)$. Let $U$ be any bounded subset of $\tilde{S}$. Combining the hypothesis on $b_n$ and Lemma 6.1.3, we have that $||D^2\bar{u}_n||_h \to 0$ in $L^2(U,h)$. As $h$ and $g_K$ are bi-Lipschitz over $U$ we have that $||D^2\bar{u}_n||_{L^2(U,g_K)} \to 0$.

By Lemma 6.1.7 there is a neighborhood $U_0$ of $x_0$ where $\bar{u}_n \to 0$ and $d\bar{u}_n \to 0$. So the set

$$\Omega = \{x \in \tilde{S} : \exists U \text{ neighborhood of } x \text{ such that } ||\bar{u}_n||_{L^2(U,g_K)} \to 0, ||d\bar{u}_n||_{L^2(U,g_K)} \to 0\}$$

is open and non-empty. Again Lemma 6.1.7 implies that this set is closed so $\Omega = \tilde{S}$. The claim easily follows. □

We are now ready to prove that the decomposition of Codazzi tensors for closed surfaces described in Proposition 5.1.5 holds for singular surfaces, when the correct behavior around the singularity is considered. For a singular metric $h$, we need to introduce the Sobolev spaces $W^{k,p}(h)$ of functions in $L^p(h)$ whose distributional derivatives up to order $k$ (computed with the Levi-Civita connection of $h$) lie in $L^p(h)$.

**Proposition 6.1.8.** *Given $b \in \mathcal{C}_2(S,h)$, a holomorphic quadratic differential $q$ and a function $u$ are uniquely determined so that*

- $b = b_q + \mathrm{Hess}\, u - u\, E$;
- *$q$ has at worst simple poles at $\mathfrak{p}$;*
- $u \in C^\infty(S \setminus \mathfrak{p}) \cap W^{2,2}(h)$.

*Such decomposition is orthogonal, in the sense that*

$$||b||^2_{L^2} = ||b_q||^2_{L^2} + ||\mathrm{Hess}u - u\mathrm{I}||^2_{L^2} \ .$$

*Moreover, if $\theta_i < \pi$ and $b$ is bounded then $u \in W^{2,\infty}(h)$.*

Again we premise an elementary Lemma of Euclidean geometry.

**Lemma 6.1.9.** *Let $(U,g)$ be a closed disc equipped with a Euclidean metric with a cone angle at $p_0$. Let $f$ be a smooth function on $U \setminus \{p_0\}$ such that $||D^2 f||$ is in $L^2(U,g)$ (resp. $L^\infty(U,g)$). Then $f$ and $||\,\mathrm{grad}\, f||$ lie in $L^2(U,g)$ (resp. $L^\infty(U,g)$).*

*Proof.* Let us consider coordinates $r,\theta$ on $U$. We may suppose that $U$ coincides with the disc of radius 1. The Taylor expansion of $f$ along the geodesics $c(t) = (1-(1-r)t,\theta)$ is

$$f(r,\theta) = f(1,\theta) - (1-r)\langle \mathrm{grad}\, f(1,\theta), \partial_r\rangle + (1-r)^2\int_0^1 (1-t)D^2 f_{(1-(1-r)t,\theta)}(\partial_r,\partial_r)dt \ .$$

Now the function $\hat{f}(r,\theta) = f(1,\theta) - (1-r)\langle \operatorname{grad} f(1,\theta), \partial_r\rangle$ is bounded and we can estimate

$$(f(r,\theta) - \hat{f}(r,\theta))^2 < \int_0^1 ||D^2 f(tr,\theta)||^2 dt\,.$$

If $D^2 f$ is bounded this formula shows that $f \in L^\infty(U,g)$. By integrating the inequality above on $U$ we also see that if $||D^2 f|| \in L^2(U,g)$ then $f$ is in $L^2(U,g)$ as well.

Cutting $U$ along a radial geodesic we get a planar domain. We can find on this domain two parallel orthogonal unitary fields $e_1, e_2$ and basically one has to prove that $f_i = \langle \operatorname{grad} f, e_i\rangle \in L^2(U,g)$ (resp. $L^\infty(U,g)$). This can be shown by a simple Taylor expansion as above noticing that $\operatorname{grad} f_i = (D^2 f)e_i$, and so $||\operatorname{grad} f_i|| \in L^2(U,g)$ (resp. $L^\infty(U,g)$). $\square$

*Proof of Proposition 6.1.8.* The proof is split in three steps:

Step 1 We prove that if $\operatorname{Hess} u - u\,E$ is in $\mathcal{C}_2(S,h)$ (resp. $\mathcal{C}_\infty(S,h)$) then $u \in W^{2,2}(S,h)$ (resp. $W^{2,\infty}(S,h)$).

Step 2 Denote by $\mathcal{C}_{\mathrm{tr}}(S,h)$ the space of trivial Codazzi tensors in $\mathcal{C}_2(S,h)$. We prove that $\mathcal{C}_{\mathrm{tr}}(S,h)^\perp$ coincides with the space of traceless Codazzi tensors.

Step 3 We prove that the orthogonal splitting holds

$$\mathcal{C}_2(S,h) = \mathcal{C}_{\mathrm{tr}}(S,h) \oplus \mathcal{C}_{\mathrm{tr}}(S,h)^\perp\ .$$

(This is not completely obvious as $\mathcal{C}_2(S,h)$ is not complete.)

To prove Step 1 first suppose $\operatorname{Hess} u - u\,E \in \mathcal{C}_2(S,h)$. We check that $u$ and $||du||_h$ lie in $L^2(S,h)$. Notice that we only need to prove integrability of $u^2$ and $||du||_h^2$ in a neighborhood $U$ of a puncture $p_i$. Consider the function $\bar{u} = (\cosh r)^{-1} u$, where $r$ is the distance from the puncture. By Lemma 6.1.3, the Hessian of $\bar{u}$ computed with respect to the Klein Euclidean metric $g_K$ is simply

$$D^2\bar{u}(\cdot,\cdot) = (\cosh r)^{-1} h((\operatorname{Hess} u - u\,E)\cdot,\cdot)\,,$$

so we see that $||D^2\bar{u}||_{g_K}$ is in $L^2(U,g_K)$. By Lemma 6.1.9 we conclude that $u$ and $||du||_{g_K}$ lie in $L^2(U,g_K)$. As $g_K$ and $h$ are bi-Lipschitz we conclude that $u$ and $||du||_h$ are in $L^2(U,h)$. The proof of Step 1 easily follows by noticing that $\operatorname{Hess} u = (\operatorname{Hess} u - u\,E) + u\,E$.

The case where $\operatorname{Hess} u - u\,E$ is in $\mathcal{C}_\infty(S,h)$ can be proved adapting the argument above in a completely obvious way.

Let us prove Step 2. Imposing the orthogonality with the trivial Codazzi tensor corresponding to $u = 1$ we see that $\mathcal{C}_{\mathrm{tr}}(S,h)^\perp$ is contained in the space of traceless Codazzi tensors.

To prove the reverse inclusion it suffices to show that

$$\int_S \operatorname{tr}(b_q(\operatorname{Hess} u - u\,E))\omega_h = 0\,.$$

for any holomorphic quadratic differential $q$ with at worst simple poles at the punctures and for any $u \in W^{2,2}(h)$.

As $b_q$ is Codazzi, $b_q\mathrm{Hess}u = \nabla(b_q \operatorname{grad} u) - \nabla_{\operatorname{grad} u} b_q$. Using that $\mathrm{tr}(b_q) = 0$, we get

$$\int_S \mathrm{tr}(b_q(\mathrm{Hess}\, u - u\, E))\omega_h = \int_S \mathrm{tr}(b_q \mathrm{Hess} u)\omega_h = \int_S \mathrm{div}(b_q \operatorname{grad} u)\omega_h\,.$$

So if $B_r$ is a neighborhood of the singular locus formed by discs of radius $r$ we have

$$\int_S \mathrm{tr}(b_q(\mathrm{Hess}\, u - u\, E))\omega_h = \lim_{r\to 0}\int_{\partial B_r} h(b_q \operatorname{grad} u, \nu)d\ell_r\,, \tag{6.4}$$

where $d\ell_r$ is the length measure element of the boundary whereas $\nu$ is the normal to the boundary pointing inside $B_r$. Now we claim that there exists a sequence $r_n \to 0$ such that

$$\int_{\partial B_{r_n}} ||b_q \operatorname{grad} u||_h d\ell_{r_n} \to 0$$

as $n \to +\infty$. Using this sequence in Equation (6.4) we get the result.

In order to prove the claim, notice that as $u \in W^{2,2}(h)$, $\operatorname{grad} u \in W^{1,2}(h)$. By Sobolev embedding, $||\operatorname{grad} u||_h \in L^p(h)$ for any $p < +\infty$. On the other hand as in the proof of Proposition 6.1.4, $||b_q|| \sim |z|^{-2\beta_i}|f|$, where $z$ is a conformal coordinate on $U$ with $z(p_i) = 0$ and we are putting $q(z) = f(z)dz^2$. As the area element of $h$ is $\sim |z|^{2\beta_i}dxdy$, using that $|f| < C|z|^{-1}$ we see that $||b_q||_h$ lies in $L^{2+\epsilon}(U,h)$ for $\epsilon < 2|\beta_i|$ for $i = 1, \dots k$.
So by a standard use of Hölder estimates we get $f = ||b_q \operatorname{grad} u||_h$ lies in $L^2(S,h)$. Now suppose that there exists a singular point $p$ and some number $a$ such that

$$\int_{\partial B_r} f d\ell_r > a$$

for any $r \in (0, r_0)$. By Schwarz inequality we get

$$a < \ell(\partial B_r)^{1/2}\left(\int_{\partial B_r} f^2\right)^{1/2}.$$

As $\ell(\partial B_r) < Cr$ for some constant $C$, we have

$$\int_{\partial B_r} f^2 > \frac{a^2}{Cr}\,.$$

Integrating this inequality and using Tonelli Theorem we obtain

$$||f||^2_{L^2} > \int_0^{r_0} dr \int_{\partial B_r} f^2 > \frac{a^2}{C}\int_0^{r_0}\frac{dr}{r}\,.$$

which gives a contradiction.

Finally we prove Step 3. Considering the completion of $\mathcal{C}_2(S,h)$ we have an orthogonal decomposition

$$\overline{\mathcal{C}_2(S,h)} = \overline{\mathcal{C}_{\mathrm{tr}}(S,h)} \oplus \overline{\mathcal{C}_{\mathrm{tr}}(S,h)^{\perp}}\,,$$

where we used a bar to denote the completed space. By Step 2 and Proposition 6.1.4 the subspace $\mathcal{C}_{\mathrm{tr}}(S,h)^\perp$ is finite dimensional, so it coincides with its completion. This implies that the splitting above induces a splitting

$$\mathcal{C}_2(S,h) = (\overline{\mathcal{C}_{\mathrm{tr}}(S,h)} \cap \mathcal{C}_2(S,h)) \oplus \mathcal{C}_{\mathrm{tr}}(S,h)^\perp .$$

By Proposition 6.1.6 we notice that $\mathcal{C}_{\mathrm{tr}}(S,h) = \delta^{-1}(0)$ is closed in $\mathcal{C}_2(S,h)$, so the first addend is actually $\mathcal{C}_{\mathrm{tr}}(S,h)$, and the proof is complete. □

Now we want to show that if $b \in \mathcal{C}_2(S,h)$ then the cocycle $\delta b$ is trivial around all the punctures, that means that for every peripheral loop $\alpha$ there exists $t_0 \in \mathbb{R}^{2,1}$ such that $(\delta b)_\alpha = \mathrm{hol}(\alpha)t_0 - t_0$.

*Remark* 6.1.10. As $\mathrm{hol}(\alpha)$ is an elliptic transformation, proving that $\delta b$ is trivial around $\alpha$ is equivalent to proving that the vector $(\delta b)_\alpha$ is orthogonal to the axis of $\mathrm{hol}(\alpha)$. The importance of this condition will be made clear in next subsection. Basically, it corresponds to being the translation part of the holonomy of a MGHF manifold with particles. Moreover, it can be checked that a cocycle trivial around the punctures, if regarded as an element in $H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1))$ by means of the isomorphism $\Lambda$, corresponds to a first-order deformation of hyperbolic metrics which preserves the cone singularity. This clarifies how Theorem 5.B can be extended to the case of manifolds with particles, which is the aim of Section 6.2.

By Proposition 6.1.8, in order to prove that $\delta b$ is trivial around the punctures, it is sufficient to consider the case $b = b_q$ where $q$ is a holomorphic quadratic differential with at most a simple pole at singular points.

**Lemma 6.1.11.** *Let $U$ be a neighborhood of a cone point $p$ in a hyperbolic surface of angle $\theta_0 \in (0, +\infty)$ and let $b$ a Codazzi operator on $U$ such that $||b(x)|| < C_0 r(x)^\alpha$ where $r(x)$ is the distance from the cone point and $\alpha$ is some fixed number bigger than $-2$. Then $\delta b$ is trivial.*

*Moreover, there exists a function $u \in W^{1,2}(h) \cap C^{0,\alpha}(U)$, smooth over $U \setminus \{p\}$, such that $b = \mathrm{Hess} u - u\mathrm{I}$. If $\alpha > -1$, then $u$ is Lipschitz continuous around $p$.*

*Proof.* As in Example 6.1.1, let $H$ be the universal cover of $\mathbb{H}^2 \setminus \{x_0\}$, and $(r,\theta)$ be global coordinates on $H$ obtained by pulling back the polar coordinates on $\mathbb{H}^2$ centered at $x_0$. We can assume $x_0 = (0,0,1)$. The cover $d : H \to \mathbb{H}^2$ is then of the form

$$d(r,\theta) = (\sinh r \cos\theta, \sinh r \sin\theta, \cosh r) .$$

Up to shrinking $U$, we may suppose that $U$ is the quotient of the region $\tilde{U} = \{(r,\theta) \,|\, r < r_0\}$ by the isometry $\tau_{\theta_0}(r,\theta) = (r, \theta + \theta_0)$.

If $\tilde{b}$ is the lifting of $U$, there is some function $u$ on $\tilde{U}$ such that $\tilde{b} = \mathrm{Hess} u - u\mathrm{I}$. Moreover, $t = (\delta b)_\alpha$ is such that

$$(u - u \circ \tau_{\theta_0}^{-1})(r,\theta) = u(r,\theta) - u(r, \theta - \theta_0) = \langle d(r,\theta), t\rangle .$$

Integrating $du$ on the path $c_r(\theta) = (r,\theta)$ with $\theta \in [0,\theta_0]$ we get

$$\int_{c_r} du = \cosh(r)\langle x_0, t\rangle + O(r) .$$

So, in order to conclude it is sufficient to prove that

$$\int_{c_r} du \to 0 \quad \text{as } r \to 0 \,. \tag{6.5}$$

To this aim we consider the Klein Euclidean metric $g_K$ on $U$ introduced in Subsection 6.1. Notice that $g_K$ is equivalent to the hyperbolic metric $h$ in $U$. In particular if $\rho$ is the Euclidean distance from $p$, we have that $\rho \sim r$. Let $\bar{u} = (\cosh r)^{-1}u$. By Lemma 6.1.3 we have $D^2\bar{u}(\bullet,\bullet) = (\cosh r)^{-1}h((\mathrm{Hess}u - u\mathrm{I})\bullet,\bullet)$ as bilinear forms, so we get $||D^2\bar{u}||_{g_K} \sim \rho^\alpha$ on $\tilde{U}$. A simple integration on vertical lines shows that $||d\bar{u}||_{g_K}(r,\theta) \leq C_0 + C_1\rho^{\alpha+1} < C_0 + C_2 r^{\alpha+1}$ for any $\theta \in [0,\theta_0]$ and $r \in [0,r_0]$, where $C_0$, $C_1$ and $C_2$ are constant depending on $\alpha$, $r_0$, $\sup_{0\leq\theta\leq\theta_0}||du||(r_0,\theta)$.

In particular $|\int_{c_r} d\bar{u}| \leq C_3(1 + r^{\alpha+1})\ell_{g_K}(c_r) \leq C_4(1 + r^{\alpha+1})r$. So if $\alpha > -2$ this integral goes to 0 as $r \to 0$. As $\int_{c_r} du = \cosh r \int_{c_r} d\bar{u}$, (6.5) follows.

We conclude that $\langle x_0, t\rangle = 0$, or equivalently $t$ can be decomposed as $t = \mathrm{hol}(\alpha)t_0 - t_0$ for some vector $t_0 \in \mathbb{R}^{2,1}$, and up to adding the linear function $f(r,\theta) = \langle d(r,\theta), t_0\rangle$, we may suppose that $u$ is $\tau_{\theta_0}$-periodic. Hence $u$ projects to a function on $U$, that with some abuse we still denote $u$, such that $b = \mathrm{Hess}\, u - u\,E$.

By the estimate on $d\bar{u}$ we also deduce that $\bar{u}$ is uniformly continuous around the singular point. More precisely the following estimate holds:

$$\begin{aligned}
|\bar{u}(r_1,\theta_1) - \bar{u}(r_2,\theta_2)| &\leq |\bar{u}(r_1,\theta_1) - \bar{u}(r_1,\theta_2)| + |\bar{u}(r_1,\theta_2) - \bar{u}(r_2,\theta_2)| \\
&\leq C_3(1 + r^{\alpha+1})r|\theta_1 - \theta_2| + C_0|r_1 - r_2| + \frac{C_2}{\alpha+2}|r_1^{\alpha+2} - r_2^{\alpha+2}| \,.
\end{aligned}$$

So $\bar{u}$ extends to a continuous function on $U$ and the same holds for $u$.

Writing $u = \cosh(r)\bar{u}$ we get $du = \sinh(r)\bar{u}dr + \cosh(r)d\bar{u}$. As $\bar{u}$ is bounded it results that $||du|| < C_5 + C_6 r^{\alpha+1}$ and this estimate shows that $||du|| \in L^2(U,h)$ and $u$ is Lipschitz if $\alpha > -1$. □

**Proposition 6.1.12.** *Let $(U,h)$ be a disc with a hyperbolic metric with a cone singularity of angle $\theta_0 \in (0,2\pi)$ at $p$. Let $q$ be a holomorphic quadratic differential with at most a simple pole in $p$. Then $\delta b_q$ is trivial. Moreover there exists a Lipschitz function $u$ over $U$ that is smooth over $U \setminus \{p\}$ such that $b = \mathrm{Hess}\, u - u\,E$.*

*Proof.* If $z$ is a conformal coordinate on $U$ with $z(p) = 0$ and $r$ is the distance from the singular point, we know that $r \sim |z|^{\beta+1}$. On the other hand if $q = f(z)dz^2$ and $h = e^{2\xi}|z|^{2\beta}|dz|^2$ we have $||b_q||^2 = e^{-4\xi}z^{-4\beta}|f(z)|^2$, so by the assumption $||b_q||^2 < C|z|^{2(-1-2\beta)}$. In particular $||b_q|| < Cr^\alpha$ with $\alpha = \frac{-1-2\beta}{1+\beta}$. As $\alpha > -1$ for any $\beta \in (-1,0)$ we can apply Lemma 6.1.11 and conclude. □

*Remark* 6.1.13. If cone angles are in bigger than $2\pi$ (but different from integer multiples of $2\pi$) the same argument shows that $b_q$ is trivial around the puncture as well. The main difference is that the exponent $\alpha$ lies in $(-2,-1]$ so Lemma 6.1.11 ensures that the function $u$ is Hölder continuous at the puncture and $du$ is only $L^2$-integrable over $U \setminus \{p\}$.

A simple corollary of Proposition 6.1.12 is that if $q$ is a quadratic differential on $(S,h)$ with at most simple poles at the punctures then the cohomology class of $\delta b_q$ can be expressed as $\delta(b)$ for some operator $b \in \mathcal{C}_\infty(S,h)$

**Corollary 6.1.14.** *Let $q$ be a quadratic differential on $S$ with at most simple poles at punctures. There exists a Lipschitz function $u$ on $S$, smooth on $S \setminus \mathfrak{p}$ such that $b = b_q - (\operatorname{Hess} u - u\,E)$ is bounded. Moreover $u$ can be chosen so that $b$ is uniformly positive definite, i.e. $b \geq a\mathrm{I}$ for some $a > 0$.*

*Proof.* By Proposition 6.1.12 around each puncture $p_i$ there exists a function $u_i$ such that $b_q = \operatorname{Hess} u_i - u_i\,E$. By a partition of the unity it is possible to construct a smooth function $u$ such that $u$ coincides with $u_i$ is some smaller neighborhood of $p_i$. In particular the support of $b' = b_q - \operatorname{Hess} u - u\,E$ is compact in $S \setminus \mathfrak{p}$, so $b'$ is bounded.

In order to get $b$ uniformly positive it is sufficient to consider the constant function $v = ||b'||_\infty + a$. Then $b = b' - \operatorname{Hess} v + v\,E = b_q - \operatorname{Hess}(u+v) + (u+v)E$ is uniformly positive since $b' - \operatorname{Hess} v + v\,E = b' + v\,E > a\,E$. □

### 6.1.2 Flat Lorentzian spacetimes with particles

We now consider maximal globally hyperbolic flat manifolds with cone singularities along timelike lines.

**Definition 6.1.15.** We say that a Lorentzian spacetime $M$ has cone singularities along timelike lines (which we call also particles) if there is a collection of lines $s_1, \ldots, s_k$ such that $M^* = M \setminus (s_1 \cup \ldots \cup s_k)$ is endowed with a flat Lorentzian metric and each $s_i$ has a neighborhood isometric to a slice in $\mathbb{R}^{2,1}$ of angle $\theta_i < 2\pi$ around a timelike geodesic, whose edges are glued by a rotation around this timelike geodesic.

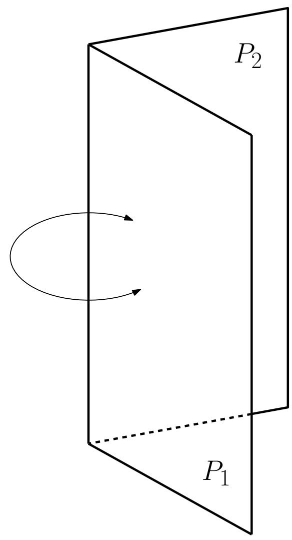



**Figure 6.2:** The model of a flat particle. The slice in $\mathbb{R}^{2,1}$ is the intersection of the half-spaces bounded by two timelike planes $P_1, P_2$. The edges of the slice are glued by a rotation fixing $P_1 \cap P_2$.

By definition it is immediate that in a neighborhood of a point $p_i \in s_i$ there are coordinates $(z, t) \in D \times \mathbb{R}$, for $D$ a disc, such that $s_i$ corresponds to the locus

$\{z = 0\}$ and the metric $g$ takes the form

$$g = |z|^{2\beta_i}|dz|^2 - dt^2, \tag{6.6}$$

where $\beta_i = \frac{\theta_i}{2\pi} - 1$. Notice that the restriction of the metric $g$ to the slices $\{t = const\}$ are isometric Euclidean metrics with cone singularities $\theta_i$.

By definition the holonomy of a loop surrounding a particle of angle $\theta$ is conjugated with a pure rotation of angle $\theta$, so the translation part of the holonomy is orthogonal to the axis of rotation.

A closed embedded surface $S \subset M$ is space-like if $S \subset M^*$ is space-like and for any point $p_0 \in S \cap s_i$, in the coordinates $(z, t)$ defined in a neighborhood of $p_0$, $S$ is the graph of a function $f(z)$. We will say that the surface is orthogonal to the singular locus at $p_0$ if $|f(z) - f(z_0)| = O(r^2)$ where $r$ is the intrinsic distance on $D$ from the puncture. As $r = \frac{1}{\beta+1}|z|^{\beta+1}$ this condition is in general not equivalent to requiring that the differential of $f$ at $z_0$ (computed with respect to the coordinates $x, y$) vanishes.

If $S$ is a space-like surface in a flat spacetime with cone singularity, then the intersection of $S$ with the singular locus is a discrete set and a Riemannian metric $I$ is defined on $S^* = S \setminus (s_1 \cup \ldots \cup s_n)$. Clearly the first fundamental form and the shape operator on $S^*$, say $(I, B)$, satisfy the Gauss-Codazzi equation.

A notion of globally hyperbolic extension of $S$ makes sense even in this singular case and the existence and uniqueness of the maximal extension can be proved by adapting *verbatim* the argument given for the Anti de-Sitter case in [BBS11].

Like the previous section, we fix a topological surface $S$ and $k$ points $\mathfrak{p} = \{p_1 \ldots, p_k\}$ and study pairs $(I, B)$ on $S \setminus \mathfrak{p}$ corresponding to embedding data of a Cauchy surface $S$ in a globally hyperbolic spacetime with particles of cone angles $\theta_1, \ldots, \theta_k$, so that the point $p_i$ lies on the particle with cone angle $\theta_i$. The divisor of $(I, B)$ is by definition $\boldsymbol{\beta} = \sum \beta_i p_i$, where we have put $\beta_i = \frac{\theta_i}{2\pi} - 1$. Again, we consider embeddings which are isotopic to $S \hookrightarrow S \times \{0\} \subset M \cong S \times \mathbb{R}$.

As in the closed case we consider uniformly convex surfaces, but we also consider an upper bound for the principal curvatures. More precisely we will assume that $B$ is a bounded and uniformly positive operator: that means that there exists a number $M$ such that $\frac{1}{M} E < B < M E$; in other words we require the eigenvalues of $B$ at every point to be bounded between $1/M$ and $M$.
We now want to show that the set

$$\mathbb{D}_{\boldsymbol{\beta}} = \left\{ (I, B) : \begin{array}{l} (I, B) \text{ embedding data of a closed uniformly convex Cauchy surface} \\ \text{in a flat Lorentzian manifold with particles of angles } \theta_1, \ldots, \theta_k \\ \text{orthogonal to singular locus, } B \text{ bounded and uniformly positive} \end{array} \right\}$$

is in bijection with

$$\mathbb{E}_{\boldsymbol{\beta}} = \left\{ (h, b) : \begin{array}{l} h \text{ hyperbolic metric on } S \text{ with cone singularities of angles } \theta_1, \ldots, \theta_k \\ b : TS^* \to TS^* \text{ self-adjoint, bounded and uniformly positive, } d_h^{\nabla} b = 0 \end{array} \right\}.$$

More precisely, we show the following relation.

**Theorem 6.A.** *The correspondence $(I, B) \to (h, b)$, where $h = I(B, B)$ and $b = B^{-1}$, induces a bijection between $\mathbb{E}_{\boldsymbol{\beta}}$ and $\mathbb{D}_{\boldsymbol{\beta}}$.*

This is a more precise version of Proposition 5.1.7, which additionally deals with the condition that $I$ and $h$ have cone singularities. The fact that the hyperbolic metric associated to $(I, B) \in \mathbb{D}_{\boldsymbol{\beta}}$ has a cone singularity at the singular points is a simple consequence of the fact that its completion is obtained by adding a point. In the opposite direction things are less clear and we will give a detailed proof of the fact that given $(h, b)$ as in the hypothesis of the theorem, the surface $S$ can be embedded in a singular flat spacetime with embedding data $(I, B)$.

We first prove two Lemmas which will be used in the proof of Theorem 6.A.

**Lemma 6.1.16.** *Let $g$ be a Euclidean metric on a disc $D$ with a cone point of angle $\theta_0 \in (0, 2\pi)$ at the point $0$. Suppose $u$ to be a $C^2$ function on the punctured disc $D^* = D \setminus \{0\}$ such that the Euclidean Hessian $\mathrm{Hess}_g u$ is bounded with respect to $g$, namely there exists a constant $a_1 > 0$ such that $\mathrm{Hess}_g u < a_1 E$. Then $||\,\mathrm{grad}_g\, u(x)|| \leq a_1 d_g(0, x)$ and $u$ extends at $0$.*

*If moreover, $\mathrm{Hess}_g u$ is uniformly positive, i.e. there is a constant $a_2$ such that $a_2 E < \mathrm{Hess}_g u$, then the metric $g'(v, w) = g(\mathrm{Hess}_g u(v), \mathrm{Hess}_g u(w))$ is a Euclidean metric on $D$ with a cone point of angle $\theta_0$ at $0$.*

*Proof.* Let $\tilde{D}^*$ be the universal cover of the punctured disc $D^*$, and let $\tilde{u}$ be the lifting of $u$ on $\tilde{D}^*$. Let dev be a developing map for the Euclidean metric on $D$; we can assume 0 is the fixed point of the holonomy of a path winding around 0 in $D$. Then we define the map $\varphi : \tilde{D}^* \to \mathbb{R}^2$ given by

$$\varphi(x) = (\mathrm{dev})_*(\mathrm{grad}_g\, \tilde{u}(x))\,,$$

where with a standard abuse we denote by $g$ also the lifting of the Euclidean metric to $\tilde{D}^*$.

By the hypothesis

$$||d\varphi(v)||_{\mathbb{R}^2} < a_1 ||v||_g$$

for any tangent vector $v$. In particular this estimate implies that $\varphi$ is $a_1$-Lipschitz, so it extends to the metric completion of $\tilde{D}^*$, which is composed by one point $\tilde{0}$ fixed by the covering transformations. As $\varphi$ conjugates the generator of the covering transformations $\tilde{D} \to D$ with the rotation of angle $\theta_0$ in $\mathbb{R}^2$, we must have $\varphi(\tilde{0}) = 0$, as $\varphi(\tilde{0})$ must be fixed by the action of the holonomy. This implies that

$$||\,\mathrm{grad}_g\, \tilde{u}(x)|| = ||\varphi(x)|| = d_{\mathbb{R}^2}(\varphi(x), \varphi(0)) \leq a_1 d_g(\tilde{0}, x)\,. \tag{6.7}$$

From the boundedness of $\mathrm{grad}\, u$ we also obtain that $u$ is Lipschitz hence $u$ extends with continuity to the metric completion $D$. This concludes the first part.

Suppose now that $\mathrm{Hess}_g u$ is uniformly positive. By construction, the pull-back $\varphi^* g_{\mathbb{R}^2} = g(\mathrm{Hess}_g \tilde{u}(\cdot), \mathrm{Hess}_g \tilde{u}(\cdot))$ is a Euclidean metric for which $\varphi$ is a developing map, and has the same holonomy as dev. We claim that $\varphi$ (suitably restricted if necessary to the lift of a smaller neighborhood of 0, which we still denote by $D$), is a covering on $U \setminus \{0\}$, where $U$ is a neighborhood of 0 in $\mathbb{R}^2$. This will show that $\varphi$ lifts to a homeomorphism $\tilde{\varphi}$ from $\tilde{D}^*$ and the universal cover of $U \setminus \{0\}$ conjugating the generator of $\pi_1(D^*)$ to an element of the isometry group $\widetilde{SO}(2)$ of this covering. Therefore $\tilde{\varphi}$ descends to an isometric homeomorphism between $D$ equipped with the metric $g'$ and a model of a Euclidean disc with a cone singularity.

To show the claim, observe that $\varphi$ is a local homeomorphism. Moreover, since $\mathrm{Hess}_g u$ is bounded by $a_2\mathrm{I}$ and $a_1\mathrm{I}$, we have

$$a_2||v||_g < ||d\varphi(v)||_{\mathbb{R}^2} < a_1|v||_g \tag{6.8}$$

for any tangent vector $v$.

We prove now that $\varphi(x) \neq 0$ for any $x \neq \tilde{0}$. Consider the geodesic path $\gamma : [0, t_0] \to \tilde{D}^*$ joining $x$ to $\tilde{0}$ parametrized by arc length: it is simply the lifting of a radial geodesic in $D^*$. Then we have that

$$g(\mathrm{grad}_g\, u(x),\, \dot{\gamma}(t_0)) - g(\mathrm{grad}_g\, u(\gamma(\epsilon)),\, \dot{\gamma}(\epsilon)) = \int_\epsilon^{t_0} g(\mathrm{Hess}_g(\dot{\gamma}(t)), \dot{\gamma}(t))dt\ .$$

Notice that $|g(\mathrm{grad}_g\, u(\gamma(\epsilon)), \dot{\gamma}(\epsilon))| \leq ||\,\mathrm{grad}_g\, u(\gamma(\epsilon))|| = ||\varphi(\gamma(\epsilon))||$. So, using that $\varphi(\gamma(\epsilon)) \to 0$ as $\epsilon \to 0$, we get

$$g(\mathrm{grad}_g\, u(x),\, \dot{\gamma}(t_0)) = \int_0^{t_0} g(\mathrm{Hess}_g(\dot{\gamma}(t)), \dot{\gamma}(t))dt\,,$$

and the integrand in the RHS is bigger than $a_2t_0$. We deduce that

$$||\varphi(x)|| = ||\,\mathrm{grad}_g\, u(x)|| \geq a_2t_0 = a_2d(x, \tilde{0})\ . \tag{6.9}$$

To conclude that $\varphi$ is a covering of $U \setminus \{0\}$ for some neighborhood of 0 in $\mathbb{R}^2$, it suffices to show that $\varphi$ has the path lifting property. Given any path $\gamma : [0, 1] \to U \setminus \{0\}$, take a small $\rho_0$ so that $\gamma$ is contained in $U \setminus B(0, \rho_0)$. Therefore by (6.7) a local lifting of $\gamma$ is contained in a region of $\tilde{D}^*$ uniformly away from $\tilde{0}$, hence metrically complete for the metric $g$. Equation (6.8) ensures that any local lifting of $\gamma$ has finite lenght and thus by standard arguments the lifting can be defined on the whole interval $[0, 1]$.

To complete the second point, we need to show that $g'$ has the same cone angle as $g$. By construction $g$ and $g'$ have the same holonomy, hence the cone angle can only differ by a multiple of $2\pi$. Consider the one-parameter family of functions $u_s : \tilde{D}^* \to \mathbb{R}$,

$$u_s(x) = su(x) + \frac{1}{2}(1 - s)d_g(0, x)^2.$$

The metrics $g_s(v, w) = g(\mathrm{Hess}_g u_s(v), \mathrm{Hess}_g u_s(w))$, constructed as above, form a one-parameter family of Euclidean metrics with cone singularities, depending smoothly on $s$. Moreover $g_0 = g$ and $g_1 = g'$. All the metrics $g_s$ have the same holonomy on a path around 0, for the same construction. Therefore, by discreteness of the possible cone angles $\{\theta_0 + 2k\pi\}$, all metrics $g_s$ must have the same cone angle $\theta_0$. □

**Lemma 6.1.17.** *Let $S$ be a surface embedded in a flat spacetime with particles. Suppose that in a cylindrical neighborhood $C = D \times (a, b)$ of a particle where the metric is of the form $g = |z|^{2\beta_i}|dz|^2 - dt^2$ as in (6.6), $S$ is the graph of a function $f : D \to (a, b)$. If the shape operator of $S$ satisfies $a_2E < s < a_1E$, then*

- *There are constants $A_1$ and $A_2$ such that $A_2 E < \mathrm{Hess}_g f < A_1 E$, where $g$ is the Euclidean singular metric on $D$.*
- $|f(x) - f(x_0)| = O(\rho^2)$ *where $\rho$ is the Euclidean distance from the singular point $x_0$.*

*Remark* 6.1.18. This lemma implies that any surface with shape operator bounded and uniformly positive is automatically orthogonal to the singular locus.

*Proof.* Let $S_C$ be the disc $S \cap C = \mathrm{graph}(f)$. Let dev be the developing map on the universal cover of $C$. We may assume that $p_0 = (0, 0, 1)$ is fixed by the holonomy of $C$, so $\mathrm{dev}(x, t) = \mathrm{dev}_0(x) + (0, 0, t)$, where $\mathrm{dev}_0$ is the developing map of the singular disc $(D, g)$.

Let $G : \tilde{S}_C \to \mathbb{H}^2$ be the Gauss map of the immersion $\mathrm{dev}|_{\tilde{S}_C}$. First we notice that $G$ is locally bi-Lipschitz by our hypothesis, and since the diameter of $S_C$ is bounded, then the image through $G$ of a fundamental domain of the covering $\tilde{S}_C \to S_C$ is contained in a hyperbolic ball $B(p_0, r_0)$. As the holonomy of $G$ is an elliptic group fixing $p_0$ we get that $G(\tilde{S}_C)$ is entirely contained in $B(p_0, r_0)$.

Now let $\pi : \mathbb{H}^2 \to \mathbb{R}^2$ be the radial projection $\pi(x, y, z) = (x/z, y/z)$. Notice that the restriction of $\pi$ over the disc $B(p_0, r_0)$ is a bi-Lipschitz diffeomorphism onto a Euclidean disc $B(p_0, \rho_0)$ where $\rho_0 = \tanh r_0$. Let $A = A(r_0)$ be the bi-Lipschitz constant of $\pi|_{B(p_0,r_0)}$.

Observe that, at a point $(x, f(x))$, the vector $\mathrm{grad}_g f(x) + \partial_t$ is a multiple of the normal vector of $S_C$. Hence the normal vector of the immersion $\mathrm{dev} : \tilde{S}_C \to \mathbb{R}^{2,1}$ at $(\tilde{x}, \tilde{f}(\tilde{x}))$ is parallel to the vector $(\mathrm{dev}_0)_*(\mathrm{grad}_g \tilde{f}(\tilde{x})) + (0, 0, 1)$. Then it is easy to check that $\pi(G(\tilde{x}, \tilde{f}(\tilde{x}))) = (\mathrm{dev}_0)_*(\mathrm{grad}_g \tilde{f}(\tilde{x}))$. So we get

$$g(\mathrm{Hess}_g \tilde{f}(v), \mathrm{Hess}_g \tilde{f}(v)) = \langle (\pi \circ G)_*(v'), (\pi \circ G)_*(v') \rangle ,$$

where $v' = v + df(v)\partial_t$. Using that $\pi$ is $A$-bi-Lipschitz and that $\langle G_*(v'), G_*(v') \rangle = I(sv', sv')$ we get on $D$

$$\frac{a_1^2}{A} I(v', v') < g(\mathrm{Hess}_g(f)v, \mathrm{Hess}_g(f)v) < A a_2^2 I(v', v') . \tag{6.10}$$

Now $I(v', v') = g(v, v) - (df(v))^2$ so it turns out that $g(\mathrm{Hess}_g(f)\cdot, \mathrm{Hess}_g(f)\cdot)$ is uniformly bounded. By the first part of Lemma 6.1.16 there is $a_0$ such that $||\,\mathrm{grad}_g f(x)|| < a_0\rho$, where $\rho$ is the distance from the singular point $x_0$. In particular, in a small neighborhood of the singular point we have $(1-\epsilon)g(v, v) \leq I(v', v')$, for a fixed small $\epsilon$. By (6.10) we conclude the proof of the first part of the lemma.

Finally, as $||\,\mathrm{grad}_g f(x)|| < a_0\rho$, a simple integration along the geodesic connecting $x$ to $x_0$ shows that $|f(x) - f(x_0)| < (a_0/2)\rho^2$. $\square$

*Proof of Theorem 6.A.* We start by showing that $I = h(b\cdot, b\cdot)$ and $B = b^{-1}$ define an embedding in a flat manifold with particles. The key point is to prove that a neighborhood $U$ of a singular point $p_0 \in \mathfrak{p}$ can be immersed as a graph in the model of the cone singularity with embedding data $(I, B)$. Once this has been proved, by a standard application of the uniqueness of the extension one sees that $S$ can be globally immersed in a spacetime with cone singularity as in Definition 6.1.15.

Let $D$ be a small disc centered at $p_0$ and let $\tilde{D}^*$ be the universal cover of the punctured disc $D^* = D \setminus \{p_0\}$. Suppose the cone angle at $p_0$ is $\theta_0$. Now we denote by $d : \tilde{D}^* \to \mathbb{H}^2$ the restriction of the developing map of $h$ to $\tilde{D}^*$; we can assume $(0,0,1)$ is the fixed point for the holonomy of a path winding around $p_0$. Consider the radial projection $\pi : \mathbb{H}^2 \to \mathbb{R}^2$ from hyperbolic plane to the horizontal plane $\{(x,y,z) \mid z = 1\}$ in $\mathbb{R}^{2,1}$, namely $\pi(x,y,z) = (x/z, y/z)$. Let $\mathrm{dev} = \pi \circ d$, which is a developing map for the Klein Euclidean metric $g_K$ on $D^*$ introduced in Subsection 6.1, which will be denoted simply by $g$ in the following.

Now let $u$ be a function on $D^*$ such that $\tilde{b} = \mathrm{Hess}_h u - u\,E$, that exists by Proposition 6.1.12. We consider the function $\bar{u}$ on $D^*$ as $\bar{u}(x) = u(x)/\cosh r$ where we have put $r = d_h(p_0, x)$. We know that $\tilde{\sigma} : \tilde{D}^* \to \mathbb{R}^{2,1}$ given by $\tilde{\sigma}(x) = d_*(\mathrm{grad}_h\, u(x)) - u(x)d(x)$ gives an immersion of $\tilde{D}^*$ with the required embedding data. Here with some abuse we denote by $u$ also the lifting of $u$ to the universal cover. By [BF14, Lemma 2.8] it turns out that the orthogonal projection of $\tilde{\sigma}(x)$ onto the horizontal plane $\mathbb{R}^2 \subset \mathbb{R}^{2,1}$ (where again we are supposing that the vertical direction is fixed by the holonomy of a loop around $p_0$) is exactly

$$\varphi(x) = (\mathrm{dev})_*(\mathrm{grad}_g\, \bar{u}(x))\,,$$

Thus $\tilde{\sigma}(x) = \varphi(x) + f(x)(0,0,1)$, for some function $f : \tilde{D}^* \to \mathbb{R}$. Notice that the holonomy of $\sigma$ is simply the elliptic rotation around the vertical line of angle $\theta_0$, so it turns out that $f$ is invariant by the action of covering transformation of $\tilde{D}^*$ so it projects to a function still denoted by $f$ on $D^*$.

Now we claim that $\varphi$ is the developing map for a Euclidean structure with cone singularity $\theta_0$ over $D^*$. In fact by Lemma 6.1.3 we know that for $v, w \in T_x D^*$

$$g(\mathrm{Hess}_g \bar{u}(v), w) = \frac{1}{\cosh r} h(b(v), w)\,. \tag{6.11}$$

Therefore, $\mathrm{Hess}_g \bar{u}$ is bounded and uniformly positive. Indeed $b$ and the factor $1/\cosh r$ appearing in Equation (6.11), as well as the metric $h$ compared to $g$, are bounded on $D^*$. Applying Lemma 6.1.16, $g' = g(\mathrm{Hess}_g \hat{u}(\cdot), \mathrm{Hess}_g \hat{u}(\cdot))$ is a Euclidean metric with cone angle $\theta_0$ and $\varphi$ coincides with its developing map.

As a consequence, if we consider on $M = D^* \times \mathbb{R}$ the flat Lorentzian metric with particle $g' - dt^2$, its developing map is $\mathrm{Dev}(x,t) = \varphi(x) + t(0,0,1)$. In particular we have that $\mathrm{Dev}(x, f(x)) = \tilde{\sigma}(x)$. So we have shown that the map $\sigma(x) = (x, f(x))$ is an immersion of $D^*$ into the spacetime with particles $M$ with embedding data $(I, s)$. The fact that the immersion is orthogonal to the singular locus follows from Lemma 6.1.17.

We prove now the opposite implication showing that if $(I, B)$ are the embedding data of a Cauchy surfaces in a spacetime $M$ with singularities of angle $\theta_i$, then $h = I(B\bullet, B\bullet)$ is a hyperbolic metric with cone points of angle $\theta_i$.

We know $h = I(B\bullet, B\bullet)$ is a hyperbolic metric, with holonomy a rotation of angle $\theta_i$ around each cone point $p_i$ of $I$. Moreover $s$ is bounded and uniformly positive, hence $h$ admits a one-point completion as $I$ does, and this is sufficient to show that $h$ has cone singularity. As $I = h(b, b)$, by applying the first part of the proof to the pair $(h, b)$ one sees that the cone angles of $h$ coincide with the cone angles of $M$. $\square$

### 6.1.3 A Corollary about metrics with cone singularities

We write here a consequence of the previous discussion, which might be of interest independently of Lorentzian geometry.

**Theorem 6.B.** *Let $h$ be a hyperbolic metric with cone singularities and let $b$ be a Codazzi, self-adjoint operator for $h$, bounded and uniformly positive. Then $I = h(b\cdot, b\cdot)$ defines a singular metric with the same cone angles as $h$. Moreover if $I = e^{2\xi}|w|^{2\beta}|dw|^2$ in a conformal coordinate $w$ around a singular point $p$, the factor $\xi$ extends to a Hölder continuous function at $p$.*

To prove Theorem 6.B, we consider the uniformly convex surface constructed in Theorem 6.A, with first fundamental form $I$. We must show that this metric has cone singularities. For every puncture of $S$, consider the singular Euclidean metric on the disc $D$, provided by Lemma 6.1.16. We now call this metric $g$ instead of $g'$. Suppose the embedding in the chart $D^* \times \mathbb{R}$ with the metric $g - dt^2$ is the graph of a function $f : D^* \to \mathbb{R}$. Hence if $z$ is a conformal coordinate for the metric $g$, the metric $I$ can be written in this coordinate as

$$I = |z|^{2\beta}|dz|^2 - df \otimes df = |z|^{2\beta}\left(|dz|^2 - \frac{df \otimes df}{|z|^{2\beta}}\right),$$

where $g = |z|^{2\beta}|dz|^2$.

**Lemma 6.1.19.** *The coefficients of the metric*

$$I' = |dz|^2 - \frac{df \otimes df}{|z|^{2\beta}} = |z|^{-2\beta} I$$

*extend to Hölder functions defined on $D$.*

*Proof.* It suffices to show that the functions $|z|^{-\beta}\partial_x f$ and $|z|^{-\beta}\partial_y f$ are Hölder functions. We will give the proof for the first function, which we denote

$$F(z) = |z|^{-\beta}\frac{\partial f}{\partial x}.$$

We split the proof in three steps. Recall from Lemmas 6.1.17 and 6.1.16 that we have $||\mathrm{Hess}_g f|| \leq C$ and $||\,\mathrm{grad}_g f||_g \leq C\rho$ for some constant $C$, where $\rho = \frac{1}{1+\beta}|z|^{1+\beta}$ is the intrinsic Euclidean distance.

First, consider the path $\gamma_1(t) = \varrho e^{it}$ for $t \in [\phi_0, \phi_1]$, so that $|\gamma_1(t)| = \varrho$ is constant. We claim that

$$|F(\gamma_1(\phi_1)) - F(\gamma_1(\phi_0))| \leq C_1 \varrho^{\beta+1}|\phi_1 - \phi_0|$$

for some constant $C_1$. Consider

$$\begin{aligned} F(\gamma_1(\phi_1)) - F(\gamma_1(\phi_0)) &= \int_{\phi_0}^{\phi_1} \frac{d(F \circ \gamma_1(t))}{dt} dt \\ &= \int_{\phi_0}^{\phi_1} \varrho^{-\beta}\left(g\left(\mathrm{Hess}_g f(\dot\gamma_1), \frac{\partial}{\partial x}\right) + df\left(\nabla_{\dot\gamma_1}\frac{\partial}{\partial x}\right)\right). \end{aligned} \tag{6.12}$$

Now we have

$$|g\left(\mathrm{Hess}_g f(\dot\gamma_1),\partial_x\right)| \le C||\dot\gamma_1||_g||\partial_x||_g = C\varrho^{2\beta+1}. \tag{6.13}$$

On the other hand, a computation (using Equation (5.16) in Remark 5.2.8) shows $\nabla_{\dot\gamma_1}\partial_x = \beta\partial_y$, hence

$$|df\left(\nabla_{\dot\gamma_1}\partial_x\right)|_g \le \beta||\,\mathrm{grad}_g f||_g||\partial_x||_g \le (|\beta|/\beta+1)C\varrho^{2\beta+1}. \tag{6.14}$$

Using (6.13) and (6.14) in (6.12), we get

$$|F(\gamma_1(\phi_1)) - F(\gamma_1(\phi_0))| \le C_1\varrho^{\beta+1}|\phi_1-\phi_0|$$

As a second step, we consider the path $\gamma_2(t) = tz_0$ with $|z_0| = 1$, for $t\in[t_0,t_1]$. We claim that

$$|F(\gamma_2(t_1)) - F(\gamma_2(t_0))| \le C_2|t_1-t_0|^{\beta+1}.$$

Since $|\gamma_2(t)| = t$, we have

$$\frac{d(F\circ\gamma_2(t))}{dt} = -\beta t^{-\beta-1}\frac{\partial f}{\partial x} + t^{-\beta}\left(g\left(\mathrm{Hess}_g f(\dot\gamma_2),\frac{\partial}{\partial x}\right) + df\left(\nabla_{\dot\gamma_2}\frac{\partial}{\partial x}\right)\right). \tag{6.15}$$

From $||\,\mathrm{grad}_g f|| \le C\rho$, we get $|\partial f/\partial x| \le (C/(\beta+1))t^{2\beta+1}$, hence the first term in (6.15) is bounded by $(|\beta|/\beta+1)Ct^\beta$. For the second term we have as above

$$|g\left(\mathrm{Hess}_g f(\dot\gamma_2),\partial_x\right)| \le C||\dot\gamma_2||_g||\partial_x||_g = Ct^{2\beta}\,, \tag{6.16}$$

whereas in this case $\nabla_{\dot\gamma_2}\partial_x = (\beta/t)\partial_x$, from which we get

$$|df\left(\nabla_{\dot\gamma_2}\partial_x\right)|_g \le |\beta/t|||\,\mathrm{grad}_g f||_g||\partial_x||_g \le (|\beta|/\beta+1)Ct^{2\beta}\,. \tag{6.17}$$

By integrating we get

$$|F(\gamma_2(t_1)) - F(\gamma_2(t_0))| \le \int_{t_0}^{t_1}\left|\frac{d(F\circ\gamma_2(t))}{dt}\right|dt \le C_2|t_1^{\beta+1}-t_0^{\beta+1}| \le C_2|t_1-t_0|^{\beta+1}.$$

We can now conclude the proof. Given two points $z_0 = \varrho_0 e^{i\phi_0}, z_1 = \varrho_1 e^{i\phi_1}\in D$, assuming $\varrho_1\ge\varrho_0$, consider the point $z_2 = \varrho_0 e^{i\phi_1}$. We have

$$\begin{aligned}|F(z_1)-F(z_0)| &\le |F(z_1)-F(z_2)| + |F(z_2)-F(z_0)|\\ &\le C_2|\varrho_1-\varrho_0|^{\beta+1} + C_1\varrho_0^{\beta+1}|\phi_1-\phi_0|\\ &\le C_3(|z_1-z_0|^{\beta+1} + \varrho_0^{\beta+1}|\phi_1-\phi_0|^{\beta+1})\\ &\le C_4|z_1-z_0|^{\beta+1}.\end{aligned}$$

In the second line we have used that $|\varrho_1-\varrho_0| \le |z_1-z_0|$ and the constant $C_3$ involves $C_1$, $C_2$ and a factor which bounds $|\phi_1-\phi_0|^{\beta+1}$ in terms of $|\phi_1-\phi_0|$, since $|\phi_1-\phi_0|\in[0,2\pi]$. In the last line, we have used $\varrho_0|\phi_1-\phi_0| \le (\pi/2)|z_1-z_0|$. □

*Proof of Theorem 6.B.* By a classical theorem of Korn and Lichtstein (see [Che55b] for a proof), Lemma 6.1.19 implies that there exists a $\mathrm{C}^{1,\alpha}$ conformal coordinate $w = w(z)$ for the metric $I'$ in a neighborhood of the point $p$. Now $I = |z|^{2\beta}I' = e^{2\xi}|w|^{2\beta}|dw|^2$ where $\xi$ is some continuous function on a neighborhood of the puncture. This concludes the proof that $I$ has a cone point of the same angle as $h$. □

### 6.1.4 Cauchy surfaces in MGHF spacetimes with particles

The purpose of this subsection is to prove a step towards the parametrization of MGHF structures with particles on $S \times \mathbb{R}$ by means of the tangent bundle of Teichmüller space of the punctured surface $S$. The parametrization will be completely achieved in Theorem 6.D.

Given two uniformly convex Cauchy surfaces in a MGHF spacetime $M$, we already know from Theorem 5.A that the holonomy of the third fundamental form of a Cauchy surface coincides with the linear holonomy of $M$. However, differently from the closed case, this is not sufficient to guarantee that the two hyperbolic metrics obtained as third fundamental form correspond to the same point in Teichmüller space. We prove this separately in Proposition 6.1.20, by using techniques similar to those developed in [Bel14].

Next, we prove a converse statement in Proposition 6.1.23. Namely, if two pairs of embedding data $(I, B)$ and $(I', B')$ are such that the third fundamental forms $h = I(B\boldsymbol{\cdot}, B\boldsymbol{\cdot})$ are isometric via an isometry isotopic to the identity, and the translation parts of the holonomy are in the same cohomology class, then $(I, B)$ and $(I', B')$ are embedding data of uniformly convex Cauchy surfaces in the same spacetime.

**Proposition 6.1.20.** *Let $(I, B), (I', B') \in \mathbb{D}_{\boldsymbol{\beta}}$ be embedding data of uniformly convex Cauchy surfaces in the same spacetime with particles $M$. Then $h = I(B\boldsymbol{\cdot}, B\boldsymbol{\cdot})$ and $h' = I'(B'\boldsymbol{\cdot}, B'\boldsymbol{\cdot})$ are isotopic.*

We first give an observation which will be used several times in the proof.

*Remark* 6.1.21. Given a uniformly convex surface $S$ with embedding data $(I, B)$, let $(h, b) \in \mathbb{E}_{\boldsymbol{\beta}}$ the corresponding pair. Let $S(t)$ be the surface obtained by the future normal flow of $S$ at time $t$. It is known that $S(t)$ corresponds to the pair $(h, b + t\mathrm{I})$, namely, the third fundamental form is constant along the normal flow. Moreover, as the first fundamental form of $S(t)$ is $I_t = h((b + t\mathrm{I})\boldsymbol{\cdot}, (b + t\mathrm{I})\boldsymbol{\cdot})$, $h$ can also be recovered as $h = \lim_{t\to\infty} \frac{1}{t^2} I_t$.

We will also use the following lemma concerning the properties of flat globally hyperbolic spacetimes.

**Lemma 6.1.22.** *Let $S_1$ and $S_2$ be uniformly convex surfaces in a MGHF spacetime with particles. If $S_2$ is contained in the future of $S_1$ and in the past of $S_1(a)$, then $S_2(t)$ is contained in the future of $S_1(t)$ and in the past of $S_1(a + t)$ for every $t > 0$.*

*Proof.* We claim that a point $x$ is in the future of $S_1(t)$ if there is a timelike path with future endpoint in $x$, of length at least $t$, entirely contained in the future of $S_1$. From this claim, the thesis follows directly.

To prove the claim, assume $x$ is in the past of $S_1(t)$. The pull-back of the Lorentzian metric of $M$ using the normal flow takes the form $-ds^2 + g_s$ , where $g_s$ are Riemannian metrics. Hence every causal path contained in the future of $S_1$ and with endpoint $x$ has length at most $t$. □

*Proof of Proposition 6.1.20.* Let $\sigma_1 : S \to M$ and $\sigma_2 : S \to M$ be embeddings of uniformly convex Cauchy surfaces $S_1$ and $S_2$ with embedding data $(I, B)$ and $(I', B')$ respectively. Assume first that $S_2$ is contained in the future of $S_1$ and in the past of $S_1(a)$. Applying Lemma 6.1.22 and [Bel14, Proposition 4.2], one sees that the projection from $S_2(t)$ to $S_1(a+t)$ obtained by following the normal flow of $S_1$ is distance-increasing, and is a diffeomorphism by the property of Cauchy surfaces. Hence we obtain a one-parameter family of 1-Lipschitz diffeomorphisms (which clearly extend to the punctures) $f_t : S_1(a+t) \to S_2(t)$.

Recall that by Remark 6.1.21, for the first fundamental form $I_t$ of $S_2(t)$, $I_t/t^2$ converges to the third fundamental form $h_1$, and analogously for $h_2$. Hence by Ascoli-Arzelà Theorem we obtain a 1-Lipschitz map $f_\infty : (S_1, h_1) \to (S_2, h_2)$ homotopic to $f_0$. Since the areas of $(S_1, h_1)$ and $(S_2, h_2)$ coincide by Gauss-Bonnet formula, $f_\infty$ is necessarily an isometry. Moreover, by construction it is clear that $(\sigma_2)^{-1} \circ f_\infty \circ \sigma_1$ is isotopic to the identity.

In the general case, given $S_1$ and $S_2$ uniformly convex, it suffices to replace $S_2$ by $S_2(k)$ for $k$ to reduce to the previous case. Indeed, we have already observed that the third fundamental forms coincide for $S_2$ and $S_2(k)$. □

We now move to the proof of a Proposition 6.1.23, which is a converse statement.

**Proposition 6.1.23.** *Let $h$ be a hyperbolic metric on $S$ with cone singularities and let $b, b' \in \mathcal{C}_\infty(S, h)$ be bounded and uniformly positive Codazzi operators. If $\delta(b) = \delta(b')$, then the pairs $(I, B)$ and $(I', B')$ corresponding to $(h, b)$ and $(h, b')$ are embedding data of uniformly convex Cauchy surfaces in the same MGHF spacetime with particles.*

Recall from Theorem 5.A that, under the hypothesis, $\delta(b)$ is the translation part of the holonomy of the spacetime $M$ provided by the embedding data $(I, B)$, and the linear part is the holonomy of $h$. The idea of the proof is to show that any small deformation of $b$ which leaves the holonomy invariant gives an embedding into the same spacetime $M$. Then, we use connectedness of the space

$$\left\{b' \in \mathcal{C}_\infty(S, h) : b' \text{ is uniformly positive and } \delta(b') = \delta(b)\right\}.$$

So we prove the first assertion, by a standard argument.

**Lemma 6.1.24.** *Let $b \in \mathcal{C}_\infty(S, h)$ be uniformly positive, and $\dot{b} \in \mathcal{C}_\infty(S, h)$ be such that $\delta(\dot{b}) = 0$. Let $M$ be the MGH spacetime obtained from the embedding data $(h, b)$. Then there exists $\epsilon > 0$ such that for $s \in (-\epsilon, \epsilon)$ every pair $(h, b + s\dot{b})$ gives an embedding of a uniformly convex spacelike surface into the spacetime $M$.*

*Proof.* Let $\tilde{\sigma} : \tilde{S} \to \mathbb{R}^{2,1}$ be the embedding constructed as in Proposition 5.1.10. We can choose a smooth path of developing maps $\mathrm{dev}_s : \tilde{S} \to \mathbb{R}^{2,1}$ having embedding data $(h, b + s\dot{b})$, for $s \in (-\epsilon, \epsilon)$. Since by linearity $\delta(b + s\dot{b}) = \delta(b)$ for all $s$, $\mathrm{dev}_s$ all have the same holonomy. We can also assume $\mathrm{dev}_0$ extends to a developing map $\mathrm{Dev}_0$ for $M$ defined on the lifting of a tubular neighborhood $\tilde{T} \cong \tilde{S} \times (-a, a)$ of $S$ in $M$ and there is a covering $\{U_\alpha\}$ of $S$ such that, for every $\alpha$, either $\mathrm{Dev}_0$ is an isometry on its image when restricted to $U_\alpha \times (-a, a)$ (if $U_\alpha$ does not contain any singular point) or there is a chart for $U_\alpha \times (-a, a)$ to a manifold $(D \times \mathbb{R}, g - dt^2)$ where $g$ is a Euclidean metric on the disc $D$ with a cone point at 0. Restricting to a smaller $\epsilon$ if necessary, and using the fact that all $\mathrm{dev}_s$ have the same holonomy, we see that $\mathrm{dev}_s$ provides an embedding of $\tilde{S}$ into the spacetime obtained by gluing the charts $\{U_\alpha \times (-a, a)\}$, for $s \in (-\epsilon', \epsilon')$. This concludes the proof. □

*Proof of Proposition 6.1.23.* Given $b$ and $b'$ as in the hypothesis, let $v = ||b'||_\infty$ be a constant function and let $b_s = b + sv\mathrm{I}$ for $s \in [0,1]$. Since $b$ is modified by adding $sv\mathrm{I} = -\mathrm{Hess}(sv) + (sv)\mathrm{I}$, $\delta(b_s) = \delta(b)$ for every $s$. Clearly, $b_s$ is bounded and uniformly positive for every $s$. By Lemma 6.1.24, $(h, b)$ and $(h, b_1)$ correspond to embeddings of uniformly convex surfaces in the same spacetime $M$. Now the same argument can be applied to $b'_s = b' + s(b_1 - b')$, which is again bounded and uniformly positive for every $s$, since by construction the eigenvalues of $b_1$ at every point are larger than the largest eigenvalue of $b'$, and $\delta(b'_s) = \delta(b')$ by linearity of $\delta$. Therefore, $b$ and $b'$ correspond to embeddings in the same spacetime $M$. $\square$

## 6.2 Relation with Teichmüller theory of a punctured surface

Let us fix a topological surface $S$, and a divisor $\boldsymbol{\beta} = \sum \beta_i p_i$, where $\beta_i \in (-1, 0)$, and put $\mathfrak{p} = \{p_1, \ldots, p_k\}$. Recall we are assuming $\chi(S, \boldsymbol{\beta}) < 0$.

We denote by $\mathcal{H}(S, \boldsymbol{\beta})$ the space of hyperbolic metrics on $S$ with cone singularity $\theta_i = 2(1+\beta_i)\pi$ at $p_i$. On the other hand, we consider the space of MGHF structures containing a uniformly convex Cauchy surface orthogonal to the singular locus with bounded second fundamental form, which we denote as follows:

$$\mathcal{S}_+(S, \boldsymbol{\beta}) = \left\{ \begin{array}{l} \text{MGHF structures on } S \times \mathbb{R} \text{ with particles on } \mathfrak{p} \times \mathbb{R} \text{ of angles } \theta_i \\ \text{containing a convex Cauchy surface orthogonal to } \mathfrak{p} \times \mathbb{R} \\ \text{having bounded and uniformly positive shape operator} \end{array} \right\} /\!\sim$$

where two structures are equivalent for the relation $\sim$ if and only if they are related by an isometry of $S \times \mathbb{R}$ isotopic to the identity fixing each particle.

In Theorem 6.A we have seen that the pairs $(h, b)$, where $h \in \mathcal{H}(S, \boldsymbol{\beta})$ and $b$ is a bounded and positive $h$-Codazzi tensor on $S$, bijectively correspond to immersion data of convex Cauchy surfaces in a MGHF spacetime, orthogonal to the singular locus. Moreover the spacetimes corresponding to $(h, b)$ and $(h', b')$ are in the same equivalence class in $\mathcal{S}_+(S, \boldsymbol{\beta})$ if and only if there is an isometry from $(S, h)$ to $(S, h')$ isotopic to the identity in $\mathrm{Homeo}^+(S, \mathfrak{p})$ and $\delta(b) = \delta(b')$.

In this section we want to use this characterization to construct a natural bijective map between $\mathcal{S}_+(S, \boldsymbol{\beta})$ and the tangent space of Teichmüller space of the punctured surface $\mathcal{T}(S, \mathfrak{p})$.

Let us recall that elements of $\mathcal{T}(S, \mathfrak{p})$ are complex structures on $S$, say $X = (S, \mathcal{A})$, where $\mathcal{A}$ is a complex atlas on $S$, considered up to isotopies of $S$ which point-wise fix $\mathfrak{p}$. Dealing with complex structures associated with singular metrics on $S \setminus \mathfrak{p}$ makes important to clarify the regularity of the complex atlas. In the classical Teichmüller theory one can deal with complex atlas whose charts are only quasi-conformal with respect to a base smooth complex structure on $S$. In this framework the group acting on this space of complex structures is the space of quasiconformal homeomorphisms of $S$, which does not depend on the complex structure chosen. We have the following lemma:

**Lemma 6.2.1.** *Let $h$ be a hyperbolic metric with cone singularities on $S$, and $b \in \mathcal{C}_\infty(S,h)$. Then there is a complex structure $X$ on $S$ such that the metric $\hat{h}_t = h((E+tb)\bullet,(E+tb)\bullet)$ is conformal for $X$.*

*Proof.* Clearly there is a smooth complex structure $\mathcal{A}_t^*$ on $S \setminus \mathfrak{p}$ for which $\hat{h}_t$ is conformal. We have to prove that $\mathcal{A}_t^*$ extends to a complex atlas over $S$.

If $t=0$ then this basically follows from the definition of metric with cone singularities.

Consider now the general case. Notice that $\mathrm{I} : (S\setminus\mathfrak{p},\mathcal{A}_0^*) \to (S\setminus\mathfrak{p},\mathcal{A}_t^*)$ is quasiconformal. In particular this implies that a neighborhood $U$ of any puncture $p_i$ with the structure inherited by $\mathcal{A}_t^*$ is quasiconformal to a punctured disc. So it is biholomorphic to a punctured disc, that is there is a biholomorphic map

$$\zeta : (U\setminus p_i,\mathcal{A}_t^*) \to \mathbb{D}^*\,,$$

where $\mathbb{D}^* = \{z\in\mathbb{C}\,|\,0<|z|<1\}$.

It is not difficult to show that $\zeta$ extends by continuity to a homeomorphism $\zeta : U \to \mathbb{D}$. This proves that the atlas $\mathcal{A}_t^*$ extends to $S$. Finally notice that the function $\zeta$ in general is not smooth at $p_i$, but, as the map

$$\zeta : (U,\mathcal{A}_0) \to \mathbb{D}$$

is quasiconformal, it has the requested regularity. □

At a point $[X]\in\mathcal{T}(S,\mathfrak{p})$ , the tangent space of $\mathcal{T}(S,\mathfrak{p})$ is identified with a quotient of the space of Beltrami differentials $\mathcal{B}(X)$. We say that a Beltrami differential $\mu$ is trivial if $\langle q,\mu\rangle = 0$ for any holomorphic quadratic differential with poles of order at worst 1 at $p_i$ (equivalently with for any holomorphic section of $K^2(\mathfrak{p})$). We will denote by $\mathcal{B}(X,\mathfrak{p})^\perp$ the subspace of trivial Beltrami differentials. The tangent space $T_{[X]}\mathcal{T}(S,\mathfrak{p})$ is naturally identified with $\mathcal{B}(X)/\mathcal{B}(X,\mathfrak{p})^\perp$. The identification works as in the case of a closed surface. The main difference is that the derivative of the Beltrami differential of the map $\mathrm{I}: X\to X_t$ is well-defined up to this more restrictive relation as we only consider homotopies which point-wise fix $\mathfrak{p}$. It turns out that if $\sigma$ is a smooth section on $K^{-1}$, then $\bar\partial\sigma$ is trivial iff $\sigma$ vanishes at punctures.

The main theorem we prove in this section is the analogue of Theorem 5.B in the closed case. Given a hyperbolic metric $h\in\mathcal{H}(S,\boldsymbol{\beta})$ and $b\in\mathcal{C}_\infty(S,h)$, the family of Riemannian metrics $\hat{h}_t = h((E+tb)\bullet,(E+tb)\bullet)$ induces a smooth family $X_t$ of complex structures by Lemma 6.2.1. As in the closed case treated in Section 5.2.4, the derivative of $X_t$ coincides with the Beltrami differential corresponding to the traceless part $b_0 = b-(\mathrm{tr}b/2)\mathrm{I}$. This leads again to the definition of a map $\Psi : \mathbb{E}_{\boldsymbol{\beta}} \to T\mathcal{T}(S,\mathfrak{p})$.

**Theorem 6.C.** *Let $\mathcal{C}_\infty(S,h)$ be the space of bounded Codazzi tensors on $(S,h)$. The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{C}_\infty(S,h) & \xrightarrow{\Lambda\circ\delta} & H^1_{Ad\circ hol}(\pi_1(S\setminus\mathfrak{p}),\mathfrak{so}(2,1)) \\ {\scriptstyle\Psi}\big\downarrow & & d\mathbf{hol}\big\uparrow \\ T_{[X]}\mathcal{T}(S,\mathfrak{p}) & \xrightarrow[\mathcal{J}]{} & T_{[X]}\mathcal{T}(S,\mathfrak{p}) \end{array}\,, \tag{6.18}$$

*where* $\Lambda : H^1_{hol}(\pi_1(S \setminus \mathfrak{p}), \mathbb{R}^{2,1}) \to H^1_{Ad\circ hol}(\pi_1(S \setminus \mathfrak{p}), \mathfrak{so}(2,1))$ *is the natural isomorphism, and* $\mathcal{J}$ *is the complex structure on* $\mathcal{T}(S, \mathfrak{p})$.

As in the closed case the proof of this theorem is based on the computation of the differential of the holonomy map, where in this case

$$\mathbf{hol} : \mathcal{T}(S, \mathfrak{p}) \to \mathcal{R}(\pi_1(S \setminus \mathfrak{p}), \mathrm{SO}(2,1)) /\!\!/ \mathrm{SO}(2,1)$$

is the map sending the marked Riemann surface $[X]$ to the holonomy of the unique hyperbolic conformal metric on $S \setminus \mathfrak{p}$ with cone singularities $\theta_i$ at $p_i$. The existence of such a metric is a corollary of a more general result [Tro91].

In [ST11] it has been proved that if $t \mapsto [X_t]$ is a smooth path in $\mathcal{T}(S, \mathfrak{p})$, then there is a smooth family of hyperbolic metrics $h_t$ on $S \setminus \mathfrak{p}$ whose underlying complex structure is isotopic to $X_t$. By Proposition 5.2.1 it turns out that the holonomy map is smooth. Now we want to compute precisely the differential of **hol**. In particular we prove the analogue of Proposition 5.2.4. Then, the proof of Theorem 6.C follows exactly as in the closed case.

**Proposition 6.2.2.** *Let* $h$ *be a hyperbolic metric in* $\mathcal{H}(S, \boldsymbol{\beta})$, $X_h \in \mathcal{T}(S, \mathfrak{p})$ *its complex structure and let* $b \in \mathcal{C}_\infty(S, h)$. *Let* $b = b_q + \mathrm{Hess} u - u\, E$ *be the decomposition of* $b$ *given in Proposition 6.1.8. Then*

$$d\mathbf{hol}_{X_h}([b_0]) = -\Lambda \delta(J b_q)\,,$$

*where* $\Lambda : H^1_{hol}(\pi_1(S \setminus \mathfrak{p}), \mathbb{R}^{2,1}) \to H^1_{Ad\circ hol}(\pi_1(S \setminus \mathfrak{p}), \mathfrak{so}(2,1))$ *is the isomorphism induced by the* $\mathrm{SO}(2,1)$*-equivariant isomorphism* $\Lambda : \mathbb{R}^{2,1} \to \mathfrak{so}(2,1)$.

The proof of this proposition follows the same line as in the closed case, but some technicalities come up. We consider the hyperbolic metric $h_t$ with cone singularities and $h_t = e^{2\psi_t} \hat{h}_t$, where $\hat{h}_t = h((E + tb)\cdot, (E + tb)\cdot)$. By Proposition 5.2.1,

$$\dot{\mathrm{hol}} = \frac{1}{2} \mathfrak{d}(h^{-1} \dot{h})\,.$$

Since $h^{-1}\dot{h} = 2((\dot{\psi} - u)E + b_q + \mathrm{Hess} u)$, putting $\phi = \dot{\psi} - u$ one gets $\Delta\phi - \phi = 0$ as in the closed case. Then proving that $\phi \equiv 0$, one concludes as in the closed case.

Notice that respect to the closed case there are two technical points.

- One has to prove that the conformal factor $\psi_t$ smoothly depends on $t$ on $S \setminus \{\mathfrak{p}\}$. The idea is to use the result of [ST11], but we emphasize that we cannot apply directly it, as the conformal structure induced by $h_t$ is not constant in a neighborhood of the punctures. So we need to use an isotopy to fix the conformal structures around the punctures.

- The second point is that the proof that $\phi$ is zero is not immediate. In fact $\phi$ is defined only on the regular part $S \setminus \mathfrak{p}$ which is not compact. Actually we will prove that $\phi$ continuously extends to the punctures and conclude by adapting the maximum principle to the context of surfaces with cone singularity. The proof that $\phi$ continuously extends to the disc needs some careful analysis around the singular points.

In the following lemma we summarize the technical construction of the isotopy $F_t$.

**Lemma 6.2.3.** *There is a smooth map $F : [0, \epsilon] \times (S \setminus \mathfrak{p}) \to (S \setminus \mathfrak{p})$ such that*

- *For any $t$ the map $F_t(\cdot) = F(t, \cdot)$ extends to a quasiconformal homeomorphism fixing $\mathfrak{p}$.*
- *$F_0$ is the identity.*
- *There is a neighborhood $U$ of $\mathfrak{p}$ such that $F_t : (U, \mathcal{A}_0) \to (U, \mathcal{A}_t)$ is conformal for any $t$.*
- *The variation field $Y = \left.\frac{dF_t(x)}{dt}\right|_{t=0}$ extends to a continuous field on $S$ with $Y(p_i) = 0$.*

*Proof.* First we construct the isotopy $F_t^{(i)}$ on a small neighborhood $U_i$ of a puncture $p_i$. Notice that the Beltrami coefficient $\mu_t$ of the identity $(S, \mathcal{A}_0) \to (S, \mathcal{A}_t)$ corresponds to the symmetric traceless tensor $\frac{tb_0}{1+t(\mathrm{tr}b/2)}$ under the usual identification, so $\mu_t$ smoothly depends on $t$. By the classical theory of Beltrami equation we can find on a disc around the puncture a family of quasiconformal maps $G_t^{(i)} : (U_i, \mathcal{A}_0) \to (U_i, \mathcal{A}_0)$ with Beltrami coefficient $\mu_t$, such that the map $G^{(i)} : (-\epsilon, \epsilon) \times U_i \to U_i$ is smooth in $t$. We may moreover suppose that $G_t^{(i)}(p_i) = p_i$ for every $t$. Regarding $G_t^{(i)}$ as a map $G_t^{(i)} : (U_i, \mathcal{A}_t) \to (U_i, \mathcal{A}_0)$, it is holomorphic, so the map defined by $F_t^{(i)} = (G_t^{(i)})^{-1}$ satisfies the requirements. Notice that the variation field $Y_t^{(i)} = \left.\frac{dF^{(i)}}{dt}\right|_{t=0}$ is defined on the whole $U_i$, is smooth outside the punctures and $Y_t(p_i) = 0$.

Now take a neighborhood $U_i'$ of $p_i$ such that $\overline{U}_i' \subset U_i$ and choose a smaller neighborhood $U_i''$ such that $F_t^{(i)}(U_i'') \subset U_i'$ for every $t$. Take a smooth function $\chi$ on $S$ which vanishes on $S \setminus (\bigcup U_i)$ and is constantly 1 over $\bigcup U_i'$. Let $Y_t$ be the field defined by $Y_t = \chi Y_t^{(i)}$ over $U_i$ and $Y_t \equiv 0$ over $S \setminus (\bigcup U_i)$. It can be readily shown that $Y_t$ generates a flow of maps $F_t \in \mathrm{Homeo}(S, \mathfrak{p}) \cap \mathrm{Diffeo}(S \setminus \mathfrak{p})$ and $F_t \equiv F_t^{(i)}$ over $U_i''$. It is easy to check that $F_t$ verifies the requirements we need. $\square$

*Proof of Proposition 6.2.2.* We consider the metrics $k_t = F_t^*(h_t)$ conformal to $F_t^*(\hat{h}_t)$. By [ST11] we know that they smoothly depend on the parameter $t$. It follows that $h_t$ also smoothly depends on $t$. Moreover we have

$$h^{-1}\dot{k} = h^{-1}\dot{h} + 2\mathbf{S}\nabla Y\,.$$

As $h^{-1}\dot{h} = (2\dot{\psi} - 2u)E + 2b_q + 2\mathrm{Hess}u = 2\phi\, E + 2b_q + 2\mathrm{Hess}u$, one deduces that $\Delta\phi - \phi = 0$ as in the closed case.

We claim that $\phi \equiv 0$. From the claim the proof follows exactly as in the closed case. To prove the claim we first check that $\phi$ continuously extends to the punctures, then we use the maximum principle adapted to the case of surfaces with cone points that we prove separately in the next Lemma (notice that here $\phi$ solves the equation

$\Delta\phi - \phi = 0$ on $S \setminus \mathfrak{p}$ but this does not imply that it is a weak solution of the equation on the closed surface).

The proof of the continuity of $\phi$ around a puncture $p_i$ is articulated in the following steps:

Step 1 We will show that around $p_i$ there exists a smooth vector field $V$ such that $h^{-1}\dot{k} = 2\mathbf{S}\nabla V$.

Step 2 Writing $b_q = J\mathrm{Hess}v - vJ$ we have that

$$\phi\mathrm{I} = \mathbf{S}\nabla(Y_1)\,,$$

where $Y_1 = V - Y - J\,\mathrm{grad}\,v - \mathrm{grad}\,u$. This implies that $\bar{\partial}Y_1 = 0$, that is $Y_1$ is a holomorphic vector field on $U \setminus \{p_i\}$ (see Remark 5.2.8). We will prove that $Y_1$ extends at $p_i$ and $Y_1(p_i) = 0$.

Step 3 As $2\phi$ is the divergence of $Y_1$, the continuity of $\phi$ will be deduced by an explicit computation where we use that $Y_1$ is analytic around the puncture with $Y_1(p_i) = 0$ (as the Christoffel symbols of the metric $h$ diverge around the puncture, the condition $Y_1(p_i) = 0$ will play a key role in the computation).

*Step 1:*

As $k_t$ smoothly depends on $t$ we can construct a smooth family of isometric embeddings

$$s_t : (U, k_t) \to (U', h)$$

where $U$ and $U'$ are fixed neighborhoods of $p_i$. Up to shrinking $U$ we may suppose that on $U$ the metric $k_t$ is conformal to $h$ for every $t$, so from the conformal point of view we have a smooth map

$$s : (-\epsilon, \epsilon) \times U \to U'$$

such that the restriction $s_t(\cdot) = s(t, \cdot)$ is holomorphic for every $t$. It follows that $V = \frac{ds_t}{dt}\big|_{t=0}$ is a holomorphic field defined on the whole $U$. As $s_t(p_i) = p_i$ we get that $V(p_i) = 0$. Finally notice that as $s_t^*(h) = k_t$ we have that $2\mathbf{S}\nabla V = h^{-1}\dot{k}$.

*Step 2:*

As we know that $Y(p_i) = 0$ and $V(p_i) = 0$, it is sufficient to prove that $\mathrm{grad}\,v$ and $\mathrm{grad}\,u$ vanish at $p_i$.

More generally we will prove that if $f$ is a function on $U$ such that $||\mathrm{Hess}f - f\,E|| < Cr^\alpha$ for some $\alpha > -1$, then $\mathrm{grad}\,f$ extends to 0 at $p_i$. This general fact implies the extendability of $\mathrm{grad}\,v$, because $\mathrm{Hess}v - v\,E = -b_{iq}$ satisfies this condition as we noted in the proof of 6.1.4. On the other hand $u$ can be regarded as the difference of the functions $u_1 - u_2$ where $b = \mathrm{Hess}u_1 - u_1\,E$ and $b_q = \mathrm{Hess}u_2 - u_2\,E$. As $b$ is bounded and $b_q$ satisfies the condition above, we conclude that $\mathrm{grad}\,u$ extends as well.

Let $g$ be the Klein Euclidean metric on $U$. As in the proof of Lemma 6.1.16, let us put $\bar{f} = (\cosh r)^{-1}f$, where $r$ is the hyperbolic distance from the cone point. By Lemma 6.1.3, $\bar{f}$ satisfies the equation $D^2_g\bar{f}(\bullet,\bullet) = (\cosh r)^{-1}h((\mathrm{Hess}\, f - f\,E)\bullet,\bullet)$, so $||D^2_g\bar{f}||_g < Cr^\alpha$. Consider on the universal cover the gradient map $\tilde{\varphi} = (\mathrm{dev})_*(\mathrm{grad}_g\,\bar{f}) : \widetilde{U\setminus\{p_i\}} \to \mathbb{R}^2$. If $\rho$ denotes the Euclidean radial coordinate and $\theta$ is the pull-back of the angular coordinate, we have

$$\begin{aligned}||\tilde{\varphi}(\rho_1,\theta_1) - \tilde{\varphi}(\rho_2,\theta_2)|| &\leq ||\tilde{\varphi}(\rho_1,\theta_1) - \tilde{\varphi}(\rho_1,\theta_2)|| + ||\tilde{\varphi}(\rho_1,\theta_2) - \tilde{\varphi}(\rho_2,\theta_2)|| \\ &\leq C(\rho_1^\alpha\rho_1||\theta_1-\theta_2|| + ||\rho_2^{\alpha+1} - \rho_1^{\alpha+1}||)\,.\end{aligned}$$

This shows that on each radial line there exists the limit $\lim_{\rho\to 0}\varphi(\rho,\theta) = \xi$. Moreover this limit does not depend on $\theta$, and the convergence is uniform as far as $\theta$ lies in some compact interval of $\mathbb{R}$. As in the proof of Lemma 6.1.16, $\varphi(\rho,\theta+\theta_0) = R_{\theta_0}\varphi(\rho,\theta)$, we deduce that $\xi$ is a fixed point of the rotation, that is $\xi = 0$. It turns out that $||\,\mathrm{grad}_g\,\bar{f}||_g \to 0$ at $p_i$. As the hyperbolic metric $h$ is equivalent to $g$ we conclude that also $||\,\mathrm{grad}_h\,f||_h \to 0$ at the puncture $p_i$.

*Step 3:*

Under the identification $K^{-1} = TS$ we have $Y_1 = f(z)\frac{\partial}{\partial z}$ and $2\phi = \mathrm{div}Y_1$. As in complex notation the connection form (compare Remark 5.2.8) is

$$\omega = 2\frac{\partial\eta}{\partial z}dz\,,$$

where $\eta$ is the conformal factor of the hyperbolic metric $h = e^{2\eta}|dz|^2$. It turns out that $\phi = \Re(f'(z) + 2\partial_z\eta f(z))$. Notice that $\eta = \beta\log|z| + \xi$ where $\xi$ is a $C^1$ function on $D$ such that $||d\xi|||z| \to 0$ (compare the explicit expression in Equation (6.2) for the hyperbolic metric in the conformal coordinate). So we get

$$\lim_{z\to 0}\phi = (1+\beta)\Re f'(0)\,,$$

and in particular $\phi$ extends to a continuous function on $S$. □

**Lemma 6.2.4.** *Let $h \in \mathcal{H}(S,\boldsymbol{\beta})$. If $\phi$ is a continuous function on $S$, and on $S\setminus\mathfrak{p}$ is a smooth solution of $\Delta_h\phi - \phi = 0$, then $\phi \equiv 0$.*

*Proof.* From the equation we know that if the maximum of $\phi$ is realized at an interior point, then it must be nonpositive. We claim that the same holds if the maximum is realized at a puncture $p_i$. From the claim we can conclude that the maximum must be nonpositive and analogously the minimum nonnegative, that is $\phi \equiv 0$.

To prove the claim we consider the function $F : [0,\epsilon) \to \mathbb{R}$ such that $F(r)$ is the average of $\phi$ over the circle centered at $p_i$ of radius $r$. We fix $\epsilon$ so that all those circles are embedded in $S$. Notice that by continuity of $\phi$ we have $\lim_{r\to 0} F(r) = \phi(p_i)$ and the assumption that $p_i$ is a maximum point for $\phi$ implies that $F(r) \leq F(0)$ for $r \geq 0$.

Now using coordinates $r,\theta$ in a neighborhood of $p_i$ we have

$$F(r) = \frac{1}{\theta_0}\int_0^{\theta_0}\phi(r,\theta)d\theta\,,$$

so

$$\dot{F}(r) = \frac{1}{\theta_0}\int_0^{\theta_0} h(\operatorname{grad}\phi(r,\theta),\nu)d\theta = \frac{1}{\theta_0 \sinh r}\int_{\partial B_r} h(\operatorname{grad}\phi(r,\theta),\nu)d\ell_r\ ,$$

where $\nu$ is the normal field on $\partial B_r$ pointing outside.

Putting $G(r) = \int_{\partial B_r} h(\operatorname{grad}\phi(r,\theta),\nu)d\ell_r$, the Divergence Theorem implies that for $s < r$

$$G(r) - G(s) = \int_{B_r\setminus B_s} \Delta_h\phi\omega_h = \int_{B_r\setminus B_s}\phi\omega_h\,.$$

As $\phi$ is bounded we have $|G(r) - G(s)| \leq K(r^2 - s^2)$ for some constant $K$. This implies that $G$ extends to 0 and, putting $C_0 = G(0)$

$$|G(r) - C_0| \leq Kr^2\,. \tag{6.19}$$

Let us show that $C_0 = 0$. If $C_0 \neq 0$, up to changing the sign of $\phi$ we may suppose $C_0 > 0$. Then by (6.19) we get

$$\left|\theta_0\dot{F}(r) - \frac{C_0}{\sinh r}\right| \leq K'r\,.$$

This implies that $\theta_0\dot{F} \geq \frac{C_0}{\sinh r} - K'r$, but this contradicts the fact that $\dot{F}$ is integrable on $[0,\epsilon)$. Thus $C_0 = 0$ so $|\dot{F}(r)| < K'r$, that implies that $\dot{F}(r) \to 0$ as $r \to 0$.

Now

$$\begin{aligned}\theta_0(\dot{F}(r) - \dot{F}(s)) &= \frac{1}{\sinh r}(G(r) - G(s)) + \left(\frac{1}{\sinh r} - \frac{1}{\sinh s}\right)G(s) = \\ &= \frac{1}{\sinh r}\int_{B_r\setminus B_s}\Delta_h\phi\omega_h + \frac{\sinh s - \sinh r}{\sinh s\sinh r}G(s)\ .\end{aligned}$$

Notice that the last addend tends to 0 as $s \to 0$, so we deduce

$$\dot{F}(r) = \frac{1}{\theta_0\sinh r}\int_{B_r}\Delta_h\phi\omega_h = \frac{1}{\theta_0\sinh r}\int_{B_r}\phi\omega_h\ .$$

Now as $F(r) \leq F(0)$ we get that $\dot{F}(r)$ must be nonpositive for small $r$, and this implies that $\phi(0)$ cannot be positive. Analogously one shows that if the minimum is achieved at a puncture, then it must be nonnegative and this concludes that $\phi \equiv 0$. □

**Theorem 6.D.** *Two embedding data $(I,B)$ and $(I',B')$ in $\mathbb{E}_{\boldsymbol{\beta}}$ correspond to Cauchy surfaces contained in the same spacetime with particles if and only if*

- *the third fundamental forms $h$ and $h'$ are isotopic;*
- *the infinitesimal variation of $h$ induced by $b = B^{-1}$ is Teichmüller equivalent to the infinitesimal variation of $h'$ induced by $b' = (B')^{-1}$.*

*The map $\Psi$ induces to the quotient a bijective map*

$$\bar{\Psi} : \mathcal{S}_+(S,\boldsymbol{\beta}) \to T\mathcal{T}(S,\mathfrak{p}).$$

*Proof.* The first part directly follows by Proposition 6.1.23. The fact that $\bar{\Psi}$ is well-defined and injective is then a consequence of commutativity of (6.18). Notice that $\Psi : \mathcal{C}(S,h) \to T_{[X]}(S,\mathfrak{p})$ is surjective by a simple dimensional argument. As for any $b \in \mathcal{C}_\infty(S,h)$ we may find a constant $M$ so that $b + M\,E$ is positive. Like in the closed case we conclude that $\bar{\Psi}$ is surjective. □

## 6.3 Symplectic structures in the singular case

In the singular case it is also possible to construct a Goldman intersection form $\omega^{\mathbf{B}}$ on the image of $d\mathbf{hol}$ in $H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1))$. Mondello [Mon10] proved that the map $d\mathbf{hol}$ is symplectic up to a factor. We will give here a different proof of that result in the analogy of the proof of Corollary 5.3.5 given in Subsection 5.3.

First let us recall some basic facts on the construction of $\omega^{\mathbf{B}}$. We denote by $H^\bullet_{\mathrm{c}}(S, F_{\mathfrak{so}(2,1)})$ the de Rham cohomology of the complex of $F_{\mathfrak{so}(2,1)}$-valued forms on $S$ with compact support, and let

$$I_* : H^1_{\mathrm{c}}(S, F_{\mathfrak{so}(2,1)}) \to H^1_{\mathrm{dR}}(S, F_{\mathfrak{so}(2,1)})$$

be the map induced by the inclusion. The image of $I_*$ will be denoted by $H^1_0(S, F_{\mathfrak{so}(2,1)})$ and contains the cohomology classes in $H^1_{\mathrm{dR}}(S, F_{\mathfrak{so}(2,1)})$ which admit a representative with compact support. Under the isomorphism $H^1_{\mathrm{Ad}\circ\mathrm{hol}}(\pi_1(S), \mathfrak{so}(2,1)) \cong H^1_{\mathrm{dR}}(S, F_{\mathfrak{so}(2,1)})$, elements of $H^1_0(S, F_{\mathfrak{so}(2,1)})$ correspond to cocycles which are trivial around the punctures.

Let $\mathbf{B}$ be the Ad-invariant product on $\mathfrak{so}(2,1)$ defined in Subsection 5.3. It induces a well-defined pairing

$$\bar{\omega}^{\mathbf{B}} : H^1_{\mathrm{c}}(S, F_{\mathfrak{so}(2,1)}) \times H^1_{\mathrm{dR}}(S, F_{\mathfrak{so}(2,1)}) \to \mathbb{R}$$

given as in the closed case by

$$\bar{\omega}^{\mathbf{B}}([\varsigma],[\sigma]) = \int_S B(\varsigma \wedge \sigma)\,.$$

This pairing is nondegenerate by Poincaré duality. Notice that if $\varsigma, \varsigma'$ are forms with compact support $\bar{\omega}^{\mathbf{B}}([\varsigma], I_*[\varsigma']) = -\bar{\omega}^{\mathbf{B}}([\varsigma'], I_*([\varsigma]))$, showing that $\ker I_*$ coincides with the orthogonal subspace of $H^1_0(S, F_{\mathfrak{so}(2,1)})$.
Thus the form $\bar{\omega}^B$ induces a symplectic form on $H^1_0(S, F_{\mathfrak{so}(2,1)})$, defined by

$$\omega^{\mathbf{B}}([\sigma],[\sigma']) = \int_S \mathbf{B}(\sigma \wedge \sigma')\ ,$$

where $\sigma$ and $\sigma'$ are representatives with compact support.

In a similar way we can define the subspace $H^1_0(S,F)$ and a symplectic form $\bar{\omega}^F$ on it, analogous to the one constructed in Subsection 5.3

By Proposition 6.1.12, the coboundary operator $\delta : \mathcal{C}_2(S,h) \to H^1_{\mathrm{dR}}(S,F)$ takes values in $H^1_0(S, F_{\mathfrak{so}(2,1)})$. The following proposition computes $\bar{\omega}^{\mathbf{B}}(\Lambda\delta b, \Lambda\delta b')$ in analogy with Proposition 5.3.2.

**Proposition 6.3.1.** *Let $\delta : \mathcal{C}_2(S,h) \to H^1_0(S,F)$. Then*

$$\bar{\omega}^F(\delta(b),\delta(b')) = \frac{1}{2}\int_S \mathrm{tr}(Jbb')\omega_h\,. \tag{6.20}$$

*or analogously*

$$\bar{\omega}^{\mathbf{B}}(\Lambda(\delta(b)),\Lambda(\delta(b'))) = \frac{1}{4}\int_S \mathrm{tr}(Jbb')\omega_h\,. \tag{6.21}$$

*Proof.* Since $\delta$ is continuous for the $L^2$-norm by Proposition 6.1.6, both LHS and RHS in (6.20) and (6.21) are continuous on $\mathcal{C}_2(S,h)\times\mathcal{C}_2(S,h)$. So by density it is sufficient to prove that (6.20) holds for $b$ and $b'$ with compact support. But in that case, the proof is the same given in the closed case, recalling that $\delta b = [\boldsymbol{\iota}_* b]$. □

Now we compute the Weil-Petersson form in terms of the map $\Phi$. Let us denote by $K^2(\mathfrak{p})$ the space of holomorphic quadratic differentials with at worst simple poles at $\mathfrak{p}$. Recall that if $q, q' \in K^2(\mathfrak{p})$ then as in the closed case one can define $g_{WP}(q,q')$ by integrating the form that in a local chart is

$$\frac{f\bar{g}}{e^{2\eta}}dx\wedge dy\,,$$

for $q = f(z)dz^2$, $q'(z) = g(z)dz^2$ and $h = e^{2\eta}|dz|^2$. In fact the integrability of that form relies on the fact that $q$ and $q'$ have at worst simple poles. We have the same result as in the closed case:

**Proposition 6.3.2.** *Given $b, b' \in \mathcal{C}_\infty(S,h)$, the following formula holds:*

$$\omega_{WP}(\Psi(b),\Psi(b')) = 2\int_S \mathrm{tr}(Jbb')\omega_h\,. \tag{6.22}$$

The computation is done as in the closed case up to a simple technical difficulty. The point is that strictly speaking if $b \in \mathcal{C}_\infty(S,h)$, then $b_q$ is only $\mathcal{C}_2(S,h)$, so in order to use the splitting $b = b_q + \mathrm{Hess}u - u\,E$ we need to extend the map $\Psi$ to $\mathcal{C}_2(S,h)$.

The reason this is possible is that the pairing

$$\mathcal{C}_\infty(S,h)\times\mathcal{K}^2(\mathfrak{p})\to\mathbb{C}\,,\quad \langle q,\Psi(b)\rangle = \int_S q\bullet\Psi(b)$$

continuously extends to a pairing $\mathcal{C}_2(S,h)\times\mathcal{K}^2(\mathfrak{p})\to\mathbb{C}$. In fact as the proof of Equation (5.22) was only local, it holds also in this singular case and we have

$$\langle q,\Psi(b)\rangle = -\int_S(\mathrm{tr}(Jb_0b_q) + i\,\mathrm{tr}(b_0b_q))\omega_h\,.$$

Moreover, the antilinear map $\mathcal{K}^2(\mathfrak{p})\to T_{X_h}\mathcal{T}(S,\mathfrak{p})$ given by the Weil-Petersson product is

$$q\mapsto\Psi(\frac{Jb_q}{2})$$

as in the closed case, which allows to recover the result. This concludes also the alternative proof of the result of Mondello ([Mon10]).

**Corollary 6.3.3.** *The Weil-Petersson symplectic form $\bar{\omega}_{WP}$ and the Goldman symplectic form $\bar{\omega}^{\mathbf{B}}$ for hyperbolic surfaces with cone points are related by:*

$$\mathbf{hol}^*(\bar{\omega}^{\mathbf{B}}) = \frac{1}{8}\bar{\omega}_{WP}\,.$$

## 6.4 Exotic structures

We now want to discuss flat Lorentzian manifolds which do not satisfy the hypothesis we considered so far.

### 6.4.1 The existence of a strictly convex Cauchy surface

Recall we are considering MGHF spacetimes on $S \times \mathbb{R}$, where $S$ is a possibly singular surface of genus $g$ and the cone angles correspond to a divisor $\boldsymbol{\beta} = \sum_{i=1}^{k} \beta_i p_i$ with $\beta_i \in (-1, 0)$. We have always assumed $\chi(S, \boldsymbol{\beta}) < 0$. In [Mes07], Mess proved that in the case of a closed surface (i.e. no cone points), every MGHF spacetime contains a strictly convex Cauchy surface provided $\chi(S) < 0$. We now show that this is not true in general if cone singularities are allowed.

*Example* 6.4.1. Let $M$ be a MGHF spacetime with a particle $s$ of angle $\theta < \pi$ which contains a uniformly convex Cauchy embedding orthogonal to the singular lines. Such spacetimes are classified in Theorem 6.C. It is easy to find another Cauchy embedding $\sigma : S \to M$ which is is flat (and thus not strictly convex) in a neighborhood of its intersection with the particle $s$. Hence there is a neighborhood $U$ of the singular point $p$ such that $\sigma(U)$ lies in the orthogonal plane to $s$ at $\sigma(x)$, so that the induced metric is Euclidean. By taking $U$ sufficiently small, we suppose $\sigma(U)$ does not intersect any other particle.

Hence we find a neighborhood $\sigma(U) \times (-\epsilon, \epsilon)$ where the metric takes the form $g_U - dt^2$, for $g_U$ is a Euclidean metric on $U$ with a cone point $p$ of angle $\theta$. We are now going to cut this neighborhood of $\sigma(U)$ and glue a germ of flat spacetime containing two cone points.

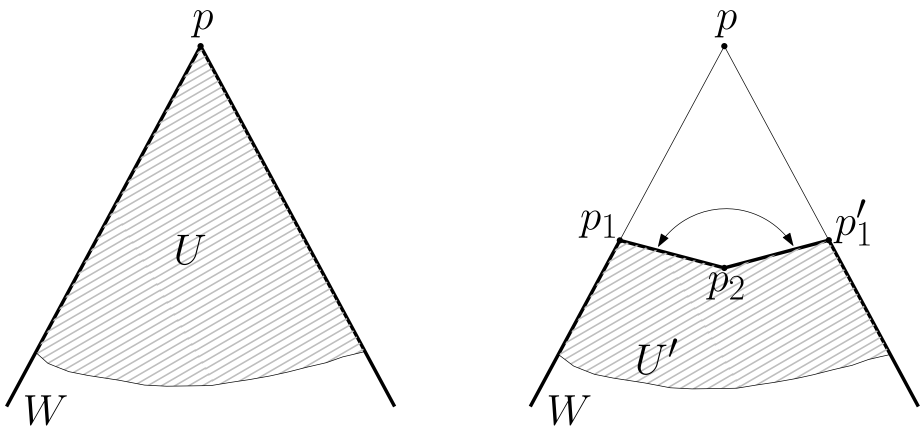



**Figure 6.3:** Replacing a neighborhood of a cone point on a Euclidean structure by a disc with two cone points. The cone angles satisfy the relation $\theta_1 + \theta_2 = 2\pi + \theta$.

Our construction is two-dimensional Euclidean geometry; see also Figure 6.3. Consider a wedge $W$ in $\mathbb{R}^2$ of angle $\theta$, which represents a model of the cone point. We choose two points $p_1$ and $p_2$ in $U$ (which will represent the new cone points), $p_1$ in the boundary of the wedge and $p_2$ in the line bisecting $W$. Let $p_1'$ be the image of $p_1$ by the rotation of angle $\theta$, namely the point identified to $p_1$ on the other edge of $W$. Connect $p_2$ to $p_1$ and $p_1'$ by geodesic segments. We remove the quadrilateral $Q = p_1pp_1'p_2$ from $W$ and glue the segments $\overline{p_1p_2}$ and $\overline{p_1'p_2}$ by a rotation around $p_2$. We keep the same gluing as before between the two edges of $W$ outside $Q$. This gives a Euclidean structure on a disc $U'$, obtained from $W \setminus Q$ by the gluing we defined, containing two cone points of angle $\theta_1$ and $\theta_2$. Observe that at least one of $\theta_1$ and $\theta_2$ has angle between $\pi$ and $2\pi$. By some simple Euclidean geometry we have that $\frac{\theta}{2\pi} - 1 = \frac{\theta_1}{2\pi} - 1 + \frac{\theta_2}{2\pi} - 1$.

We extend this operation to $U' \times (-\epsilon, \epsilon)$ in the obvious way and glue the new structure to a tubular neighborhood of $\sigma(S) \setminus \sigma(U)$ in $M \setminus (\sigma(U) \times (-\epsilon, \epsilon))$. By taking the maximal extension, we obtain a spacetime $M'$ with two particles $s_1$ and $s_2$ of angles $\theta_1$ and $\theta_2$. Notice that the Euler characteristic of $M'$ equals that of $M$ so it is negative. It is also clear that $M'$ cannot contain any strictly convex Cauchy surface, as the requirement of being orthogonal to the singularities forces a spacelike surface to be flat (i.e. the shape operator has a null eigenvalue) at some points. More precisely, $s_1$ and $s_2$ are connected by a geodesic segment entirely contained in $S$.

Observe that the holonomy of $M'$ of a path winding around both $s_1$ and $s_2$ is the same as the holonomy of $M$ of a path around $s$. Moreover, on peripheral paths around $s_1$ and $s_2$, the linear part of the holonomy of $M'$ fixes the same point in $\mathbb{H}^2$.

# Part III

# Negative constant curvature geometries

# Chapter 7

# Minimal surfaces in hyperbolic space

In this chapter we will study minimal surfaces in $\mathbb{H}^3$ in relation with the Teichmüller theory of its asymptotic boundary, which will be supposed to be a quasicircle. A special case of this setting is the lifting to the universal cover of a minimal surface in a quasi-Fuchsian manifolds (the latter were introduced in Subsection 1.2.3). Indeed, an application of the results we are going to present is obtained by restricting to the case of quasi-Fuchsian manifolds, and is the content of Section 7.D.

It was proved by Anderson ([And83, Theorem 4.1]) that for every Jordan curve $\Gamma$ in $\partial_\infty\mathbb{H}^3$ there exists a minimal embedded disc $S$ such that its boundary at infinity coincides with $\Gamma$. It can be proved that if the supremum $||\lambda||_\infty$ of the principal curvatures of $S$ is in $(-1,1)$, then $\Gamma = \partial_\infty S$ is a quasicircle. The reader can compare with [Eps86] and also Remark 8.2.2 later.

However, uniqueness does not hold in general. For instance, countexamples were constructed in the case of the lift to the universal cover of quasi-Fuchsian manifolds. Anderson proved the existence of a curve at infinity $\Gamma$ invariant under the action of a quasi-Fuchsian group (hence a quasicircle) spanning several distinct minimal embedded discs, see [And83, Theorem 5.3]. More recently in [HW13a] invariant curves spanning an arbitrarily large number of minimal discs were constructed. On the other hand, if the supremum of the principal curvatures of a minimal embedded disc $S$ satisfies $||\lambda||_\infty \in (-1,1)$, by an application of the maximum principle, then $S$ is the unique minimal disc asymptotic to the given quasicircle $\Gamma$.

The aim of Section 7.1 is to study the supremum $||\lambda||_\infty$ of the principal curvatures of a minimal embedded disc, in relation with the norm of the quasicircle at infinity, in the sense of universal Teichmüller space (as discussed in Subsection 2.3.4). The relations we obtain are interesting for "small" quasicircles, that are close in universal Teichmüller space to a circle. The main result of Section 7.1 is the following:

**Theorem 7.A.** *There exist universal constants $K_0$ and $C$ such that every minimal embedded disc in $\mathbb{H}^3$ with boundary at infinity a $K$-quasicircle $\Gamma \subset \partial_\infty(\mathbb{H}^3)$, with $K \leq K_0$, has principal curvatures bounded by*

$$||\lambda||_\infty \leq C \log K \,.$$

Since the minimal disc with prescribed quasicircle at infinity is unique if $||\lambda||_\infty < 1$, we can draw the following consequence, by choosing $K_0' < \min\{K_0, 1/C\}$:

**Corollary 7.B.** *There exists a universal constant $K_0'$ such that every $K$-quasicircle $\Gamma \subset \partial_\infty(\mathbb{H}^3)$ with $K \leq K_0'$ is the boundary at infinity of a unique minimal embedded disc.*

Moreover, in the case of quasi-Fuchsian manifold, we obtain the following Corollary.

**Corollary 7.C.** *If the Teichmüller distance between the conformal metrics at infinity of a quasi-Fuchsian manifold $M$ is smaller than a universal constant $d_0$, then $M$ is almost-Fuchsian.*

## 7.1 Minimal surfaces in $\mathbb{H}^3$

We recall here some know properties of minimal surfaces. First, the definition of minimal surface, which has been given in Subsection 1.2.1.

**Definition 7.1.1.** An embedded surface $S$ in $\mathbb{H}^3$ with shape operator $B$ is minimal if $\mathrm{tr}(B) = 0$. Equivalently, the principal curvatures are opposite to one another, and they will be denoted by $\lambda > 0$ and $-\lambda$.

The shape operator is symmetric with respect to the first fundamental form of the surface $S$; hence the condition of minimality and maximality amounts to the fact that the principal curvatures (namely, the eigenvalues of $B$) are opposite at every point.

An embedded disc in $\mathbb{H}^3$ is said to be area minimizing if any compact subdisc is locally the smallest area surface among all surfaces with the same boundary. It is well-known that area minimizing surfaces are minimal. The problem of existence for minimal surfaces with prescribed curve at infinity was solved by Anderson; see [And83] for the original source and [Cos13] for a survey on this topic.

**Theorem 7.1.2** ([And83])**.** *Given a simple closed curve $\Gamma$ in $\partial_\infty\mathbb{H}^3$, there exists a complete area minimizing embedded disc $S$ with $\partial_\infty S = \Gamma$.*

The following property is a well-known application of the maximum principle.

**Proposition 7.1.3.** *If a simple closed curve $\Gamma$ in $\partial_\infty\mathbb{H}^3$ spans a minimal disc $S$ with principal curvatures in $[-1+\epsilon, 1-\epsilon]$, then $S$ is the unique minimal surface with boundary at infinity $\Gamma$.*

A key property used in this paper is that minimal surfaces in $\mathbb{H}^3$ with boundary at infinity a Jordan curve $\Gamma$ are contained in the convex hull of $\Gamma$. Although this fact is known, we prove it here by applying maximum principle to a simple linear PDE describing minimal and maximal surfaces.

**Definition 7.1.4.** Given a curve $\Gamma$ in $\partial_\infty\mathbb{H}^3$ (or $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$), the convex hull of $\Gamma$, which we denote by $\mathcal{CH}(\Gamma)$, is the intersection of half-spaces bounded by planes $P$ such that $\partial_\infty P$ does not intersect $\Gamma$, and the half-space is taken on the side of $P$ containing $\Gamma$.

Hereafter $\operatorname{Hess} u$ denotes the Hessian of a smooth function $u$ on the surface $S$, i.e.the (1,1) tensor

$$\operatorname{Hess} u(v) = \nabla^S_v \operatorname{grad}(u).$$

Sometimes the Hessian is also considered as a (2,0) tensor, which we denote (in the rare occurrences) with

$$\nabla^2 u(v, w) = \langle \operatorname{Hess} u(v), w\rangle.$$

Finally, $\Delta_S$ denotes the Laplace-Beltrami operator of $S$, which can be defined as

$$\Delta_S u = \operatorname{tr}(\operatorname{Hess} u).$$

**Proposition 7.1.5.** *Given a minimal surface $S \subset \mathbb{H}^3$ and a plane $P$, let $u : S \to \mathbb{R}$ be the function $u(x) = \sinh d_{\mathbb{H}^3}(x, P)$, let $N$ be the unit normal to $S$ and $B = -\nabla N$ the shape operator. Then*

$$\operatorname{Hess} u - u\, E = \sqrt{1 + u^2 - ||\operatorname{grad} u||^2}B \tag{7.1}$$

*as a consequence, $u$ satisfies*

$$\Delta_S u - 2u = 0\,. \tag{L}$$

*Proof.* Consider the hyperboloid model for $\mathbb{H}^3$. Let us assume $P$ is the plane dual to the point $p \in \mathrm{d}\mathbb{S}^3$, meaning that $P = p^\perp \cap \mathbb{H}^3$. Then $u$ is the restriction to $S$ of the function $U(x) = \sinh d_{\mathbb{H}^3}(x, P) = \langle x, p\rangle$, for $x \in \mathbb{H}^3 \subset \mathbb{R}^{3,1}$. Let $N$ be the unit normal vector field to $S$; we compute $\operatorname{grad} u$ by projecting the gradient $\nabla U$ of $U$ to the tangent plane to $S$:

$$\nabla U = p + \langle p, x\rangle x \tag{7.2}$$

$$\operatorname{grad} u(x) = p + \langle p, x\rangle x - \langle p, N\rangle N \tag{7.3}$$

Now $\operatorname{Hess} u(v) = \nabla^S_v \operatorname{grad} u$, where $\nabla^S$ is the Levi-Civita connection of $S$, namely the projection of the flat connection of $\mathbb{R}^{3,1}$, and so

$$\operatorname{Hess} u(x)(v) = \langle p, x\rangle v - \langle p, N\rangle \nabla^S_v N = u(x)v + \langle \nabla U, N\rangle B(v).$$

Moreover, $\nabla U = \operatorname{grad} u + \langle \nabla U, N\rangle N$ and thus

$$\langle \nabla U, N\rangle^2 = \langle \nabla U, \nabla U\rangle - ||\operatorname{grad} u||^2 = 1 + u^2 - ||\operatorname{grad} u||^2$$

which proves (7.1). By taking the trace, (L) follows. □

**Corollary 7.1.6.** *Let $S$ be a minimal surface in $\mathbb{H}^3$, with $\partial_\infty(S) = \Gamma$ a Jordan curve. Then $S$ is contained in the convex hull $\mathcal{CH}(\Gamma)$.*

*Proof.* If $\Gamma$ is a circle, then $S$ is a totally geodesic plane which coincides with the convex hull of $\Gamma$. Hence we can suppose $\Gamma$ is not a circle. Consider a plane $P_-$ which does not intersect $\Gamma$ and the function $u$ defined as in Proposition 7.1.5, with respect to $P_-$. Suppose their mutual position is such that $u \geq 0$ in the region close to the boundary at infinity (i.e. in the complement of a compact set). If there exists some point where $u < 0$, then at a minimum point $\Delta_S u = 2u < 0$, which gives a contradiction. The proof is analogous for a plane $P_+$ on the other side of $\Gamma$, by switching the signs. Therefore every convex set containing $\Gamma$ contains also $S$. □

### 7.1.1 A sketch of the proof of Theorem 7.A

The goal of this section is to prove Theorem 7.A. The proof is divided into several steps, whose general idea is the following:

1. Given $\Psi \in QD(\mathbb{D})$, if $||\Psi||_{\mathcal{B}}$ is small, then there is a foliation of a convex subset $\mathcal{C}$ of $\mathbb{H}^3$ by equidistant surfaces, which extends to $\partial_\infty \mathbb{H}^3$ with boundary at infinity the quasicircle $\Gamma = \Psi(S^1)$. Hence the convex hull of $\Gamma$ is trapped between two parallel surfaces, whose distance is estimated in terms of $||\Psi||_{\mathcal{B}}$.

2. As a consequence of point (1), given a minimal surface $S$ in $\mathbb{H}^3$ with $\partial_\infty(S) = \Gamma$, for every point $x \in S$ there is a geodesic segment through $x$ of small length orthogonal at the endpoints to two planes $P_-$,$P_+$ which do not intersect $\mathcal{C}$. Moreover $S$ is contained between $P_-$ and $P_+$.

3. Since $S$ is contained between two parallel planes close to $x$, the principal curvatures of $S$ in a neighborhood of $x$ cannot be too large. In particular, we use Schauder theory to show that the principal curvatures of $S$ at a point $x$ are uniformly bounded in terms of the distance from $P_-$ of points in a neighborhood of $x$.

4. Finally, the distance from $P_-$ of points of $S$ in a neighborhood of $x$ is estimated in terms of the distance of points in $P_+$ from $P_-$, hence is bounded in terms of the Bers norm $||\Psi||_{\mathcal{B}}$.

It is important to remark that the estimates we give are uniform, in the sense that they do not depend on the point $x$ or on the surface $S$, but just on the Bers norm of the quasicircle at infinity. The above heuristic arguments are formalized in the following subsections.

### 7.1.2 Description from infinity

The main result of this part is the following. See Figure 7.1.

**Proposition 7.1.7.** *Let $A < 1/2$. Given an embedded minimal disc $S$ in $\mathbb{H}^3$ with boundary at infinity a quasicircle $\partial_\infty S = \Psi(S^1)$ with $||\Psi||_{\mathcal{B}} \leq A$, every point of $S$ lies on a geodesic segment of length at most* $\operatorname{arctanh}(2A)$ *orthogonal at the endpoints to two planes $P_-$ and $P_+$, such that the convex hull $\mathcal{CH}(\Gamma)$ is contained between $P_-$ and $P_+$.*

We review here some important facts on the so-called description from infinity of surfaces in hyperbolic space. For details, see [Eps84] and [KS08]. Given an embedded surface $S$ in $\mathbb{H}^3$ with bounded principal curvatures, let $I$ be its first fundamental form and $I\!I$ the second fundamental form. Recall we defined $B = -\nabla N$ its shape operator, for $N$ the oriented unit normal vector field (we fix the convention that $N$ points towards the $x_4 > 0$ direction), so that $I\!I = I(B\cdot, \cdot)$. Denote by $E$ the identity operator. Let $S_\rho$ be the $\rho$-equidistant surface from $S$ (where the sign of $\rho$ agrees with the choice of unit normal vector field to $S$). For small $\rho$, there is a map from $S$ to $S_\rho$ obtained following the geodesics orthogonal to $S$ at every point.

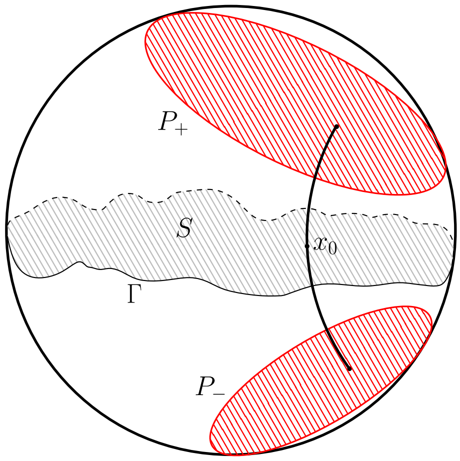



**Figure 7.1:** The statement of Proposition 7.1.7. The geodesic segment through $x_0$ has length $\leq w$, for $w = \operatorname{arctanh}(2||\Psi||_{\mathcal{B}})$, and this does not depend on $x_0 \in S$.

**Lemma 7.1.8.** *Given a smooth surface $S$ in $\mathbb{H}^3$, let $S_\rho$ be the surface at distance $\rho$ from $S$, obtained by following the normal flow at time $\rho$. Then the pull-back to $S$ of the induced metric on the surface $S_\rho$ is given by:*

$$I_\rho = I((\cosh(\rho)E - \sinh(\rho)B)\cdot, (\cosh(\rho)E - \sinh(\rho)B)\cdot)\,. \tag{7.4}$$

*The second fundamental form and the shape operator of $S_\rho$ are given by*

$$I\!I_\rho = I((-\sinh(\rho)E + \cosh(\rho)B)\cdot, (\cosh(\rho)E - \sinh(\rho)B)\cdot) \tag{7.5}$$

$$B_\rho = (\cosh(\rho)E - \sinh(\rho)B)^{-1}(-\sinh(\rho)E + \cosh(\rho)B)\,. \tag{7.6}$$

*Proof.* In the hyperboloid model, let $\sigma : \mathbb{D} \to \mathbb{H}^2$ be the minimal embedding of the surface $S$, with oriented unit normal $N$. The geodesics orthogonal to $S$ at a point $x$ can be written as

$$\gamma_x(\rho) = \cosh(\rho)\sigma(x) + \sinh(\rho)N(x)\,.$$

Then we compute

$$\begin{aligned} I_\rho(v,w) =& \langle d\gamma_x(\rho)(v), d\gamma_x(\rho)(w)\rangle \\ =& \langle \cosh(\rho)d\sigma_x(v) + \sinh(\rho)dN_x(v), \cosh(r)d\sigma_x(w) + \sinh(\rho)dN_x(w)\rangle \\ =& I(\cosh(\rho)v - \sinh(\rho)B(v), \cosh(\rho)w - \sinh(\rho)B(w))\,. \end{aligned}$$

The formula for the second fundamental form follows from the fact that $I\!I_\rho = -\frac{1}{2}\frac{dI_\rho}{d\rho}$. □

It follows that, if the principal curvatures of a minimal surface $S$ are $\lambda$ and $-\lambda$, then the principal curvatures of $S_\rho$ are

$$\lambda_\rho = \frac{\lambda - \tanh(\rho)}{1 - \lambda\tanh(\rho)} \qquad \lambda'_\rho = \frac{-\lambda - \tanh(\rho)}{1 + \lambda\tanh(\rho)}\,.$$

In particular, if $-1 \leq \lambda < 1$, then $I_\rho$ is a non-singular metric for every $\rho$ and the foliation extends to all of $\mathbb{H}^3$.

We now define the first, second and third fundamental form at infinity associated to $S$. Recall the second and third fundamental form of $S$ are $I\!I = I(B\cdot,\cdot)$ and $I\!I\!I = I(B\cdot, B\cdot)$.

$$I^* = \lim_{\rho\to\infty} 2e^{-2\rho} I_\rho = \frac{1}{2} I((E-B)\cdot,(E-B)\cdot) = \frac{1}{2}(I - 2I\!I + I\!I\!I) \tag{7.7}$$

$$B^* = (E-B)^{-1}(E+B) \tag{7.8}$$

$$I\!I^* = \frac{1}{2} I((E+B)\cdot,(E-B)\cdot) = I^*(B^*\cdot,\cdot) \tag{7.9}$$

$$I\!I\!I^* = I^*(B^*\cdot, B^*\cdot) \tag{7.10}$$

We observe that the metric $I_\rho$ and the second fundamental form can be recovered as

$$I_\rho = \frac{1}{2}e^{2\rho} I^* + I\!I^* + \frac{1}{2}e^{-2\rho} I\!I\!I^* \tag{7.11}$$

$$I\!I_\rho = -\frac{1}{2}\frac{dI_\rho}{d\rho} = \frac{1}{2} I^*((e^\rho E + e^{-\rho}B^*)\cdot,(-e^\rho E + e^{-\rho}B^*)\cdot) \tag{7.12}$$

$$B_\rho = (e^\rho E + e^{-\rho}B^*)^{-1}(-e^\rho E + e^{-\rho}B^*) \tag{7.13}$$

The following relation can be proved by some easy computation:

**Lemma 7.1.9** ([KS08, Remark 5.4 and 5.5])**.** *The embedding data at infinity* $(I^*, B^*)$ *associated to an embedded surface* $S$ *in* $\mathbb{H}^3$ *satisfy the equation*

$$tr(B^*) = -K_{I^*}\,, \tag{7.14}$$

*where* $K_{I^*}$ *is the curvature of* $I^*$*. Moreover,* $B^*$ *satisfies the Codazzi equation with respect to* $I^*$*:*

$$d^{\nabla_{I^*}} B^* = 0\,. \tag{7.15}$$

A partial converse of this fact, which can be regarded as a fundamental theorem from infinity, is the following theorem. This follows again by the results in [KS08], although it is not stated in full generality here.

**Theorem 7.1.10.** *Given a Jordan curve* $\Gamma \subset \partial_\infty \mathbb{H}^3$*, let* $(I^*, B^*)$ *be a pair of a metric in the conformal class of a connected component of* $\partial_\infty\mathbb{H}^3 \setminus \Gamma$ *and a self-adjoint* $(1,1)$*-tensor, satisfiying the conditions* (7.14) *and* (7.15) *as in Lemma 7.1.9. Assume the eigenvalues of* $B^*$ *are positive at every point. Then there exists a foliation of* $\mathbb{H}^3$ *by equidistant surfaces* $S_\rho$*, for which the first fundamental form at infinity (with respect to* $S = S_0$*) is* $I^*$ *and the shape operator at infinity is* $B^*$*.*

We want to give a relation between the Bers norm of the quasicircle $\Gamma$ and the existence of a foliation of (part of) $\mathbb{H}^3$ by equidistant surfaces with boundary $\Gamma$, containing both convex and concave surfaces. We identify $\partial_\infty\mathbb{H}^3$ to $\widehat{\mathbb{C}}$ by means of the stereographic projection, so that $\mathbb{D}$ correponds to the lower hemisphere of the sphere at infinity. The following property will be used, see [ZT87] or [KS08, Appendix A].

**Theorem 7.1.11.** *Let $\Gamma \subset \partial_\infty \mathbb{H}^3$ be a Jordan curve. If $I^*$ is the complete hyperbolic metric in the conformal class of a connected component $\Omega$ of $\partial_\infty \mathbb{H}^3 \setminus \Gamma$, and $I\!I_0^*$ is the traceless part of the second fundamental form at infinity $I\!I^*$, then $-I\!I_0^*$ is the real part of the Schwarzian derivative of the isometry $\Psi : \mathbb{D}^* \to \Omega$, namely the map $\Psi$ which uniformizes the conformal structure of $\Omega$:*

$$I\!I_0^* = -Re(S_\Psi)\,. \tag{7.16}$$

We now derive, by straightforward computation, a useful relation.

**Lemma 7.1.12.** *Let $\Gamma = \Psi(S^1)$ be a quasicircle, for $\Psi \in QD(\mathbb{D})$. If $I^*$ is the complete hyperbolic metric in the conformal class of a connected component $\Omega$ of $\partial\mathbb{H}^3 \setminus \Gamma$, and $B_0^*$ is the traceless part of the shape operator at infinity $B^*$, then*

$$\sup_{z\in\Omega} |\det B_0^*(z)| = ||\Psi||_{\mathcal{B}}^2\,. \tag{7.17}$$

*Proof.* From Theorem 7.1.11, $B_0^*$ is the real part of the holomorphic quadratic differential $-S_\Psi$. In complex conformal coordinates, we can assume that

$$I^* = e^{2\eta}|dz|^2 = \begin{pmatrix} 0 & \frac{1}{2}e^{2\eta} \\ \frac{1}{2}e^{2\eta} & 0 \end{pmatrix}$$

and $S_\Psi = h(z)dz^2$, so that

$$I\!I_0^* = -\frac{1}{2}(h(z)dz^2 + \overline{h(z)}d\bar{z}^2) = -\begin{pmatrix} \frac{1}{2}h & 0 \\ 0 & \frac{1}{2}\bar{h} \end{pmatrix}$$

and finally

$$B_0^* = (I^*)^{-1} I\!I_0^* = -\begin{pmatrix} 0 & e^{-2\eta}\bar{h} \\ e^{-2\eta}h & 0 \end{pmatrix}.$$

Therefore $|\det B_0^*(z)| = e^{-4\eta(z)}|h(z)|^2$. Moreover, by definition of Bers embedding, $\mathcal{B}([\Psi]) = S_\Psi$, because $\Psi$ is a holomorphic map from $\mathbb{D}^*$ which maps $S^1 = \partial\mathbb{D}$ to $\Gamma$. Since

$$||\Psi||_{\mathcal{B}}^2 = \sup_{z\in\Omega}(e^{-4\eta(z)}|h(z)|^2)\,,$$

this concludes the proof. □

We are finally ready to prove Proposition 7.1.7.

*Proof of Proposition 7.1.7.* Suppose again $I^*$ is a hyperbolic metric in the conformal class of $\Omega$. We can write $B^* = B_0^* + (1/2)E$, where $B_0^*$ is the traceless part of $B^*$, since $tr(B^*) = 1$ by Lemma 7.1.9. The symmetric operator $B^*$ is diagonalizable; therefore we can suppose its eigenvalues at every point are $(a+1/2)$ and $(-a+1/2)$, where $a$ is a positive number depending on the point. Hence $\pm a$ are the eigenvalues of the traceless part $B_0^*$.

By using Equation (7.17) of Lemma 7.1.12, and observing that $|\det B_0^*| = a^2$, one obtains $||\Psi||_{\mathcal{B}} = ||a||_\infty$. Since this quantity is less than $A < 1/2$ by hypothesis, at every point $a < 1/2$, and therefore $B^*$ is positive at every point.

By Theorem 7.1.10 there exists a smooth foliation of $\mathbb{H}^3$ by equidistant surfaces $S_\rho$, whose first fundamental form and shape operator are as in equations (7.11) and (7.13) above. We are going to compute

$$\rho_1 = \inf\{\rho : B_\rho \text{ is non-singular and negative definite}\}$$

and

$$\rho_2 = \sup\{\rho : B_\rho \text{ is non-singular and positive definite}\}.$$

Hence $S_{\rho_1}$ is concave and $S_{\rho_2}$ is convex; by Corollary 7.1.6, it suffices to consider $\rho_1 - \rho_2$, since a minimal surface $S$ is necessarily contained between $S_{\rho_1}$ and $S_{\rho_2}$. From the expression (7.13), the eigenvalues of $B_\rho$ are

$$\lambda_\rho = \frac{-2e^{2\rho} + (2a+1)}{2e^{2\rho} + (2a+1)}$$

and

$$\lambda'_\rho = \frac{-2e^{2\rho} + (1-2a)}{2e^{2\rho} + (1-2a)}.$$

Since $a < 1/2$, the denominators of $\lambda_\rho$ and $\lambda'_\rho$ are always positive; one has $\lambda_\rho < 0$ if and only if $e^{2\rho} > a + 1/2$, whereas $\lambda'_\rho < 0$ if and only if $e^{2\rho} > -a + 1/2$. Therefore

$$\rho_1 - \rho_2 = \frac{1}{2}\left(\log\left(A + \frac{1}{2}\right) - \log\left(-A + \frac{1}{2}\right)\right) = \frac{1}{2}\log\left(\frac{1+2A}{1-2A}\right) = \operatorname{arctanh}(2A).$$

This shows that every point $x$ on $S$ lies on a geodesic orthogonal to the leaves of the foliation, and the distance between the concave surface $S_{\rho_1}$ and the convex surface $S_{\rho_2}$, on the two sides of $x$, is less than $\operatorname{arctanh}(2A)$. □

*Remark* 7.1.13. The proof relies on the observation - given in [KS08] and expressed here implicitly in Theorem 7.1.10 - that if the shape operator at infinity is positive definite, then one reconstructs the shape operator $B_\rho$ as in Equation (7.13), and for $\rho = 0$ the principal curvatures are in $(-1, 1)$. Hence from our argument it follows that, if the Bers norm $||\Psi||_{\mathcal{B}}$ is less than $1/2$, then one finds a surface $S$ with $\partial_\infty S = \Psi(S^1)$, with principal curvatures in $(-1, 1)$. This is a special case of the results in [Eps86], where the existence of such surface is used to prove (using techniques of hyperbolic geometry) a generalization of the univalence criterion of Nehari.

### 7.1.3 Boundedness of curvature

Recall that the curvature of a minimal surface $S$ is given by $K_S = -1 - \lambda^2$, where $\pm\lambda$ are the principal curvatures of $S$. We will need to show that the curvature of a complete minimal surface $S$ is also bounded below in a uniform way, depending only on the complexity of $\partial_\infty S$. This is the content of Lemma 7.1.16.

We will use a conformal identification of $S$ with $\mathbb{D}$. Under this identification the metric takes the form $g_S = e^{2f}|dz|^2$, $|dz|^2$ being the Euclidean metric on $\mathbb{D}$. The following uniform bounds on $f$ are known (see [Ahl38]).

**Lemma 7.1.14.** *Let $g = e^{2f}|dz|^2$ be a conformal metric on $\mathbb{D}$. Suppose the curvature of $g$ is bounded above, $K_g < -\epsilon^2 < 0$. Then*

$$e^{2f} < \frac{4}{\epsilon^2(1-|z|^2)^2}\,. \tag{7.18}$$

*Analogously, if $-\delta^2 < K_g$, then*

$$e^{2f} > \frac{4}{\delta^2(1-|z|^2)^2}\,. \tag{7.19}$$

*Remark* 7.1.15. A consequence of Lemma 7.1.14 is that, for a conformal metric $g = e^{2f}|dz|^2$ on $\mathbb{D}$, if the curvature of $g$ is bounded from above by $K_g < -\epsilon^2 < 0$, then a Euclidean ball $B_0(0,R)$ of radius $R$ centered at 0 is contained in the geodesic ball of radius $R'$ centered at the same point, where $R'$ only depends from $R$. This can be checked by a simple integration argument, and $R'$ is actually obtained by multiplying $R$ for the square root of the constant in the RHS of Equation (7.18). Analogously, a lower bound on the curvature, of the form $-\delta^2 < K_g$, ensures that the geodesic ball of radius $R$ centered at 0 is contained in the conformal ball $B_0(0,R')$, where $R'$ depends on $R$ and $\delta$.

**Lemma 7.1.16.** *For every $K_0 > 1$, there exists a constant $\Lambda_0 > 0$ such that all minimal surfaces $S$ with $\partial_\infty S$ a $K$-quasicircle, $K \leq K_0$, have principal curvatures bounded by $||\lambda||_\infty < \Lambda_0$.*

We will prove Lemma 7.1.16 by giving a compactness argument. It is known that a conformal embedding $\sigma : \mathbb{D} \to \mathbb{H}^3$ is harmonic if and only if $\sigma(\mathbb{D})$ is a minimal surface, see [ES64]. The following Lemma is proved in [Cus09] in the more general case of CMC surfaces. We give a sketch of the proof here for convenience of the reader.

**Lemma 7.1.17.** *Let $\sigma_n : \mathbb{D} \to \mathbb{H}^3$ a sequence of conformal harmonic maps such that $\sigma(0) = x_0$ and $\partial_\infty(\sigma_n(\mathbb{D})) = \Gamma_n$ is a Jordan curve, $\Gamma_n \to \Gamma$ in the Hausdoff topology. Then there exists a subsequence $\sigma_{n_k}$ which converges $C^\infty$ on compact subsets to a conformal harmonic map $\sigma_\infty$ with $\partial_\infty(\sigma_\infty(\mathbb{D})) = \Gamma$.*

*Sketch of proof.* Consider the coordinates on $\mathbb{H}^3$ given by the Poincaré model, namely $\mathbb{H}^3$ is the unit ball in $\mathbb{R}^3$. Let $\sigma_n^l$, for $l = 1,2,3$, be the components of $\sigma_n$ in such coordinates. Fix $R > 0$ for the moment.

Since the curvature of the minimal surfaces $\sigma_n(\mathbb{D})$ is less than $-1$, from Lemma 7.1.14 (setting $\epsilon = 1$) and Remark 7.1.15, for every $n$ we have that $\sigma_n(B_0(0,2R))$ is contained in a geodesic ball for the induced metric of fixed radius $R'$ centered at $x_0$. In turn, the geodesic ball for the induced metric is clearly contained in the ball $B_{\mathbb{H}^3}(x_0,R')$, for the hyperbolic metric of $\mathbb{H}^3$. We remark that the radius $R'$ only depends on $R$.

We will apply standard Schauder theory (compare also similar applications in Sections 7.1.4 and 8.1.4) to the harmonicity condition

$$\Delta_0 \sigma_n^l = -\left(\Gamma^l_{jk}\circ\sigma\right)\left(\frac{\partial\sigma_i^j}{\partial x^1}\frac{\partial\sigma_i^k}{\partial x^1} + \frac{\partial\sigma_i^j}{\partial x^2}\frac{\partial\sigma_i^k}{\partial x^2}\right) =: h_n^l \tag{7.20}$$

for the Euclidean Laplace operator $\Delta_0$, where $\Gamma^l_{jk}$ are the Christoffel symbols of the hyperbolic metric in the Poincaré model.

The RHS in Equation (7.20), which is denoted by $h^l_n$, is uniformly bounded on $B_0(0,2R)$. Indeed Christoffel symbols are uniformly bounded, since $\sigma_n(B_0(0,2R))$ is contained in a compact subset of $\mathbb{H}^3$, as already remarked. The partial derivatives of $\sigma^l_n$ are bounded too, since one can observe that, if the induced metric on $S$ is $e^{2f}|dz|^2$, then $2e^{2f} = ||d\sigma||^2$, where $||d\sigma||^2$ equals:

$$\frac{4}{(1-\Sigma_i(\sigma^i_n)^2)^2}\left(\left(\frac{\partial\sigma^1_n}{\partial x}\right)^2+\left(\frac{\partial\sigma^2_n}{\partial x}\right)^2+\left(\frac{\partial\sigma^3_n}{\partial x}\right)^2+\left(\frac{\partial\sigma^1_n}{\partial y}\right)^2+\left(\frac{\partial\sigma^2_n}{\partial y}\right)^2+\left(\frac{\partial\sigma^3_n}{\partial y}\right)^2\right).$$

Hence from Lemma 7.1.14 and again the fact that $\sigma_n(B_0(0,2R))$ is contained in a compact subset of $\mathbb{H}^3$, all partial derivatives of $\sigma_n$ are uniformly bounded.

The Schauder estimate of Theorem 3.1.1 for the equation $\Delta_0\sigma^l_n = h^l_n$ give (for every $\alpha\in(0,1)$) a constant $C_1$ such that:

$$||\sigma^l_n||_{C^{1,\alpha}(B_0(0,R_1))} \leq C_1\left(||\sigma^l_n||_{C^0(B_0(0,2R))} + ||h^l_n||_{C^0(B_0(0,2R))}\right).$$

Hence one obtains uniform $C^{1,\alpha}(B_0(0,R_1))$ bounds on $\sigma^l_n$, where $R < R_1 < 2R$, and this provides $C^{0,\alpha}(B_0(0,R_1))$ bounds on $h^l_n$. Then the following estimate of Schauder-type, recall Theorem 3.1.2, provide

$$||\sigma^l_n||_{C^{2,\alpha}(B_0(0,R_2))} \leq C_2\left(||\sigma^l_n||_{C^0(B_0(0,R_1))} + ||h^l_n||_{C^{0,\alpha}(B_0(0,R_1))}\right)$$

provide $C^{2,\alpha}$ bounds on $B_0(0,R_2)$, for $R < R_2 < R_1$. By a boot-strap argument which repeats this construction, uniform $C^{k,\alpha}(B_0(0,R))$ for $\sigma^l_n$ are obtained for every $k$.

By Ascoli-Arzelà theorem, one can extract a subsequence of $\sigma_n$ converging uniformly in $C^{k,\alpha}(B_0(0,R))$ for every $k$. By applying a diagonal procedure one can find a subsequence converging $C^\infty$. One concludes the proof by a diagonal process again on a sequence of compact subsets $B_0(0,R_n)$ which exhausts $\mathbb{D}$.

The limit function $\sigma_\infty : \mathbb{D}\to\mathbb{H}^3$ is conformal and harmonic, and thus gives a parametrization of a minimal surface. It remains to show that $\partial_\infty(\sigma_\infty(\mathbb{D})) = \Gamma$. Since each $\sigma_n(\mathbb{D})$ is contained in the convex hull of $\Gamma_n$, the Hausdorff convergence on the boundary at infinity ensures that $\sigma_\infty(\mathbb{D})$ is contained in the convex hull of $\Gamma$, and thus $\partial_\infty(\sigma_\infty(\mathbb{D}))\subseteq\Gamma$.

For the other inclusion, assume there exists a point $p\in\Gamma$ which is not in the boundary at infinity of $\sigma_\infty(\mathbb{D})$. Then there is a neighborhood of $p$ which does not intersect $\sigma_\infty(\mathbb{D})$, and one can find a totally geodesic plane $P$ such that a half-space bounded by $P$ intersects $\Gamma$ (in $p$, for instance), but does not intersect $\sigma_\infty(\mathbb{D})$. But such half-space intersects $\sigma_n(\mathbb{D})$ for large $n$ and this gives a contradiction. □

*Proof of Lemma 7.1.16.* We argue by contradiction. Suppose there exists a sequence of minimal surfaces $S_n$ bounded by $K$-quasicircles $\Gamma_n$, with $K\leq K_0$, with curvature in a point $K_{S_n}(x_n)\leq -n$. Let us consider isometries $T_n$ of $\mathbb{H}^3$, so that $T_n(x_n) = x_0$.

Using the fact that the point $x_0$ is contained in the convex hull of $T_n(\Gamma_n)$ for every $n$, it is easy to see that the quasicircles $T_n(\Gamma_n)$ can be assumed to be the image of $K_0$-quasiconformal maps $\Psi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, such that $\Psi_n$ maps three points of $S^1$ (say 1, $i$ and $-1$) to points of $T_n(\Gamma_n)$ at uniformly positive distance from one another. By the compactness property in Theorem 2.3.2, there exists a subsequence $T_{n_k}(\Gamma_{n_k})$ converging to a $K$-quasicircle $\Gamma_\infty$, with $K \leq K_0$. By Lemma 7.1.17, the minimal surfaces $T_{n_k}(S_{n_k})$ converge $C^\infty$ on compact subsets (up to a subsequence) to a smooth minimal surface $S_\infty$ with $\partial_\infty(S_\infty) = \Gamma_\infty$. Hence the curvature of $T_{n_k}(S_{n_k})$ at the point $x_0$ converges to the curvature of $S_\infty$ at $x_0$. This contradicts the assumption that the curvature at the points $x_n$ goes to infinity. $\square$

It follows that the curvature of $S$ is bounded by $-\delta^2 < K_S < -\epsilon^2$, where $\delta$ is some constant, whereas we can take $\epsilon = 1$.

*Remark* 7.1.18. The main result of this section, Theorem 7.A, is indeed a quantitative version of Lemma 7.1.16, which gives a control of how an optimal constant $\Lambda_0$ would vary if $K_0$ is chosen close to 0.

### 7.1.4 Schauder estimates

By using equation (7.1), we will eventually obtain bounds on the principal curvatures of $S$. For this purpose, we need bounds on $u = \sinh d_{\mathbb{H}^3}(\cdot, P_-)$ and its derivatives. Schauder theory plays again an important role: since $u$ satisfies the equation

$$\Delta_S u - 2u = 0\,. \tag{L}$$

we will use uniform estimates of the form

$$||u||_{C^2(B_0(0,\frac{R}{2}))} \leq C||u||_{C^0(B_0(0,R))}$$

for the function $u$, written in a suitable coordinate system. This will follow from Theorem 3.1.3. The main difficulty is basically to show that the operators

$$u \mapsto \Delta_S u - 2u$$

are strictly elliptic and have uniformly bounded coefficients.

**Proposition 7.1.19.** *Let $K_0 > 1$ and $R > 0$ be fixed. There exist a constant $C > 0$ (only depending on $K_0$ and $R$) such that for every choice of:*

- *A minimal embedded disc $S \subset \mathbb{H}^3$ with $\partial_\infty S$ a $K$-quasicircle, with $K \leq K_0$;*
- *A point $x \in S$;*
- *A plane $P_-$;*

*the function $u(\cdot) = d_{\mathbb{H}^3}(\cdot, P_-)$ expressed in terms of normal coordinates of $S$ centered at $x$, namely*

$$u(z) = \sinh d_{\mathbb{H}^3}(\exp_x(z), P_-)$$

*where $\exp_x : \mathbb{R}^2 \cong T_xS \to S$ denotes the exponential map, satisfies the Schauder-type inequality*

$$||u||_{C^2(B_0(0,\frac{R}{2}))} \leq C||u||_{C^0(B_0(0,R))}\,. \tag{7.21}$$

*Proof.* This will be again an argument by contradiction, using the compactness property.

Suppose our assertion is not true, and find a sequence of minimal surfaces $S_n$ with $\partial_\infty(S_n) = \Gamma_n$ a $K$-quasicircle ($K \le K_0$), a sequence of points $x_n \in S_n$, and a sequence of planes $P_n$ as in the third hypothesis, such that the functions $u_n(z) = \sinh d_{\mathbb{H}^3}(\exp_{x_n}(z), P_n)$ have the property that

$$||u_n||_{C^2(B_0(0,\frac{R}{2}))} \ge n||u||_{C^0(B_0(0,R))} .$$

We can compose with isometries $T_n$ of $\mathbb{H}^3$ so that $T_n(x_n) = x_0$ for every $n$ and the tangent plane to $T_n(S_n)$ at $x_0$ is a fixed plane. Let $S'_n = T_n(S_n)$, $\Gamma'_n = T_n(\Gamma_n)$ and $P'_n = T_n(P_n)$. Note that $\Gamma'_n$ are again $K$-quasicircles, for $K \le K_0$, and the convex hull of each $\Gamma'_n$ contains $x_0$.

Using this fact, it is then easy to see - as in the proof of Lemma 7.1.16 - that one can find $K_0$-quasiconformal maps $\Psi_n$ such that $\Psi_n(S^1) = \Gamma'_n$ and $\Psi_n(1)$, $\Psi_n(i)$ and $\Psi_n(-1)$ are at uniformly positive distance from one another. Therefore, using Theorem 2.3.2 there exists a subsequence of $\Psi_n$ converging uniformly to a $K_0$-quasiconformal map. This gives a subsequence $\Gamma'_{n_k}$ converging to $\Gamma'_\infty$ in the Hausdorff topology.

By Lemma 7.1.17, considering $S'_n$ as images of conformal harmonic embeddings $\sigma'_n : \mathbb{D} \to \mathbb{H}^3$, we find a subsequence of $\sigma'_{n_k}$ converging $C^\infty$ on compact subsets to the conformal harmonic embedding of a minimal surface $S'_\infty$. Moreover, by Lemma 7.1.16 and Remark 7.1.15, the convergence is also $C^\infty$ on the image under the exponential map of compact subsets containing the origin of $\mathbb{R}^2$.

It follows that the coefficients of the Laplace-Beltrami operators $\Delta_{S'_n}$ on a Euclidean ball $B_0(0, R)$ of the tangent plane at $x_0$, for the coordinates given by the exponential map, converge to the coefficients of $\Delta_{S'_\infty}$. Therefore the operators $\Delta_{S'_n} - 2$ are uniformly strictly elliptic with uniformly bounded coefficients. Using these two facts, one can apply Schauder estimates (Theorem 3.1.3) to the functions $u_n$, which are solutions of the equations $\Delta_{S'_n}(u_n) - 2u_n = 0$. See again for a reference. We deduce that there exists a constant $c$ such that

$$||u_n||_{C^2(B_0(0,\frac{R}{2}))} \le c||u_n||_{C^0(B_0(0,R))}$$

for all $n$, and this gives a contradiction. □

### 7.1.5 Principal curvatures

We can now proceed to complete the proof of Theorem 7.A. Fix some $w > 0$. We know from Section 7.1.2 that if the Bers norm is smaller than the constant $(1/2)\tanh(w)$, then every point $x$ on $S$ lies on a geodesic segment $l$ orthogonal to two planes $P_-$ and $P_+$ at distance $d_{\mathbb{H}^3}(P_-, P_+) < w$. Obviously the distance is achieved along $l$.

Fix a point $x \in S$. Denote again $u = \sinh d_{\mathbb{H}^3}(\cdot, P_-)$. By Proposition 7.1.19, first and second partial derivatives of $u$ in normal coordinates on a geodesic ball $B_S(x, R)$ of fixed radius $R$ are bounded by $C||u||_{C^0(B_S(x,R))}$. The last step for the proof is an estimate of the latter quantity in terms of $w$.

We first need a simple lemma which controls the distance of points in two parallel planes, close to the common orthogonal geodesic. Compare Figure 7.2.

**Lemma 7.1.20.** *Let $p \in P_-$, $q \in P_+$ be the endpoints of a geodesic segment $l$ orthogonal to $P_-$ and $P_+$ of length $w$. Let $p' \in P_-$ a point at distance $r$ from $p$ and let $d = d_{\mathbb{H}^3}((\pi|_{P_+})^{-1}(p'), P_-)$. Then*

$$\tanh d = \cosh r \tanh w \tag{7.22}$$

$$\sinh d = \cosh r \frac{\sinh w}{\sqrt{1 - (\sinh r)^2 (\sinh w)^2}} \,. \tag{7.23}$$

*Proof.* This is easy (2-dimensional) hyperbolic trigonometry; however we give a short proof as this formula will be extended to the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ context later on. In the hyperboloid model, we can assume $P_-$ is the plane $x_3 = 0$, $p = (0, 0, 0, 1)$ and the geodesic $l$ is given by $l(t) = (0, 0, \sinh t, \cosh t)$. Hence $P_+$ is the plane orthogonal to $l'(w) = (0, 0, \cosh w, \sinh w)$ passing through $l(w) = (0, 0, \sinh w, \cosh w)$. The point $p'$ has coordinates

$$p' = (\cos\theta \sinh r, \sin\theta \sinh r, 0, \cosh r)$$

and the geodesic $l_1$ orthogonal to $P_-$ through $p'$ is given by

$$l_1(d) = (\cosh d)(p') + (\sinh d)(0, 0, 1, 0) \,.$$

We have $l_1(d) \in P_+$ if and only if $\langle l_1(d), l'(w)\rangle = 0$, which is satisfied for

$$\tanh d = \cosh r \tanh w \,,$$

provided $\cosh r \tanh w < 1$. The second expression follows straightforwardly. $\square$

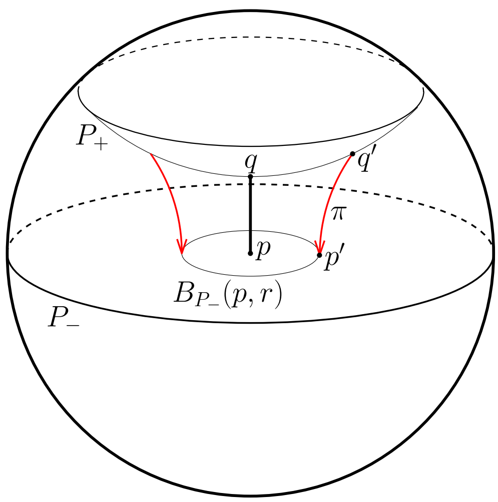



**Figure 7.2:** The setting of Lemma 7.1.20. Here $d_{\mathbb{H}^3}(p, p') = r$ and $q' = (\pi|_{P_+})^{-1}(p')$.

We are finally ready to prove Theorem 7.D. The key point for the proof is that all the quantitative estimates previously obtained in this section are independent on the point $x \in S$.

**Theorem 7.D.** *There exist constants $K_0 > 1$ and $C > 4$ such that the principal curvatures $\pm\lambda$ of every minimal surface $S$ in $\mathbb{H}^3$ with $\partial_\infty S = \Gamma$ a $K$-quasicircle, with $K \leq K_0$, are bounded by:*

$$||\lambda||_\infty \leq \frac{C||\Psi||_{\mathcal{B}}}{\sqrt{1 - C||\Psi||^2_{\mathcal{B}}}} \tag{7.24}$$

*where $\Gamma = \Psi(S^1)$, for $\Psi \in QD(\mathbb{D})$.*

*Proof.* Fix $K_0 > 1$. Let $S$ a minimal surface with $\partial_\infty S$ a $K$-quasicircle, $K \leq K_0$. Let $x \in S$ an arbitrary point on a minimal surface $S$. By Proposition 7.1.7, we find two planes $P_-$ and $P_+$ whose common orthogonal geodesic passes through $x$, and has length $w = \operatorname{arctanh}(2||\Psi||_{\mathcal{B}})$.

Now fix $R > 0$. By Proposition 7.1.19, applied to the point $x$ and the plane $P_-$, we obtain that the first and second derivatives of the function

$$u = \sinh d_{\mathbb{H}^3}(\exp_x(\cdot), P_-)$$

on a geodesic ball $B_S(x, R/2)$ for the induced metric on $S$, are bounded by the supremum of $u$ itself, on the geodesic ball $B_S(x, R)$, multiplied by a universal constant $C = C(K_0, R)$.

Let $\pi : \mathbb{H}^3 \to P_-$ the orthogonal projection to the plane $P_-$. The map $\pi$ is contracting distances, by negative curvature in the ambient manifold. Hence $\pi(B_S(x, R))$ is contained in $B_{P_-}(\pi(x), R)$. Moreover, since $S$ is contained in the region bounded by $P_-$ and $P_+$, clearly $\sup\{u(x) : x \in B_S(0, R)\}$ is less than the hyperbolic sine of the distance of points in $(\pi|_{P_+})^{-1}(B_{P_-}(\pi(x), R))$ from $P_-$. See Figure 7.3.

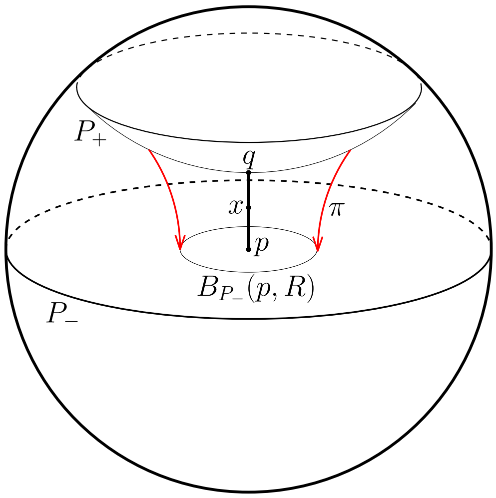



**Figure 7.3:** Projection to a plane $P_-$ in $\mathbb{H}^3$ is distance contracting. The dash-dotted ball schematically represents a geodesic ball of $\mathbb{H}^3$.

Hence, using Proposition 7.1.20 (in particular Equation (7.23)), we get:

$$||u||_{C^0(B_S(x,R))} \leq \cosh R \frac{\sinh w}{\sqrt{1 - (\sinh R)^2 (\sinh w)^2}}\,, \tag{7.25}$$

where we recall that $w = \operatorname{arctanh}(2||\Psi||_{\mathcal{B}})$.

We finally give estimates on the principal curvatures of $S$, in terms of the complexity of $\partial_\infty(S) = \Psi(S^1)$. We compute such estimate only at the point $x \in S$; by the independence of all the above construction from the choice of $x$, the proof will be concluded. From Equation (7.1), we have

$$B = \frac{1}{\sqrt{1 + u^2 - ||\operatorname{grad} u||^2}} (\operatorname{Hess} u - u\,E)\,.$$

Moreover, in normal coordinates centered at the point $x$, the expression for the Hessian and the norm of the gradient at $x$ are just

$$(\operatorname{Hess} u)_i^j = \frac{\partial^2 u}{\partial x^i \partial x^j}\,, \qquad ||\operatorname{grad} u||^2 = \left(\frac{\partial u}{\partial x^1}\right)^2 + \left(\frac{\partial u}{\partial x^2}\right)^2 .$$

It then turns out that the principal curvatures $\pm\lambda$ of $S$, i.e. the eigenvalues of $B$, are bounded by

$$|\lambda| \leq \frac{C_1 ||u||_{C^0(B_S(x,R))}}{\sqrt{1 - C_1 ||u||^2_{C^0(B_S(x,R))}}}\,. \tag{7.26}$$

The constant $C_1$ involves the constant $C$ of Equation (7.21) in the statement of Proposition 7.1.19. Substituting Equation (7.25) into Equation (7.26), with some manipulation one obtains

$$||\lambda||_\infty \leq \frac{C_1 (\cosh R)(\tanh w)}{\sqrt{1 - (1 + C_1)(\cosh R)^2 (\tanh w)^2}}\,. \tag{7.27}$$

On the other hand $\tanh w = 2||\Psi||_{\mathcal{B}}$. Upon relabelling $C$ with a larger constant, the inequality

$$||\lambda||_\infty \leq \frac{C||\Psi||_{\mathcal{B}}}{\sqrt{1 - C||\Psi||^2_{\mathcal{B}}}}$$

is obtained. □

*Remark* 7.1.21. Actually, the statement of Theorem 7.D is true for any choice of $K_0 > 1$ (and the constant $C$ varies accordingly with the choice of $K_0$). However, the estimate in Equation (7.24) does not make sense when $||\Psi||^2 \geq 1/C$. Indeed, our procedure seems to be quite uneffective when the quasicircle at infinity is "far" from being a circle - in the sense of universal Teichmüller space. Applying Theorem 2.3.15, this possibility is easily ruled out, by replacing $K_0$ in the statement of Theorem 7.D with a smaller constant.

Observe that the function $x \mapsto Cx/\sqrt{1 - Cx^2}$ is differentiable with derivative $C$ at $x = 0$. As a consequence of Theorem 2.3.15, there exists a constant $L$ (with respect to the statement of Theorem 2.3.15 above, $L = 1/b_1$) such that $||\Psi||_{\mathcal{B}} \leq L d_{\mathcal{T}}([\Psi], [\mathrm{id}])$ if $d_{\mathcal{T}}([\Psi], [\mathrm{id}]) \leq r$ for some small radius $r$. Then the proof of Theorem 7.A follows, replacing the constant $C$ by a larger constant if necessary.

**Theorem 7.A.** *There exist universal constants $K_0$ and $C$ such that every minimal embedded disc in $\mathbb{H}^3$ with boundary at infinity a $K$-quasicircle $\Gamma \subset \partial_\infty \mathbb{H}^3$, with $K \leq K_0$, has principal curvatures bounded by*

$$||\lambda||_\infty \leq C \log K\,.$$

*Remark* 7.1.22. With the techniques used in this paper, it seems difficult to give explicit estimates for the best possible value of the constant $C$ of Theorem 7.A. In our argument, this constant actually depends on several choices, one of which is the choice of the radius $R$ in Subsection 7.1.4 (see Proposition 7.1.19).

## 7.2 A consequence for quasi-Fuchsian manifolds

A quasi-Fuchsian manifold contaning a closed minimal surface with principal curvatures in $(-1, 1)$ is called almost-Fuchsian, according to the definition given in [KS07]. The minimal surface in an almost-Fuchsian manifold is unique, as first observed by Uhlenbeck ([Uhl83]). Hence, applying Theorem 7.A to the case of quasi-Fuchsian manifolds, the following Corollary is proved.

**Corollary 7.C.** *If the Teichmüller distance between the conformal metrics at infinity of a quasi-Fuchsian manifold $M$ is smaller than a universal constant $d_0$, then $M$ is almost-Fuchsian.*

*Proof.* Choose $d_0 = (1/2)\log K_0'$ where $K_0'$ is the universal constant of Corollary 7.B. (see the beginning of this chapter). Under the hypothesis, the Teichmüller map from one hyperbolic end of $M$ to the other is $K$-quasiconformal for $K \leq K_0'$, hence the lift to the universal cover $\mathbb{H}^3$ of any closed minimal surface in $M$ is a minimal embedded disc with boundary at infinity a $K$-quasicircle (again with $K \leq K_0'$), namely the limit set of the corresponding quasi-Fuchsian group. It follows from Theorem 7.A that the principal curvatures of such closed minimal surface are in $(-1, 1)$. □

*Remark* 7.2.1. We remark that Theorem 7.A, when restricted to the case of quasi-Fuchsian manifolds, is a partial converse of results presented in [GHW10], giving a bound on the Teichmüller distance between the hyperbolic ends of an almost-Fuchsian manifold in terms of the maximum of the principal curvatures. Another invariant which has been studied in relation with the properties of minimal surfaces in hyperbolic space is the Hausdorff dimension of the limit set. Theorem 7.A and Corollary 7.C can be compared with the following Theorem given in [San14]: for every $\epsilon$ and $\epsilon_0$ there exists a constant $\delta = \delta(\epsilon, \epsilon_0)$ such that any stable minimal surface with injectivity radius bounded by $\epsilon_0$ in a quasi-Fuchsian manifold $M$ are in $(-\epsilon, \epsilon)$ provided the Hausdorff dimension of the limit set of $M$ is at most $1 + \delta$. In particular, $M$ is almost Fuchsian if one chooses $\epsilon < 1$. Conversely, in [HW13b] the authors give an estimate of the Hausdorff dimension of the limit set in an almost-Fuchsian manifold $M$ in terms of the maximum of the principal curvatures of the (unique) minimal surface.

# Chapter 8

# Maximal surfaces in Anti-de Sitter space

In this chapter we will discuss an application of similar techniques of Chapter 7 to Anti-de Sitter geometry. Consider a maximal surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with boundary at infinity the graph of a quasisymmetric homeomorphism $\phi : S^1 \to S^1$. We will prove the following inequality for the supremum of the principal curvatures of a maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with boundary at infinity the graph of a quasisymmetric homeomorphism $\phi : S^1 \to S^1$.

$$||\lambda||_\infty \leq C(\sinh ||\phi||_{cr}/2) \tag{8.1}$$

The cross-ratio norm $||\phi||_{cr}$ of $\phi$ was introduced in 2.3.2. To some extent, this result can be considered the analogue of Theorem 7.A in the Anti-de Sitter setting. The proof of the inequality (8.1) is the content of Section 8.1.

Recall that a maximal surface $S$ provides a minimal Lagrangian extension $\Phi : \mathbb{D} \to \mathbb{D}$ of $\phi$. Such minimal Lagrangian map is obtained by composition $(\Phi_l)^{-1} \circ \Phi_r$ of the left and right projections $\Phi_l, \Phi_r : S \to \mathbb{H}^2$, defined in Subsection 1.3.1. This observation is used by Bonsante and Schlenker to prove the existence and uniqueness of a minimal Lagrangian quasiconformal extension to the disc of any quasisymmetric homeomorphism of $S^1$. In Section 8.2 we then apply inequality (8.1) to obtain an estimate of the maximal dilatation of the minimal Lagrangian extension of $\phi : S^1 \to S^1$ which only depends on the cross-ratio norm of $\phi$.

**Theorem 8.A.** *There exist universal constants $\delta$ and $C$ such that, for any quasisymmetric homeomorphism $\phi$ of $S^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian quasiconformal extension $\Phi : \mathbb{D} \to \mathbb{D}$ has maximal distortion $K(\Phi)$ bounded by the relation*

$$\log K(\Phi) < C||\phi||_{cr} .$$

As for Theorem 7.A, the proof is composed of several steps. Given a convex subset $\mathcal{C}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ spanning a curve in the boundary at infinity, the *width* of $\mathcal{C}$ is defined as the supremum of the lenghts of all timelike curves entirely contained in $\mathcal{C}$ (see Definition 8.1.8). The first step towards the proof is the following relation:

**Proposition 8.B.** *Given any quasisymmetric homeomorphism $\phi$, let $w$ be the width of the convex hull of the graph of $\phi$ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$. Then*

$$\tanh\left(\frac{||\phi||_{cr}}{4}\right) \leq \tan(w) \leq \sinh\left(\frac{||\phi||_{cr}}{2}\right).$$

The second part is a use of Schauder estimates, as in the hyperbolic case, to provide bounds on the principal curvatures of the maximal surface $S$.

**Theorem 8.C.** *There exists a constant $C$ such that, for every maximal surface $S$ with bounded principal curvatures $\pm\lambda$ and width $w = w(\mathcal{CH}(\partial_\infty S))$,*

$$||\lambda||_\infty \leq C \tan w\,.$$

The differential of the minimal Lagrangian extension of $\phi$ can be expressed (as noted in [BS10] and [KS07]) in terms of the shape operator of $S$. This fact is used in Section 8.2 to finally obtain the proof of Theorem 8.A.

By applying opposite estimates at each step, we obtain also an opposite inequality, giving a uniform bound from below of the maximal dilatation of the quasiconformal minimal Lagrangian extension.

**Theorem 8.D.** *There exist universal constants $\delta$ and $C_0$ such that, for any quasisymmetric homeomorphism $\phi$ of $S^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian quasiconformal extension $\Phi : \mathbb{D} \to \mathbb{D}$ has maximal dilatation $K(\Phi)$ bounded by the relation*

$$C_0 ||\phi||_{cr} \leq \log K(\Phi)\,.$$

*The constant $C_0$ can be taken arbitrarily close to* $1/2$.

Although investigation of the best value of the constant $C$ in Theorem 8.A was not pursued in this work, this shows that $C$ cannot be taken smaller than $1/2$.

## 8.1 Maximal surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$

The definition of maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is the analogue of Definiton 8.1.1 for $\mathbb{H}^3$.

**Definition 8.1.1.** An embedded surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with shape operator $B$ is maximal if $\mathrm{tr}(B) = 0$.

An existence result for maximal surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ was given by Bonsante and Schlenker.

**Theorem 8.1.2** ([BS10])**.** *Given a spacelike curve $\Gamma$ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$, there exists a complete maximal embedded disc $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ such that $\partial_\infty S = \Gamma$.*

Moreover, when the curve at infinity $\Gamma$ is the graph of a quasisymmetric homeomorphism (see Definition 2.3.3 below), boundedness of curvature and uniqueness were proved.

**Theorem 8.1.3** ([BS10]). *Given a quasisymmetric homeomorphism $\phi : S^1 \to S^1$, there exists a unique maximal embedded compression disc $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with bounded principal curvatures such that $\partial_\infty S = gr(\phi)$. Moreover, the principal curvatures are in $[-1+\epsilon, 1-\epsilon]$ for some $\epsilon > 0$.*

*Remark* 8.1.4. A consequence of the results proved in [BS10] is that the maximal surface $S$ with bounded principal curvatures, spanning the graph of a quasisymmetric homeomorphism, is complete. In fact, there is a bi-Lipschitz homeomorphism from $S$ to $\mathbb{H}^2$, and $\mathbb{H}^2$ is complete. Such homeomorphism is described also in Subsection 8.2.

The following proposition can be proved in a very similar fashion of Proposition 7.1.5, with little adaptations to the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ case. Compare also [BS10, Lemma 4.1] and the proof of Lemma 8.1.15 below.

**Proposition 8.1.5.** *Given a maximal surface $S \subset \mathbb{A}\mathrm{d}\mathbb{S}^3$ and a plane $P$, let $u : S \to \mathbb{R}$ be the function $u(x) = \sin d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P)$, let $N$ be the future unit normal to $S$ and $B = \nabla N$ the shape operator. Then*

$$\mathrm{Hess} u - u\mathrm{I} = \sqrt{1 - u^2 + ||\,\mathrm{grad}\, u||^2} B \tag{8.2}$$

*as a consequence, $u$ satisfies*

$$\Delta_S u - 2u = 0\,. \tag{L}$$

This is an important property to show that a maximal surface with boundary at infinity a weakly spacelike curve $\Gamma$ (the graph of a homeomorphism of $S^1$) is contained in the convex hull of $\Gamma$. We give here the definition of convex hull, completely similar to Definition 7.1.4.

**Definition 8.1.6.** Given a weakly spacelike curve $\Gamma$ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$, the convex hull of $\Gamma$, which we denote by $\mathcal{CH}(\Gamma)$, is the intersection of half-spaces bounded by planes $P$ such that $\partial_\infty P$ does not intersect $\Gamma$, and the half-space is taken on the side of $P$ containing $\Gamma$.

It can be proved that the convex hull of $\Gamma$, which is well-defined in $\mathbb{R}P^3$, is contained in $\mathbb{A}\mathrm{d}\mathbb{S}^3 \cup \partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$.

**Corollary 8.1.7.** *Let $S$ be a minimal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, with $\partial_\infty(S) = \Gamma$ a graph. Then $S$ is contained in the convex hull $\mathcal{CH}(\Gamma)$.*

It will also be important to use the notion of width of the convex hull, as defined in [BS10]. We introduce the definiton and give a short discussion about its properties, which will be of use in the following.

**Definition 8.1.8.** Given a homeomorphism $\phi : S^1 \to S^1$, we define the width of the convex hull $\mathcal{CH}(gr(\phi))$ as the supremum of the lenght of a timelike geodesic contained in $\mathcal{CH}(gr(\phi))$.

*Remark* 8.1.9. Recall from the Preliminaries that for totally geodesic spacelike plane $Q$, time distances in $\mathbb{A}\mathrm{d}\mathbb{S}^3 \backslash Q$ (which we denote bt $d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}$) satisfy the inverse triangular inequality and the distance between two points $p$ and $q \in \mathrm{I}^+(p)$ is achieved along the geodesic line passing through $p$ and $q$. The width can be defined as (setting $\mathcal{C} = \mathcal{CH}(gr(\phi))$)

$$w(\mathcal{CH}(gr(\phi))) = \sup_{p \in \partial_- \mathcal{C}, q \in \partial_+ \mathcal{C}} d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p,q) = \sup_\gamma \int ||\dot\gamma||_{\mathbb{A}\mathrm{d}\mathbb{S}^3}. \tag{8.3}$$

where the supremum in the RHS is taken over all timelike curves $\gamma$ connecting $\partial_-\mathcal{C}$ and $\partial_+\mathcal{C}$. In particular, we note that

$$w(\mathcal{C}) = \sup_{x \in \mathcal{C}} \left( d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, \partial_- \mathcal{C}) + d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, \partial_+ \mathcal{C}) \right). \tag{8.4}$$

To stress once more the meaning of this equality, note that the supremum in (8.4) cannot be achieved on a point $x$ such that the two segments realizing the distance from $x$ to $\partial_-\mathcal{C}$ and $\partial_+\mathcal{C}$ are not part of a unique geodesic line. Indeed, if at $x$ the two segments form an angle, the piecewise geodesic can be made longer by avoiding the point $x$, as in Figure 8.1. We also remark that if the distance between a point $x$ and $\partial_\pm\mathcal{C}$ is achieved along a geodesic segment $l$, then the maximality condition imposes that $l$ must be orthogonal to a support plane to $\partial_\pm\mathcal{C}$ at $\partial_\pm\mathcal{C} \cap l$.

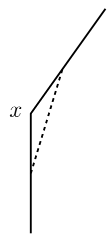



**Figure 8.1:** A path through $x$ which is not geodesic does not achieve the maximum distance.

### 8.1.1 A sketch of the proof of the inequality (8.1)

Again, the proof is divided into several steps, in a similar way to the hyperbolic case treated in the previous section. We resume here the main steps:

1. Given a quasisymmetric homeomorphism $\phi \in \mathcal{T}(\mathbb{D})$, we can estimate the width $w = w(\mathcal{CH}(gr(\phi)))$ in terms of the cross-ratio norm $||\phi||_{cr}$.

2. Given a maximal surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $\partial_\infty(S) = gr(\phi)$, for every point $x \in S$ there are two geodesic timelike segments starting from $x$ orthogonal to two planes $P_-$,$P_+$ which do not intersect $\mathcal{CH}(gr(\phi))$; the sum of the lengths of the two segments is less than the width $w$ of $\mathcal{CH}(gr(\phi))$. Moreover $S$ is contained between $P_-$ and $P_+$.

3. Since $S$ is contained between two disjoint planes close to $x$, the principal curvatures of $S$ in a neighborhood of $x$ cannot be too large. In particular, we use Schauder theory to show that the principal curvatures of $S$ at a point $x$ are bounded in terms of the distance from $P_-$ of points in a neighborhood of $x$.

4. The distance from $P_-$ of points in a neighborhood of $x$ is estimated in terms of the width $w$.

5. Finally, we estimate the quasiconformal coefficient of the minimal Lagrangian extension of $\phi$ in terms of the principal curvatures of $S$.

### 8.1.2 Cross-ratio norm and width

In this subsection, we will prove a relation between the cross-ratio norm of a quasisymmetric homeomorphism $\phi$ and the width $w(\mathcal{CH}(gr(\phi)))$.

**Proposition 8.B.** *Given any quasisymmetric homeomorphism $\phi$, let $w = w(\mathcal{CH}(gr(\phi)))$ the width of the convex hull of $gr(\phi)$. Then*

$$\tanh\left(\frac{||\phi||_{cr}}{4}\right) \leq \tan(w) \leq \sinh\left(\frac{||\phi||_{cr}}{2}\right). \tag{8.5}$$

*Proof.* We first prove the upper bound on the width. Suppose the width of the convex hull $\mathcal{C}$ of $gr(\phi)$ is $w \in (0, \pi/2)$; let $k = ||\phi||_{cr}$. We can find a sequence of pairs $(p_n, q_n)$ such that $d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p_n, q_n) \nearrow w$, with $p_n \in \partial_-\mathcal{C}$, $q_n \in \partial_+\mathcal{C}$. We can assume the geodesic connecting $p_n$ and $q_n$ is orthogonal to $\partial_-\mathcal{C}$ at $p_n$; indeed one can replace $p_n$ with a point in $\partial_-\mathcal{C}$ which maximizes the distance from $q_n$, if necessary. Let us now apply isometries $T_n$ so that $T_n(p_n) = p = [\hat{p}] \in \mathbb{A}\mathrm{d}\mathbb{S}^3$, for $\hat{p} = (0, 0, 1, 0) \in \widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$, and $T_n(q_n)$ lies on the timelike geodesic through $p$ orthogonal to $P_- = (0, 0, 0, 1)^\perp$.

The curve at infinity $gr(\phi)$ is mapped by $T_n$ to a curve $gr(\phi_n)$, where $\phi_n$ is obtained by pre-composing and post-composing $\phi$ with Möbius transformations (this is easily seen from the description of $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ as $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$). Hence $\phi_n$ is still quasisymmetric with norm $||\phi_n||_{cr} = ||\phi||_{cr} = k$.

It is easy to see that $\phi_n$ cannot converge to a map sending the complement of a point in $\mathbb{R}P^1$ to a single point of $\mathbb{R}P^1$. Indeed, the curves $gr(\phi_n)$ are all contained between $P_-$ and a spacelike plane $P_n$ disjoint from $P_-$, which contains the point $T_n(q_n)$. Moreover the distance of $p$ from $T_n(q_n) \in P_n$ is at most $w$. This shows that the curves $gr(\phi_n)$ all lie in a bounded region in an affine chart of $\mathbb{A}\mathrm{d}\mathbb{S}^3$; this would not be the case if $\phi_n$ were converging on the complement of one point to a constant map. See Figure 8.2.

Hence, by the convergence property of $k$-quasisymmetric homeomorphisms (Theorem 2.3.5), $\phi_n$ converges to a $k$-quasisymmetric homeomorphism $\phi_\infty$, so that $w = w(\mathcal{CH}(gr(\phi_\infty)))$. Denote $\mathcal{C}_\infty = \mathcal{CH}(gr(\phi_\infty))$.

We will mostly refer to the coordinates in the affine chart $\{x^3 \neq 0\}$, namely $(x, y, z) = (x^1/x^3, x^2/x^3, x^4/x^3)$. Our assumption is that the point $p$ has coordinates $(0, 0, 0)$ and $P_- = \{(x, y, 0) : x^2 + y^2 < 1\}$ is the totally geodesic plane through $p$ which is a support plane for $\partial_-\mathcal{C}_\infty$. The geodesic line $l$ through $p$ orthogonal to $P_-$ is $\{(0, 0, z)\}$. By construction, the width of $\mathcal{C}_\infty$ equals $d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p, q)$, where $q = (0, 0, h) = l \cap \partial_+\mathcal{C}_\infty$. It is then an easy computation to show that $h = \tan w$. Hence the plane $P_+ = \{(x, y, h) : x^2 + y^2 < 1 + h^2\}$, which is the plane orthogonal to $l$ through $q$, is a support plane for $\partial_+\mathcal{C}_\infty$. See Figure 8.3.

Since $\partial_-\mathcal{C}_\infty$ and $\partial_+\mathcal{C}_\infty$ are pleated surfaces, $\partial_-\mathcal{C}_\infty$ contains an ideal triangle $T_-$, such that $p \in T_-$ (possibly $p$ is on the boundary of $T_-$). The ideal triangle might

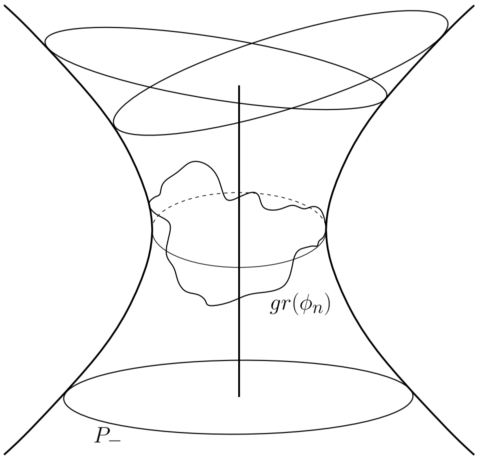



**Figure 8.2:** The curves $gr(\phi_n)$ are contained in a bounded region in an affine chart, hence they cannot diverge to a constant map.

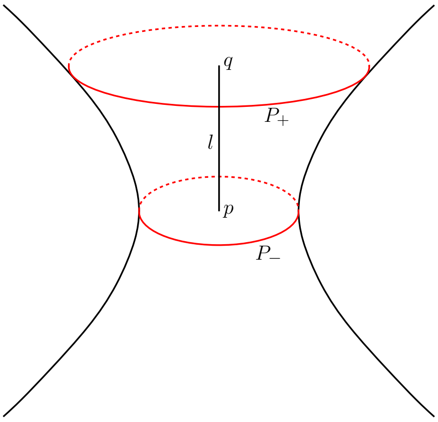



**Figure 8.3:** The setting of the proof of Proposition 8.B.

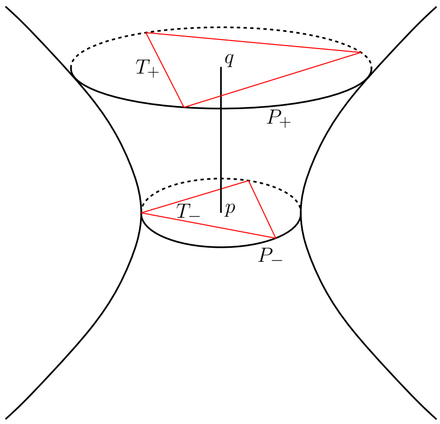



**Figure 8.4:** The point $p$ is contained in the convex envelope of three (or two) points in $\partial_\infty(P_-)$; analogously $q$ in $P_+$.

also be degenerate if $p$ is contained in an entire geodesic, but this will not affect the argument. Hence we can find three geodesic half-lines in $P_-$ connecting $p$ to $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ (or an entire geodesic connecting $p$ to two opposite points in the boundary, if $T_-$ is degenerate). Analogously we have an ideal triangle $T_+$ in $P_+$, compare Figure 8.4. The following Lemma will provide constraints on the position the half-geodesics in $P_+$ can assume. See Figure 8.5 and 8.6 for a picture of the "sector" described in Lemma 8.1.10.

**Sublemma 8.1.10.** *Suppose $\partial_-\mathcal{C}_\infty \cap P_-$ contains a half-geodesic*

$$g = \{t(\cos\theta, \sin\theta, 0) : t \in [0, 1)\}$$

*from $p$, asymptotic to the point at infinity $\eta = (\cos\theta, \sin\theta, 0)$. Then $\partial_+\mathcal{C}_\infty \cap P_+$ must be contained in $P_+ \setminus S(\eta)$, where $S(\eta)$ is the sector $\{x\cos\theta + y\sin\theta > 1\}$.*

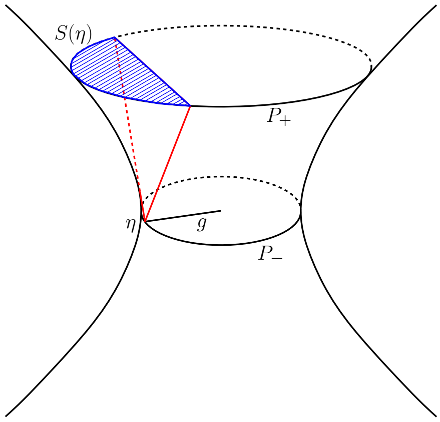



**Figure 8.5:** The sector $S(\eta)$ as in Sublemma 8.1.10.

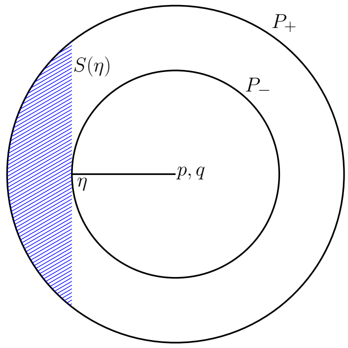



**Figure 8.6:** The $(x, y)$-plane seen from above. The sector $S(\eta)$ is bounded by the chord in $P_+$ tangent to the concentric circle, which projects vertically to $P_-$

*Proof.* The computation will be carried out in the double cover $\widehat{\mathbb{A}d\mathbb{S}^3}$ of $\mathbb{A}d\mathbb{S}^3$. It suffices to check the assertion when $\theta = \pi$, since in the statement there is a rotational symmetry along the vertical axis. The half-geodesic $g$ is parametrized in $\widehat{\mathbb{A}d\mathbb{S}^3} \subset \mathbb{R}^{2,2}$ by $g(t) = (\sinh(t), 0, \cosh(t), 0)$, for $t \in (-\infty, 0]$. Since the width is less than $\pi/2$, every point in $\partial_+\mathcal{C}_\infty \cap P_+$ must lie in the region bounded by $P_-$ and the dual plane $g(t)^\perp$. Indeed for every $t$, $g(t)^\perp$ is the locus of points at timelike distance $\pi/2$ from $g(t)$. We have

$$P_+ = \{(\cos\alpha\sinh r, \sin\alpha\sinh r, \cos w\cosh r, \sin w\cosh r) : r > 0, \alpha \in [0, 2\pi)\} .$$

Hence the intersection $P_+ \cap g(t)^\perp$ is given by the condition

$$\sinh(t)\cos(\alpha)\sinh(r) = \cosh(t)\cos(w)\cosh(r)$$

and thus is composed (in the affine coordinates of $\{x^3 \neq 0\}$) by the points of the form

$$\left(\frac{1}{\tanh(t)}, \frac{\tan(\alpha)}{\tanh(t)}, \tan(w)\right) .$$

Therefore, points in $\partial_+\mathcal{C}_\infty \cap P_+$ need to have $x \geq 1/\tanh(t)$, and since this holds for every $t \leq 0$, we have $x \geq -1$. □

By the previous Sublemma, if $p$ is contained in the convex envelope of three points $\eta_1, \eta_2, \eta_3$ in $\partial_\infty(P_-)$, then any point at infinity of $\partial_+\mathcal{C}_\infty \cap P_+$ is necessarily contained in $P_+ \setminus (S(\eta_1) \cup S(\eta_2) \cup S(\eta_3))$. We will use this fact to choose two pairs of points, $\eta, \eta'$ in $\partial_\infty(P_-)$ and $\xi, \xi'$ in $\partial_\infty(P_+)$, in a convenient way. This is the content of next sublemma. See Figure 8.7.

**Sublemma 8.1.11.** *Suppose $p$ is contained in the convex envelope of three points $\eta_1, \eta_2, \eta_3$ in $\partial_\infty(P_-)$. Then $gr(\phi_\infty)$ must contain (at least) two points $\xi, \xi'$ of $\partial_\infty(P_+)$ which lie in different connected components of $\partial_\infty(P_+) \setminus (S(\eta_1) \cup S(\eta_2) \cup S(\eta_3))$.*

*Proof.* The proof is simple 2-dimensional Euclidean geometry. Recall that the point $q$, which is the "center" of the plane $P_+$, is in the convex hull of $gr(\phi_\infty)$. If the claim were false, then one connected component of $\partial_\infty(P_+) \setminus (S(\eta_1) \cup S(\eta_2) \cup S(\eta_3))$ would contain a sector $S_0$ of angle $\geq \pi$. But then the points $\eta_1, \eta_2, \eta_3$ would all be contained in the complement of $S_0$. This contradicts the fact that $p$ is in the convex hull of $\eta_1, \eta_2, \eta_3$. $\square$

*Remark* 8.1.12. If $p$ is in the convex envelope of only two points at infinity, which means that $P_-$ contains an entire geodesic, the previous statement is simplified, see Figure 8.8.

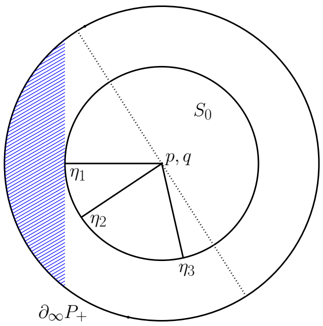
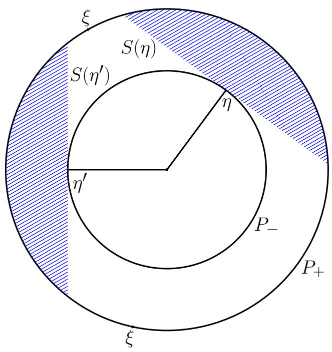



**Figure 8.7:** The proof of Sublemma 8.1.11. Below, the choice of points $\eta, \eta', \xi, \xi'$.



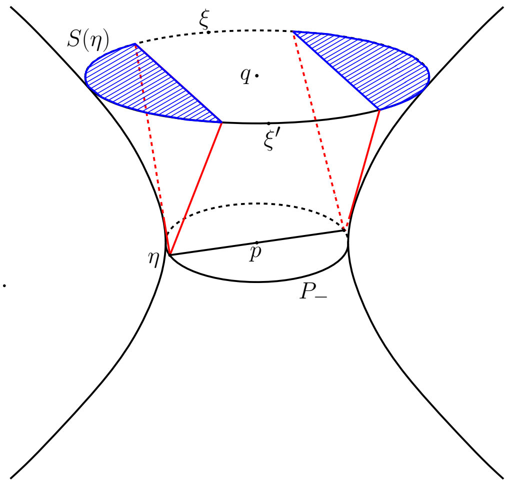




**Figure 8.8:** The same statement of Sublemma 8.1.11 is simpler if $p$ is contained in an entire geodesic line contained in $P_-$.

Let us now choose two points $\eta, \eta' \in \partial_\infty(P_-)$ among $\eta_1, \eta_2, \eta_3$, and $\xi, \xi' \in \partial_\infty(P_+)$ in such a way that $\xi$ and $\xi'$ lie in two different connected components of $\partial_\infty(P_+) \setminus (S(\eta_1) \cup S(\eta_2))$. The strategy will be to use this quadruple to show that the cross-ratio distortion of $\phi_\infty$ is not too small, depending on the width $w$. However, such quadruple is not symmetric in general. Hence $\xi'$ will be replaced later by another point $\xi''$. First we need some tool to compute the left and right projections to $\partial_\infty\mathbb{H}^2$ of the chosen points.

We use the plane $P_-$ to identify $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $\partial_\infty\mathbb{H}^2 \times \partial_\infty\mathbb{H}^2$. Let $\pi_l$ and $\pi_r$ denote left and right projection to $\partial_\infty(P_-)$, following the left and right ruling of $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$. In what follows, angles like $\theta_l$, $\theta_r$ and similar symbols will always be considered in $(-\pi, \pi]$.

**Sublemma 8.1.13.** *Suppose $\xi \in \partial_\infty(P_+)$, where the length of the timelike geodesic segment orthogonal to $P_-$ and $P_+$ is $w$. If $\pi_l(\xi) = (\cos(\theta_l), \sin(\theta_l), 0)$, then $\pi_r(\xi) = (\cos(\theta_l - 2w), \sin(\theta_l - 2w), 0)$.*

*Proof.* By the description of the left ruling (see Section 1.3), recalling $h = \tan(w)$, it is easy to check that

$$\begin{aligned}\xi =&(\cos(\theta_l), \sin(\theta_l), 0) + h(\sin(\theta_l), -\cos(\theta_l), 1)\\ =&(\cos(\theta_l) + h\sin(\theta_l), \sin(\theta_l) - h\cos(\theta_l), h)\\ =&(\sqrt{1+h^2}\cos(\theta_l - w), \sqrt{1+h^2}\sin(\theta_l - w), h)\,.\end{aligned}$$

By applying the same argument to the right projection, the claim follows. □

We can assume $\eta' = (-1, 0, 0)$, namely $\eta'$ corresponds to $(-1, -1) \in \partial_\infty\mathbb{H}^3 \times \partial_\infty\mathbb{H}^3$. Let $\eta = (e^{i\theta_0}, e^{i\theta_0})$; by symmetry, we can assume $\theta_0 \in [0, \pi)$; in this case we need to consider the point $\xi = (e^{i\theta_l}, e^{i\theta_r})$ constructed above, with $\theta_r \in [\theta_0, \pi)$. More precisely, Sublemma 8.1.13 shows $\theta_r = \theta_l - 2w$; by Sublemma 8.1.10 we must have $\theta_l - w \notin (\theta_0 - w, \theta_0 + w) \cup (\pi - w, \pi) \cup (-\pi, -\pi + w)$ and thus, by choosing $\xi$ in the correct connected component (i.e. switching $\xi$ and $\xi'$ if necessary), necessarily $\theta_l \in [\theta_0 + 2w, \pi]$ (see Figure 8.9). We remark again that the quadruple $Q = \pi_l(\xi', \eta, \xi, \eta')$ will not be symmetric in general, so we need to consider a point $\xi''$ instead of $\xi'$ so as to obtain a symmetric quadruple. However, if $\theta_0 \in (-\pi, 0)$, then one would consider the point $\xi'$ in the connected component having $\theta_r \in (-\pi, \theta_0)$ - and then a point $\xi''$ in the other connected component so as to have a symmetric quadruple - and obtain the same final estimate.

So let $\xi'' = (e^{i\theta''_l}, e^{i\theta''_r})$ be a point on $gr(\phi)$ so that the quadruple $Q = \pi_l(\xi'', \eta, \xi, \eta')$ is symmetric; we are going to compute the cross-ratio of $\phi(Q) = \pi_r(\xi'', \eta, \xi, \eta')$. However, in order to avoid dealing with complex numbers, we first map $\partial_\infty\mathbb{H}^3 = \partial_\infty(P_-)$ to $\mathbb{R} \cup \{\infty\}$ using the Möbius transformation

$$z \mapsto \frac{z-1}{i(z+1)}$$

which maps $e^{i\theta}$ to $\tan(\theta/2) \in \mathbb{R}$ if $\theta \neq \pi$, and $-1$ to $\infty$. We need to compute

$$|\log|cr(\phi(Q))|| = \left|\log\left|\frac{\tan(\theta_r/2) - \tan(\theta_0/2)}{\tan(\theta_0/2) - \tan(\theta''_r/2)}\right|\right| \tag{8.6}$$

and in particular we want to show this is uniformly away from 1. By construction $\theta_r < \theta_l$ (see also Figure 8.10), and since $P_-$ does not disconnect $gr(\phi)$, also $\theta''_r < \theta''_l$. We have

$$\tan(\theta_0/2) - \tan(\theta''_r/2) \geq \tan(\theta_0/2) - \tan(\theta''_l/2)\,. \tag{8.7}$$

The condition that $(\theta''_l, \theta_0, \theta_l, \infty)$ forms a symmetric quadruple translates on $\mathbb{R}$ to the condition that

$$\tan(\theta_0/2) - \tan(\theta''_l/2) = \tan(\theta_l/2) - \tan(\theta_0/2)\,. \tag{8.8}$$

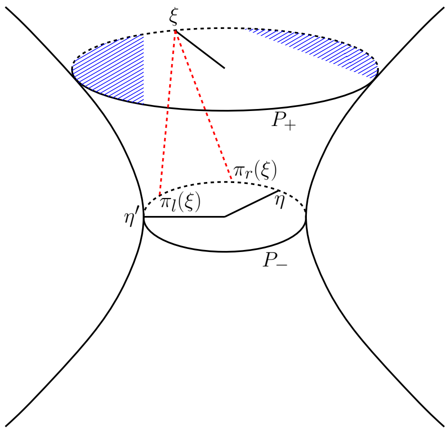



**Figure 8.9:** The choice of points $\eta, \xi, \eta'$ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$, endpoints at infinity of geodesic half-lines in the boundary of the convex hull.

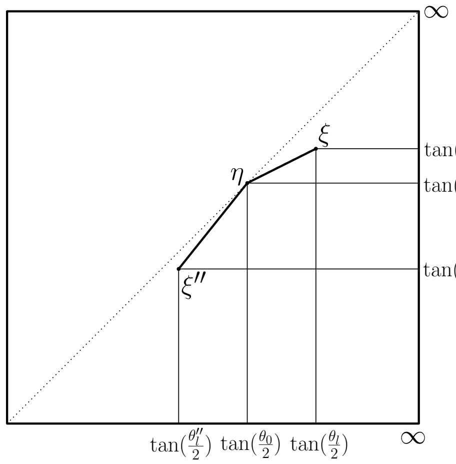



**Figure 8.10:** We give an upper bound on the ratio between the slopes of the two thick lines. The dotted line represents the plane $P_-$.

Using (8.7) and (8.8) in the argument of the logarithm in (8.6), we obtain:

$$\frac{\tan(\theta_r/2) - \tan(\theta_0/2)}{\tan(\theta_0/2) - \tan(\theta''_r/2)} \leq \frac{\tan((\theta_l/2) - w) - \tan(\theta_0/2)}{\tan(\theta_l/2) - \tan(\theta_0/2)} =: S(\theta_l).$$

Note that $S(\theta_l) < 1$ on $[\theta_0 + 2w, \pi]$ and $S(\theta_l) \to 0$ when $\theta_l \to \theta_0 + 2w$ or $\theta_l \to \pi$: this corresponds to the fact that $gr(\phi_\infty)$ tends to contain a lightlike segment. On the other hand $S(\theta_l)$ is positive on $[\theta_0 + 2w, \pi]$ and the maximum $S_{max}$ is achieved at some interior point of the interval. A computation gives

$$|cr(\phi(Q))| \leq S_{max} = \left(\frac{\cos(\theta_0/2 + w)}{\cos(\theta_0/2) + \sin(w)}\right)^2 .$$

The RHS quantity depends on $\theta_0$, but is maximized on $[0, \pi - 2w]$ for $\theta_0 = 0$, where it assumes the value $(1 - \sin(w))/(1 + \sin(w))$. This gives

$$e^{||\phi_\infty||_{cr}} \geq \left|\frac{1}{cr(\phi(Q))}\right| \geq \frac{1 + \sin(w)}{1 - \sin(w)} .$$

From this we deduce

$$\sin(w) \leq \frac{e^{||\phi_\infty||_{cr}} - 1}{e^{||\phi_\infty||_{cr}} + 1} = \tanh \frac{||\phi_\infty||_{cr}}{2}$$

or equivalently

$$\tan(w) \leq \sinh \frac{||\phi_\infty||_{cr}}{2} .$$

Since $||\phi_\infty||_{cr} \leq ||\phi||_{cr}$, the first part of the proof is concluded.

It remains to show the other inequality. This will follow more easily from the above construction. Suppose $||\phi||_{cr} > k$. Then we can find a quadruple of symmetric points $Q$ such that $|cr(\phi(Q))| = e^k$. Consider the points $\xi', \eta, \xi, \eta'$ on $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ such that their left and right projection are $Q$ and $\phi(Q)$, respectively.

Recall that the isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ act on $\partial_\infty(\mathbb{A}\mathrm{d}\mathbb{S}^3) \cong S^1 \times S^1$ as a pair of Möbius transformations, therefore they preserve the cross-ratio of both $Q$ and $\phi(Q)$. Thus we can suppose $Q = (-1, 0, 1, \infty)$ and $\phi(Q) = (-e^{k/2}, 0, e^{-k/2}, \infty)$ when the quadruples are regarded as composed of points on $\mathbb{R} \cup \{\infty\}$.

Passing to the coordinates in $S^1$ (by the map $\theta \in S^1 \mapsto \tan(\theta/2) \in \mathbb{R}$) for this quadruple of points at infinity, it is easy to see that - in the affine chart $\{x^3 \neq 0\}$ - the position of the four points has an order 2 symmetry obtained by rotation around the $z$-axis. See Figure 8.11. This is ensured by the special renormalization chosen for $Q$ and $\phi(Q)$.

Hence the geodesic line $g_1$ with endpoints at infinity $\eta$ and $\eta'$ is contained in the plane $P_-$ as in the first part of the proof. More precisely, in the usual affine chart $\{x^3 \neq 0\}$,

$$g_1 = \{(\tanh(t), 0, 0) : t \in \mathbb{R}\}\,.$$

The geodesic line $g_2$ connecting $\xi$ and $\xi'$ has the form

$$g_2(s) = \left\{ \left( \frac{\cos(\alpha)\tanh(s)}{\cos(w')}, \frac{\sin(\alpha)\tanh(s)}{\cos(w')}, \tan(w') \right) : s \in \mathbb{R} \right\}\,.$$

The lines $g_1$ and $g_2$ are in $\mathcal{CH}(gr(\phi))$ and have the common orthogonal segment $l$ which lies in the $z$-axis in the usual affine chart (Figure 8.11), the feet of $l$ being achieved for $t = 0$ and $s = 0$.

The distance between $g_1$ and $g_2$ is achieved along this common orthgonal geodesic and its value is $w'$. Recalling Sublemma 8.1.13 and the computation in its proof, we find $\alpha = \theta_l - w' = \pi/2 - w'$ and $\theta_r = \theta_l - 2w'$. Since $\tan(\theta_r/2) = e^{-k/2}$ and $\theta_l = \pi/2$, one can compute

$$w' = \pi/4 - \arctan(e^{-k/2})\,.$$

It follows that

$$\tan(w) \geq \tan(w') = \frac{1 - e^{-k/2}}{1 + e^{-k/2}} = \tanh\left(\frac{k}{4}\right)\,.$$

Since this is true for an arbitrary $k \leq ||\phi||_{cr}$, the inequality $\tan(w) \geq \tanh(||\phi||_{cr}/4)$ holds. $\square$

### 8.1.3 Uniform gradient estimates

Let $S$ a maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Let $P_-$ be a spacelike plane which does not intersect the convex hull. As in the hyperbolic setting, we now want to use the fact that the function $u(x) = \sin d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_-)$, satisfies the equation

$$\Delta_S u - 2u = 0\,. \tag{L}$$

given in Proposition 8.1.5. This will enable us to use Equation (8.2) to give estimates on the principal curvatures of $S$. Note that, by Gauss equation in the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ setting, a maximal surface with principal curvatures $\pm\lambda$ has curvature given by $K_S = -1 + \lambda^2$. It is proved in [BS10] that, if $\partial_\infty(S)$ is the graph of a quasisymmetric homeomorphism and the principal curvatures of $S$ are bounded, then $K_S$ is

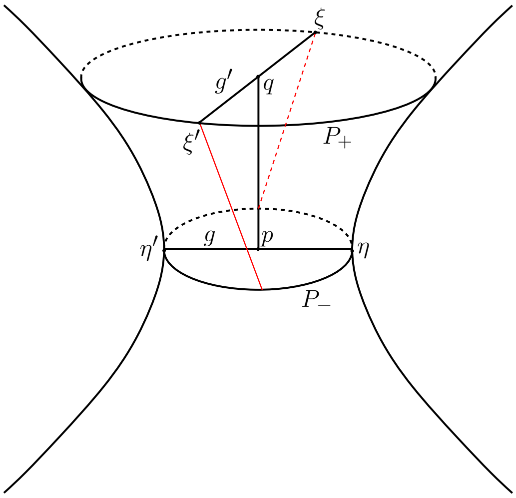



**Figure 8.11:** The distance between the two lines $g$ and $g'$ is achieved along the common orthogonal geodesic.

uniformly negative, which means that $||\lambda||_\infty < 1$. This is a substantial difference with the case of hyperbolic minimal surfaces, where the principal curvatures can be larger than 1.

From this point, we will always assume that $S$ is a maximal surface spanning the graph of a quasisymmetric homeomorphism, which is a compression disc for $\mathbb{A}\mathrm{d}\mathbb{S}^3$, with bounded principal curvatures; hence $S$ is complete (recall Remark 8.1.4) and the curvature is bounded by $-1 \leq K_S < 0$. However, when $||\lambda||_\infty$ approaches 1, the curvature becomes close to 0. Therefore we will not be able to use uniform bounds on the metric provided by upper bound on the curvature, as in the hyperbolic case (Subsection 7.1.3). Instead, we will use uniform estimates on the norm of the gradient of $u$.

**Lemma 8.1.14.** *The universal constant* $L = \sqrt{2(1+\sqrt{2})}$ *is such that, for every point* $x$ *on a maximal surface in* $\mathbb{A}\mathrm{d}\mathbb{S}^3$ *with nonpositive curvature,* $||\operatorname{grad} u|| < L$.

*Proof.* Let $\gamma$ be a path on $S$ obtained by integrating the gradient vector field; more precisely, we impose $\gamma(0) = x$ and

$$\gamma'(t) = -\frac{\operatorname{grad} u}{||\operatorname{grad} u||}\,.$$

Observe that

$$\begin{aligned} u(\gamma(t)) - u(x) &= \int_0^t du(\gamma'(s))ds = \int_0^t -\langle \operatorname{grad} u(s), \frac{\operatorname{grad} u(s)}{||\operatorname{grad} u(s)||}\rangle ds \\ &= -\int_0^t ||\operatorname{grad} u(s)||ds\,. \end{aligned}$$

We denote $y(s) = ||\operatorname{grad} u(s)||$. We will show that $y(0)$ is bounded by a universal constant, since $u(\gamma(t))$ cannot become negative on $S$ (recall Corollary 8.1.7). We have

$$\left.\frac{d}{dt}\right|_{t=0} y(t)^2 = 2\langle \nabla_{\gamma'(t)} \operatorname{grad} u(\gamma(t)), \operatorname{grad} u(\gamma(t))\rangle = 2\nabla^2 u(\gamma'(t), \operatorname{grad} u(\gamma(t))) \quad (8.9)$$

Since, by equation (7.1), $\nabla^2 u - uI = \sqrt{1-u^2+||\operatorname{grad} u||^2}\,I\!I$ and $||B(v)|| \leq ||v||$,

$$-\left.\frac{d}{dt}\right|_{t=0} y(t)^2 \leq \left|\left.\frac{d}{dt}\right|_{t=0} y(t)^2\right| \leq 2\left(u(\gamma(t) + \sqrt{1-u(\gamma(t))^2+y(t)^2}\right) y(t)$$

and therefore

$$-\left.\frac{d}{dt}\right|_{t=0} y(t) \leq \sqrt{2}\sqrt{1+y(t)^2}. \quad (8.10)$$

It follows that

$$y(t) \geq y(0)\cosh(\sqrt{2}t) - \sqrt{1+y(0)^2}\sinh(\sqrt{2}t) \quad (8.11)$$

since the RHS of (8.11) is the solution of (8.10) with inequality replaced by equality. Now

$$u(\gamma(t)) - u(x) = -\int_0^t y(s)ds \leq \frac{1}{\sqrt{2}}\left(-y(0)\sinh(\sqrt{2}t) + \sqrt{1+y(0)^2}(\cosh(\sqrt{2}t)-1)\right).$$

Let us define $F(t)$ the RHS of the above inequality. We must have $u(\gamma(t)) \geq 0$ for every $t$; so we impose that $F(t) \geq -u(x)$ for every $t$. The minimum of $F$ is achieved for

$$\tanh(\sqrt{2}t_{min}) = \frac{y(0)}{\sqrt{1+y(0)^2}}.$$

Therefore

$$F(t_{min}) = -\frac{1}{\sqrt{2}}\left(1+\sqrt{1+y(0)^2}\right) \geq -u(x)$$

which is equivalent to $y(0)^2 \leq 2(u(x)^2 + \sqrt{2}u(x))$. Recalling $u \in [-1,1]$, $||\operatorname{grad} u(x)||^2 \leq 2(1+\sqrt{2})$ independently on the maximal surface $S$ and on the support plane $P_-$. □

We now apply the above uniform gradient estimate to prove a fact which will be of use shortly. Given two unit timelike vectors $v, v' \in T_x\mathbb{A}\mathrm{d}\mathbb{S}^3$, we define the hyperbolic angle between $v$ and $v'$ as the number $\alpha \geq 0$ such that $\cosh\alpha = \langle v, v'\rangle$. Compare with Figure 8.14 below.

**Lemma 8.1.15.** *There exists a constant $\bar{\alpha}$ such that the following holds for every maximal surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and every totally geodesic plane $P_-$ in the past of $S$ which does not intersect $S$. Let $l$ be a geodesic line orthogonal to $P_-$ and let $x = l \cap S$. Suppose $x$ is at distance less than $\pi/4$ from $P_-$. Then the hyperbolic angle $\alpha$ at $x$ between $l$ and the normal vector to $S$ is bounded by $\alpha \leq \bar{\alpha}$.*

*Proof.* We use the same notation as Proposition 7.1.5 and 8.1.5. It is clear that the tangent direction to $l$ is given by the vector $\nabla U$, where $U(x) = \sin d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_-) = \langle x, p\rangle$ is defined on the entire $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and $p$ is the point dual to $P_-$. Recall $u$ is the restriction of $U$ to $S$. In the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ setting, we have the formulae $\nabla U(x) = p + \langle p, x\rangle x$ and $\langle \nabla U, \nabla U\rangle = -1 + u^2 = ||\operatorname{grad} u||^2 - \langle \nabla U, N\rangle^2$. It follows that the angle $\alpha$ at $x$

between the normal to the maximal surface $S$ and the geodesic $l$ can be computed as

$$(\cosh\alpha)^2 = \langle \frac{\nabla U(x)}{||\nabla U(x)||}, N\rangle^2 = \frac{1 - u(x)^2 + ||\operatorname{grad} u(x)||^2}{1 - u(x)^2}$$

and so $\alpha$ is bounded by Lemma 8.1.14 and the assumption that $u(x)^2 \leq 1/2$. □

### 8.1.4 Schauder estimates

As in Subsection 7.1.4 for the hyperbolic case, we now want to give Schauder-type estimates on the derivatives of the function $u = \sin d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(\cdot, P_-)$, expressed in suitable coordinates, of the form

$$||u||_{C^2(B_0(0,\frac{R}{2}))} \leq C||u||_{C^0(B_0(0,R))}$$

where the constant does not depend on $S$ and $P_-$. We again prove this estimate by using a compactness argument.

The following Lemma is proved in [BS10, Lemma 5.1]. Given a spacelike plane $P_0$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and a point $x_0 \in P_0$, let $l$ be the timelike geodesic through $x_0$ orthogonal to $P_0$. We define the cylinder $Cl(x_0, P_0, R_0)$ of radius $R_0$ above $P_0$ centered at $x_0$ as the set of points $x \in \mathbb{A}\mathrm{d}\mathbb{S}^3$ which lie on a spacelike plane $P_x$ orthogonal to $l$ such that $d_{P_x}(x, l \cap P_x) \leq R_0$. See also Figure 8.12.

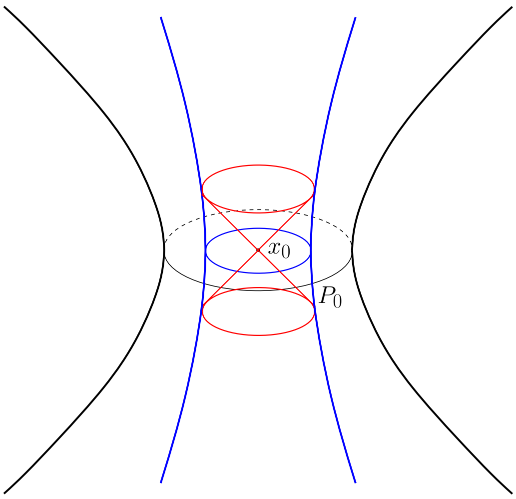



**Figure 8.12:** The cylinder $Cl(x_0, P_0, R_0)$ (blue) and its intersection with $\mathrm{I}^+(x_0)$ and $\mathrm{I}^-(x_0)$ (red).

**Lemma 8.1.16** ([BS10])**.** *There exists a radius $R_0$ such that, for every spacelike plane $P_0$ and every point $x_0 \in P_0$, every sequence $S_n$ of maximal surfaces tangent to $P_0$ at $x_0$ admits a subsequence converging $C^\infty$ on the cylinder $Cl(x_0, P_0, R_0)$ to a maximal surface.*

Denote by $w = w(\partial_\infty S)$ the width of the convex hull of the asymptotic boundary of $S$; we have $w(\partial_\infty S) \leq \pi/2$ (see [BS10, Lemma 4.16]). Let $x$ be a point of

$S$; by Remark 8.1.9, we have that $d_{\mathbb{A}d\mathbb{S}^3}(x, \partial_-\mathcal{C}) + d_{\mathbb{A}d\mathbb{S}^3}(x, \partial_+\mathcal{C}) \leq w$, therefore one among $d_{\mathbb{A}d\mathbb{S}^3}(x, \partial_-\mathcal{C})$ and $d_{\mathbb{A}d\mathbb{S}^3}(x, \partial_+\mathcal{C})$ must be smaller than $\pi/4$. Composing with an isometry of $\mathbb{A}d\mathbb{S}^3$ (which possibly reverses time-orientation), we can assume $d_{\mathbb{A}d\mathbb{S}^3}(x, \partial_-\mathcal{C}) \leq d_{\mathbb{A}d\mathbb{S}^3}(x, \partial_+\mathcal{C})$, which implies that $x$ has distance less than $\pi/4$ from $P_-$. This assumption will be very important in the following.

**Proposition 8.1.17.** *There exists a radius $R > 0$ and a constant $C > 0$ such that for every choice of:*

- *A maximal surface $S \subset \mathbb{A}d\mathbb{S}^3$ with $\partial_\infty S$ the graph of an orientation-preserving homeomorphism;*
- *A point $x \in S$;*
- *A plane $P_-$ disjoint from $S$ with $d_{\mathbb{A}d\mathbb{S}^3}(x, P_-) \leq \pi/4$,*

*the function $u(\cdot) = d_{\mathbb{H}^3}(\cdot, P_-)$ expressed in terms of normal coordinates centered at $x$, namely*

$$u(z) = \sin d_{\mathbb{A}d\mathbb{S}^3}(\exp_x(z), P_-)$$

*where $\exp_x : \mathbb{R}^2 \cong T_xS \to S$ denotes the exponential map, satisfies the Schauder-type inequality*

$$||u||_{C^2(B_0(0,\frac{R}{2}))} \leq C||u||_{C^0(B_0(0,R))} \,. \tag{8.12}$$

*Proof.* Let $R_0$ be the universal constant appearing in Lemma 8.1.16. First, we show that there exists a radius $R$ such that the image of the Euclidean ball $B_0(0, R)$ under the exponential map at every point $x \in S$, for every surface $S$, is contained in the cylinder $Cl(x, T_xS, R_0)$. Indeed, suppose this does not hold, namely

$$\inf_{x \in S} \sup \{R : \exp_x(B_0(0, R)) \subset Cl(x, T_xS, R_0)\} = 0 \,. \tag{8.13}$$

Then one can find a sequence $S_n$ of maximal surfaces and points $x_n$ such that the supremum $R_n$ of radii $R$ for which $\exp_{x_n}(B_0(0, R))$ is contained in the respective cylinder of radius $R_0$ goes to zero. We can compose with isometries of $\mathbb{A}d\mathbb{S}^3$ so that all points $x_n$ are sent to the same point $x_0$ and all surfaces are tangent at $x_0$ to the same plane $P_0$. By Lemma 8.1.16, there exists a subsequence converging inside $Cl(x_0, P_0, R_0)$ to a maximal surface $S_\infty$. Therefore the infimum in the LHS of Equation (8.13) cannot be zero, since for the limiting surface $S_\infty$ there is a radius $R_\infty$ such that $\exp_x(B_0(0, R_\infty)) \subset Cl(x, T_xS, R_0)$.

We use a similar argument to prove the main statement. We can consider $P_-$ a fixed plane, and a point $x \in S$ lying on a fixed geodesic $l$ orthogonal to $P_-$. Suppose the claim does not hold, namely there exists a sequence of surfaces $S_n$ in the future of $P_-$ such that for the function $u_n(z) = \sin d_{\mathbb{A}d\mathbb{S}^3}(\exp_{x_n}(z), P_n)$,

$$||u_n||_{C^2(B_0(0,\frac{R}{2}))} \geq n||u||_{C^0(B_0(0,R))} \,.$$

Let us compose each $S_n$ with an isometry $T_n \in \mathrm{Isom}(\mathbb{A}d\mathbb{S}^3)$ so that $S_n' = T_n(S_n)$ is tangent to a fixed plane $P_0$ at a fixed point $x_0$, whose normal unit vector is $N_0$. We claim that the sequence of isometries $T_n$ is bounded in $\mathrm{Isom}(\mathbb{A}d\mathbb{S}^3)$, since $T_n^{-1}$

maps the element $(x_0, N_0)$ of the tangent bundle $T\mathbb{A}\mathrm{d}\mathbb{S}^3$ to a bounded region of $T\mathbb{A}\mathrm{d}\mathbb{S}^3$. Indeed, by our assumptions, $T_n^{-1}(x_0) = x_n$ lies on a geodesic $l$ orthogonal to $P_-$ and has distance less than $\pi/4$ (in the future) from $P_-$; moreover by Lemma 8.1.15 the vector $(dT_n)^{-1}(N_0)$ forms a bounded angle with $l$. By Lemma 8.1.16, up to extracting a subsequence, we can assume $S'_n \to S'_\infty$ on $Cl(x_0, P_0, R_0)$ with all derivatives. Since we can also extract a converging subsequence from $T_n$, we assume $T_n \to T_\infty$, where $T_\infty$ is an isometry of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Therefore $T_n(P_-)$ converges to a totally geodesic plane $P_\infty$.

Using the first part of this proof and Lemma 8.1.16, on the image under the exponential map of $S'_n$ of the ball $B_0(0, R)$ the coefficients of the Laplace-Beltrami operators $\Delta_{S'_n}$ (in normal coordinates on $B_0(0, R)$) converge to the coefficients of $\Delta_{S'_\infty}$. As in the hyperbolic case, the operators $\Delta_{S'_n} - 2$ are uniformly strictly elliptic with uniformly bounded coefficients. By the Schauder estimate of Theorem 3.1.3, using the fact that $u_n$ solves the equation $\Delta_{S'_n}(u_n) - 2u_n = 0$, there exists a constant $c$ such that

$$||u_n||_{C^2(B_0(0,\frac{R}{2}))} \leq c||u_n||_{C^0(B_0(0,R))} ,$$

for every $n$. This gives a contradiction. □

*Remark* 8.1.18. The statements of Lemma 8.1.15, Lemma 8.1.16 and Proposition 8.1.17 (and also Proposition 8.1.19 below) could be improved so as to be stated in terms of the choice of any radius $R > 0$, any number $w_0 < \pi/2$ (replacing $\pi/4$), where the constant $C$ would depend on such choices. However, these details would not improve the final statement of Theorem 8.C and thus are not pursued here. The reader can compare with Proposition 7.1.19 and the lemmata used in the proof.

Let us remark that in Anti-de Sitter space the projection from a spacelike curve or surface to a totally geodesic spacelike plane is not distance-contracting. Hence we need to give an additional computation in order to ensure (by substituting the radius $R$ in Proposition 8.1.17 by a smaller one if necessary) that the projection from the geodesic balls $B_S(x, R)$ to $P_-$ has image contained in a uniformly bounded set - which was obtained for free in the case of hyperbolic geometry. This is proved in the next Proposition, see also Figure 8.13.

**Proposition 8.1.19.** *There exist constant radii $R'_0$ and $R'$ such that for every maximal surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, every point $x_0 \in S$ and every totally geodesic plane $P_-$ which does not intersect $S$, such that the distance of $x_0$ from $P_-$ is at most $\pi/4$, the orthogonal projection $\pi|_S : S \to P_-$ maps $S \cap Cl(x_0, T_{x_0}S, R'_0)$ to $B_{P_-}(\pi(x_0), R')$.*

*Proof.* We can suppose $T_{x_0}S$ is the intersection of the plane $\{x_4 = 0\}$ with $\mathbb{A}\mathrm{d}\mathbb{S}^3 \subset \mathbb{R}P^3$ and $x_0 = [\widehat{x}_0]$ with $\widehat{x}_0 = (0, 0, 1, 0)$. Therefore - doing as usual the computation in the double cover $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ inside $\mathbb{R}^{2,2}$ - the points $x$ in $Cl(x_0, T_{x_0}S, R'_0)$ have coordinates

$$x = (\cos\theta \sinh r, \sin\theta \sinh r, \cos m \cosh r, \sin m \cosh r)$$

for $r \leq R'_0$. Let us denote by $\mathrm{I}^+(p)$ (resp. $\mathrm{I}^-(x_0)$) the future (resp. past) of a point $p$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3 \setminus Q$, where $Q$ is the plane at infinity in the affine chart.

Since $S$ is spacelike, $S \cap Cl(x_0, T_{x_0}S, R_0')$ is contained in $Cl(x_0, T_{x_0}S, R_0') \setminus (\mathrm{I}^+(x_0) \cup \mathrm{I}^-(x_0))$. Hence $|\langle x, x_0 \rangle| > 1$ (recall Equation (1.12) in the Prelimiaries), which is equivalent to

$$|\cos m| > \frac{1}{\cosh r}\,. \tag{8.14}$$

Let $l$ be the geodesic through $x_0$ orthogonal to $P_-$. We can assume $l$ has normal vector at $x_0$ given by $l'(0) = (\sinh\alpha, 0, 0, \cosh\alpha)$, where of course $\alpha$ is the angle between $l$ and the normal to $S$ at $x_0$. Therefore

$$l(t) = (\cos t)x_0 + (\sin t)l'(0) = (\sin t \sinh\alpha, 0, \cos t, \sin t \cosh\alpha)\,.$$

Let $w_1 = d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x_0, P_-)$, so $P_- = p^\perp$ is the plane orthogonal to

$$p = l'(-w_1) = (\cos w_1 \sinh\alpha, 0, \sin w_1, \cos w_1 \cosh\alpha)\,.$$

The projection of $x$ to $P_-$ is given by

$$\pi(x) = \frac{x + \langle x, p\rangle p}{\sqrt{1 - \langle x, p\rangle^2}}$$

provided $\langle x, p\rangle^2 < 1$, which is the condition for $x$ to be in the domain of dependence of $P_-$. The distance $d$ between $\pi(x)$ and $\pi(x_0) = l(-w_1)$ is given by the expression

$$\cosh d = |\langle \pi(x), l(-w_1)\rangle| = \left| \frac{\langle x, l(-w_1)\rangle}{\sqrt{1 - \langle x, p\rangle^2}} \right|\,. \tag{8.15}$$

Now, we have

$$\begin{aligned} |\langle x, p\rangle| =& |\cos\theta \sinh r \cos w_1 \sinh\alpha - \cos m \cosh r \sin w_1 - \sin m \cosh r \cos w_1 \cosh\alpha| \\ &\leq \sinh r \sinh\alpha + \frac{\sqrt{2}}{2}\cosh r + \sinh r \cosh\alpha = \frac{\sqrt{2}}{2}\cosh r + (\sinh r)e^\alpha\,. \end{aligned}$$

In the last line, we have used that $|\sin m| = \sqrt{1 - (\cos m)^2} \leq \tanh r$, by Equation (8.14), and that $\sin w_1 < \sqrt{2}/2$. Since the hyperbolic angle $\alpha$ is uniformly bounded by Lemma 8.1.15 (Figure 8.14), it follows that if $r \leq R_0'$ for $R_0'$ sufficiently small, $\sqrt{1 - \langle x, p\rangle^2}$ is uniformly bounded below. Moreover,

$$\begin{aligned} &|\langle x, l(-w_1)\rangle| = \\ &= |-\cos\theta \sinh r \sin w_1 \sinh\alpha - \cos m \cosh r \cos w_1 + \sin m \cosh r \sin w_1 \cosh\alpha| \\ &\qquad\qquad\qquad\qquad \leq \sinh r \sinh\alpha + \cosh r + \sinh r \cosh\alpha \end{aligned}$$

is uniformly bounded. This shows, from Equation (8.15), that $\cosh d \leq \cosh R'$ for some constant radius $R'$ (depending on $R_0'$). This concludes the proof. □

Therefore, replacing $R_0$ in Lemma 8.1.16 with $\min\{R_0, R_0'\}$, we have that the geodesic balls of radius $R$ ($R$ as in Proposition 8.1.17) on $S$ centered at $x$ project to $P_-$ with image contained in $B_{P_-}(\pi(x), R')$. The radii $R$ and $R'$ are fixed, not depending on $S$.

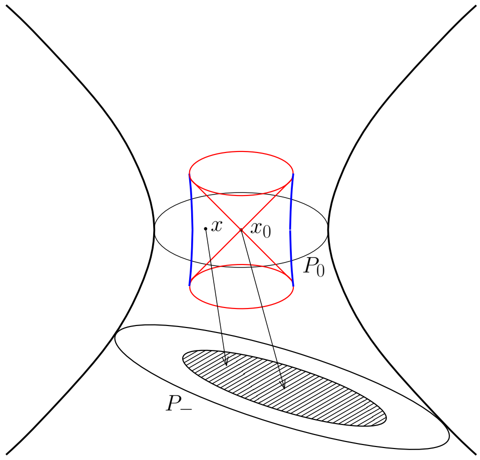



**Figure 8.13:** Projection from points in $Cl(x_0, T_{x_0}S, R_0')$ which are connected to $x_0$ by a spacelike geodesic have bounded image.

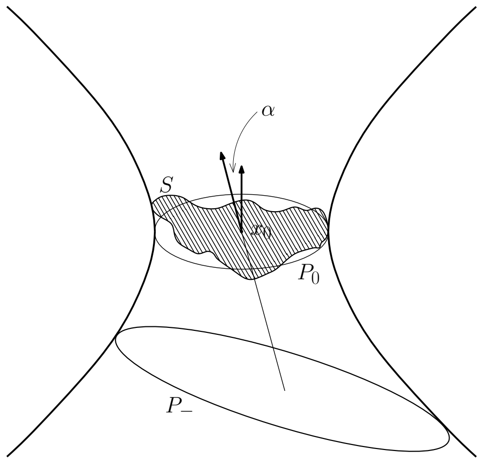



**Figure 8.14:** The key point is that the hyperbolic angle $\alpha$ is uniformly bounded, by Lemma 8.1.15.

### 8.1.5 Principal curvatures

In this subsection we will prove the estimate on the supremum of the principal curvatures of $S$ in terms of the width. In particular, we prove the following theorem.

**Theorem 8.C.** *There exists a constant $C$ such that, for every maximal surface $S$ with bounded principal curvatures $\pm\lambda$ and width $w = w(\mathcal{CH}(\partial_\infty S))$,*

$$||\lambda||_\infty \leq C \tan w \,.$$

*Remark* 8.1.20. Of course, the result in Theorem 8.C does give a new estimate only for $w \leq w_0$ for some $w_0$, as it is already known that every maximal surface with bounded principal curvatures has curvatures in $[-1, 1]$. However, this gives a good description of the behavior of principal curvatures for a maximal surface "close" to being a totally geodesic plane.

We take an arbitrary point $x \in S$. By Remark 8.1.9, we know that there are two disjoint planes $P_-$ and $P_+$ with $d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_-) + d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_+) = w_1 + w_2 \leq w$ where $w$ is the width. As in the previous subsection, we will assume $P_-$ is a fixed plane in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, upon composing with an isometry. Figure 8.15 gives a picture of the situation of the following lemma.

**Lemma 8.1.21.** *Let $p \in P_-$, $q \in P_+$ be the endpoints of geodesic segments $l_1$ and $l_2$ from $x \in S$ orthogonal to $P_-$ and $P_+$ of length $w_1$ and $w_2$, with $w_1 \leq w_2$. Let $p' \in P_-$ a point at distance $R'$ from $p$ and let $d = d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}((\pi|_{P_+})^{-1}(p'), P_-)$. Then*

$$\tan d \leq (1 + \sqrt{2}) \cosh R' \tan(w_1 + w_2). \tag{8.16}$$

*Proof.* As usual, we do the computation in $\widehat{\mathbb{A}\mathrm{d}\mathbb{S}^3}$. We assume $x = (0, 0, 1, 0)$ and $l_1$ is the geodesic segment parametrized by $l_1(t) = (\cos t)x - (\sin t)(0, 0, 0, 1)$, so that the plane $P_-$ is dual to $p_- = (0, 0, \sin w_1, \cos w_1)$. Points on the plane $P_-$ at distance $R'$ from $\pi(x) = l_1(w_1) = (0, 0, \cos w_1, -\sin w_1)$ have coordinates

$$p' = (\cos\theta \sinh R', \sin\theta \sinh R', \cosh R' \cos w_1, -\cosh R' \sin w_1)\,.$$

We also assume $l_2$ has initial tangent vector $l_2'(0) = (\sinh\alpha, 0, 0, \cosh\alpha)$, where $\alpha$ is the hyperbolic angle between $(0, 0, 0, 1)$ and $l_2'(0)$, so that

$$l_2(t) = (\cos t)x + (\sin t)(\sinh\alpha, 0, 0, \cosh\alpha)\,.$$

Note that

$$l_2'(w_2) = (\cos w_2 \sinh\alpha, 0, -\sin w_2, \cos w_2 \cosh\alpha) =: p_+$$

is the unit vector orthogonal to $P_+$, by construction.

We derive a condition which must necessarily be satisfied by $\alpha$, because $P_-$ and $P_+$ are disjoint. Indeed, we must have

$$|\langle p_-, p_+\rangle| = -\sin w_1 \sin w_2 + \cos w_1 \cos w_2 \cosh\alpha \leq 1$$

which is equivalent to

$$\cosh\alpha < \frac{1 + \sin w_1 \sin w_2}{\cos w_1 \cos w_2}\,. \tag{8.17}$$

Let us now write

$$(\tanh\alpha)^2 = \left(1 + \frac{1}{\cosh\alpha}\right)\left(1 - \frac{1}{\cosh\alpha}\right) \leq 2(\cosh\alpha - 1)$$

and therefore, using (8.17),

$$(\tanh\alpha)^2 < 2\left(\frac{1 - \cos(w_1 + w_2)}{\cos w_1 \cos w_2}\right) \leq 2\left(\frac{1 - (\cos(w_1 + w_2))^2}{\cos w_1 \cos w_2}\right) \leq 2\frac{(\sin(w_1 + w_2))^2}{\cos w_1 \cos w_2}\,. \tag{8.18}$$

To compute $d$, we now write explicitly the geodesic $\gamma$ starting from $p'$ and orthogonal to $P_-$. We find $d$ such that $\gamma(d) \in P_+$ and this will give the expected inequality. We have

$$\gamma(d) = (\cos d)p' + (\sin d)(0, 0, \sin w_1, \cos w_1)$$

and $\gamma(d) \in P_+$ if and only if $\langle\gamma(d), p_+\rangle = 0$, which gives the condition

$$\begin{aligned}\cos d(\cosh R'(\cos w_1 \sin w_2 + \cos w_2 \sin w_1 \cosh\alpha) + \sinh R'(\cos\theta \cos w_2 \sinh\alpha))\\ + \sin d(\sin w_1 \sin w_2 - \cos w_1 \cos w_2 \cosh\alpha) = 0\,.\end{aligned}$$

We express

$$\begin{aligned}\tan d = {}&\cosh R' \frac{\cos w_1 \sin w_2 + \cos w_2 \sin w_1 \cosh\alpha}{\cos w_1 \cos w_2 \cosh\alpha - \sin w_1 \sin w_2}\\ &+ \sinh R' \frac{\cos\theta \cos w_2 \sinh\alpha}{\cos w_1 \cos w_2 \cosh\alpha - \sin w_1 \sin w_2}\end{aligned}$$

The first term in the RHS is easily seen to be less than $\cosh R' \tan(w_1 + w_2)$. We turn to the second term. Using (8.18), it is bounded by

$$\sinh R' \tanh \alpha \frac{\cos w_2}{\cos(w_1 + w_2)} \leq \sqrt{2} \sinh R' \tan(w_1 + w_2) \left( \frac{\cos w_2}{\cos w_1} \right)^{\frac{1}{2}} .$$

In conclusion, having assumed $w_1 \leq w_2$, we can put $\cos(w_2)/\cos(w_1) \leq 1$, sum the two terms and get

$$\tan d \leq (1 + \sqrt{2}) \cosh R' \tan(w_1 + w_2) .$$

This concludes the proof. □

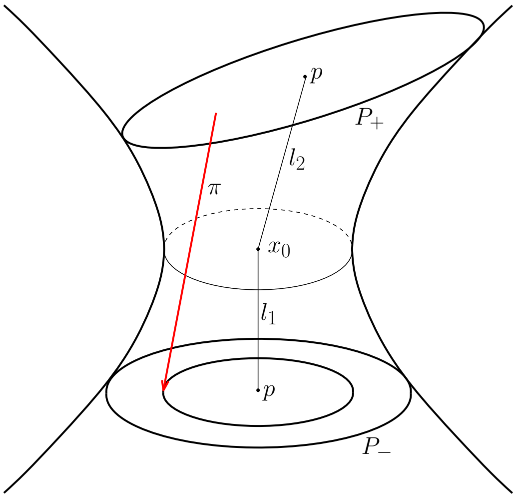



**Figure 8.15:** The setting of Lemma 8.1.21. We assume $w_1 = d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x_0, p) < d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x_0, q) = w_2$.

*Proof of Theorem 8.C.* Let $x \in S$ and consider the point $x_-$ of $\partial_- \mathcal{C}$ which minimizes the distance from $x$, where $\mathcal{C}$ is the convex hull of $S$. Let $P_-$ be the plane through $x_-$ orthogonal to the geodesic line containing $x$ and $x_-$ (recall Remark 8.1.9). The plane $P_-$ is then a support plane of $\partial_- \mathcal{C}$. We construct analogously the support plane $P_+$ for $\partial_+ \mathcal{C}$. As discussed in Remark 8.1.9,

$$d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_-) + d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_+) \leq w .$$

Moreover, we can assume (upon composing with a time-orientation-reversing isometry, if necessary) that $d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_-) \leq d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_+)$. As a consequence, $d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(x, P_-) \leq \pi/4$.

Let us now consider the function

$$u = \sinh d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(\exp_x(\cdot), P_-) .$$

By Equation (8.2), we have the following expression for the shape operator of $S$:

$$B = \frac{1}{\sqrt{1 - u^2 + ||\operatorname{grad} u||^2}} (\operatorname{Hess} u - u\, E) .$$

In normal coordinates at $x$ the Hessian of $u$ is given just by the second derivatives of $u$; in Proposition 8.1.17 we showed the second derivatives of $u$ are bounded, up to a factor, by $||u||_{C^0(B_S(x,R))}$. By Proposition 8.1.19, $||u||_{C^0(B_S(x,R))}$ is smaller than the supremum of the hyperbolic sine of the distance $d$ from $P_-$ of points of $S$ which project to $B_{P_-}(\pi(x), R')$. Therefore we have the following estimate for the principal curvatures at $x$:

$$|\lambda| \leq C_2 \frac{||u||}{\sqrt{1-||u||^2}} \leq C_2 \tan\left(\sup\{d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p, P_-) : p \in (\pi_S)^{-1}(B_{P_-}(\pi(x), R'))\}\right) .$$

The quantity in brackets in the RHS is certainly less than

$$\left(\sup\{d_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(p, P_-) : p \in (\pi_{P_+})^{-1}(B_{P_-}(\pi(x), R'))\}\right) .$$

Thus, applying Lemma 8.1.21 we obtain:

$$||\lambda||_\infty \leq C \tan w\, .$$

The constant $C_2$ involves the constant which appears in Equarion (8.12) in Proposition 8.1.17. The constant $C$ then involves $C_2$ and $\cosh R'$. Such inequality holds independently on the point $x$ and thus concludes the proof. □

To conclude the subsection, we prove a converse estimate, in fact we express an upper bound on the width when a bound on the principal curvatures is known. The following is the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ analogue of Lemma 7.1.8; see [KS07].

**Lemma 8.1.22.** *Given a smooth spacelike surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, let $S_\rho$ be the surface at timelike distance $\rho$ from $S$, obtained by following the normal flow. Then the pull-back to $S$ of the induced metric on the surface $S_\rho$ is given by*

$$I_\rho = I((\cos(\rho)E + \sin(\rho)B)\cdot, (\cos(\rho)E + \sin(\rho)B)\cdot)\, . \tag{8.19}$$

*The second fundamental form and the shape operator of $S_\rho$ are given by*

$$I\!I_\rho = I((-\sin(\rho)E + \cos(\rho)B)\cdot, (\cos(\rho)E - \sin(\rho)B)\cdot)\, , \tag{8.20}$$

$$B_\rho = (\cos(\rho)E + \sin(\rho)B)^{-1}(-\sin(\rho)E + \cos(\rho)B)\, . \tag{8.21}$$

*Proof.* Compare also the proof of Lemma 7.1.8. The geodesics orthogonal to $S$ at a point $x$ can be written as

$$\gamma(x)(\rho) = \cos(r)\sigma(x) + \sin(\rho)N(x)\, .$$

One obtains the thesis since in this case $B = \nabla N$. The formula for the second fundamental form follows from the fact that $I\!I_\rho = \frac{1}{2}\frac{dI_\rho}{d\rho}$. □

It follows that, if the principal curvatures of a maximal surface $S$ are $\lambda \in [0, 1)$ and $\lambda' = -\lambda$, then the principal curvatures of $S_\rho$ are

$$\lambda_\rho = \frac{\lambda - \tan(\rho)}{1 + \lambda\tan(\rho)} = \tan(\rho_0 - \rho)\, ,$$

where $\tan \rho_0 = \lambda$, and

$$\lambda'_\rho = \frac{-\lambda - \tan(-\rho)}{1 - \lambda \tan(\rho)} = \tan(-\rho_0 - \rho)\,.$$

In particular $\lambda_\rho$ and $\lambda'_\rho$ are non-singular for every $\rho$ between $-\pi/4$ and $\pi/4$.

It turns out that $S_\rho$ is convex at every point for $\rho < -||\rho_0||_\infty$, and concave for $\rho > ||\rho_0||_\infty$. Observe that the surfaces $S_\rho$ all have the same boundary at infinity, say $\Gamma = gr(\phi)$, and foliate the domain of dependence of $\Gamma$. The following is then proved:

**Proposition 8.1.23.** *Let $S$ be a maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with principal curvatures $\pm\lambda$ and $||\lambda||_\infty \leq 1$. Then*

$$w(\mathcal{CH}(\partial_\infty S)) \leq 2 \arctan ||\lambda||_\infty\,.$$

## 8.2 An application: Minimal Lagrangian extensions of quasisymmetric homeomorphisms

In this section we will give a relation between the principal curvatures of $S$ and the quasiconformal coefficient of $\Phi$. First, we prove an easy proposition.

The key observation here, given in [BS10], is that, for $\phi \in \mathcal{T}(\mathbb{D})$ a fixed quasisymmetric homeomorphism of the circle, the (unique) maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $\partial_\infty S = gr(\phi)$ corresponds to the minimal Lagrangian extension $\Phi$ of $\phi$. Such extension is given geometrically as

$$\Phi = (\Phi_l)^{-1} \circ \Phi_r\,.$$

where $\Phi_l$ and $\Phi_r$ are the projections defined in Subsection 1.3.1

In [KS07, Lemma 3.16] it is shown that the pull-back of the hyperbolic metric $h$ of $P$ on $S$ by means of $\Phi_r$ and $\Phi_l$ is given by

$$\Phi_l^* h = I((E + JB)\cdot, (E + JB)\cdot)\,, \tag{8.22}$$

and by

$$\Phi_r^* h = I((E - JB)\cdot, (E - JB)\cdot)\,, \tag{8.23}$$

where $I$ is the first fundamental form of $S$, $J$ is the almost-complex structure of $S$, $B$ the shape operator and $E$ the identity. We are now ready to give a relation between the principal curvatures of $S$ and the quasiconformal distortion of $\Phi$:

**Proposition 8.2.1.** *Given a maximal surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, with principal curvatures $\pm\lambda$, the quasiconformal distortion of the minimal Lagrangian map $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ at a point $x$ is given by*

$$K(\Phi_l(x)) = \left(\frac{1 + \lambda(x)}{1 - \lambda(x)}\right)^2\,.$$

*Therefore, by taking $K = \sup_x K(\Phi_l(x))$, namely $K$ is the maximal dilatation of $\Phi$, the following holds:*

$$K = \left(\frac{1 + ||\lambda||_\infty}{1 - ||\lambda||_\infty}\right)^2\,.$$

*Proof.* Let $h$ be the hyperbolic metric of $P$; it follows from the above description that

$$\Phi^* h = h((E+JB)^{-1}(E-JB)\cdot, (E+JB)^{-1}(E-JB)\cdot).$$

The quasiconformal distortion of $\Phi$ at a fixed point $x$ can be computed as the ratio between $\sup ||\Phi_*(v)||$ and $\inf ||\Phi_*(v)||$ where the supremum and the infimum are taken over all tangent vectors $v \in T_x P$ with $||v|| = 1$. Since $B$ is diagonalizable with eigenvalues $\pm\lambda$, $(E+JB)^{-1}(E-JB)$ can be diagonalized to be of the form

$$\begin{pmatrix} \frac{1-\lambda}{1+\lambda} & 0 \\ 0 & \frac{1+\lambda}{1-\lambda} \end{pmatrix}$$

hence the quasiconformal distortion is given by

$$K(\Phi_l(x)) = \left( \frac{\lambda(x)+1}{\lambda(x)-1} \right)^2 .$$

□

*Remark* 8.2.2. The same relation holds in $\mathbb{H}^3$ for $S$ a minimal surface and $\Phi$ is obtained by composing the hyperbolic Gauss maps from the surface to the two connected components of $\partial_\infty \mathbb{H}^3 \setminus \partial_\infty S$. Indeed, we have analogue formulae for the pull-back by $\Phi$, where $E \pm JB$ is replaced by $E \pm B$, recall the definition of first fundamental form at infinity in Subsection 7.1.2. This gives a quantitative proof of the fact that a minimal surface $S$ with principal curvatures in $[-1+\epsilon, 1-\epsilon]$ has boundary at infinity a quasicircle.

This concludes the proof of Theorem 8.A. More precisely, putting together the inequalities in Proposition 8.B, Theorem 8.C and Proposition 8.2.1, we obtain the following:

**Theorem 8.E.** *There exists a constant $C$ such that the minimal Lagrangian quasiconformal extension $\Phi : \mathbb{D} \to \mathbb{D}$ of a quasisymmetric homeomorphism $\phi$ of $S^1$ has quasiconformal coefficient*

$$K(\Phi) \leq \left( \frac{1 + C\sinh(\frac{||\phi||_{cr}}{2})}{1 - C\sinh(\frac{||\phi||_{cr}}{2})} \right)^2$$

*provided $||\phi||_{cr}$ is sufficiently small so that $1 - C\sinh(\frac{||\phi||_{cr}}{2}) > 0$.*

Indeed, by studying the behaviour of the RHS of the inequality of Theorem 8.E, we prove the main result of Section 8.1:

**Theorem 8.A.** *There exist universal constants $\delta$ and $C$ such that, for any quasisymmetric homeomorphism $\phi$ of $S^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian quasiconformal extension $\Phi : \mathbb{D} \to \mathbb{D}$ has maximal dilatation $K(\Phi)$ bounded by the relation*

$$\log K(\Phi) < C ||\phi||_{cr} .$$

As in the hyperbolic case, the arguments of this paper do not provide any explicit value of the constant $C$ in Theorem 8.A.

On the other hand, by using the inequalities in Proposition 8.B, Proposition 8.1.23 and Proposition 8.2.1, we obtain the following estimate in the other direction:

**Theorem 8.F.** *If the quasiconformal coefficient $K = K(\Phi)$ of the minimal Lagrangian extension $\Phi : \mathbb{D} \to \mathbb{D}$ of a quasisymmetric homeomorphism $\phi$ of $S^1$ is in $[1, (1+\sqrt{2})^2)$, then*

$$||\phi||_{cr} \leq 2 \log \left( \frac{(\sqrt{K}+1-\sqrt{2})(\sqrt{K}+1+\sqrt{2})}{(\sqrt{K}-1+\sqrt{2})(1+\sqrt{2}-\sqrt{K})} \right).$$

Let us observe that the function

$$K \mapsto 2 \log \left( \frac{(\sqrt{K}+1-\sqrt{2})(\sqrt{K}+1+\sqrt{2})}{(\sqrt{K}-1+\sqrt{2})(1+\sqrt{2}-\sqrt{K})} \right),$$

which appears in the RHS of Theorem 8.F, is differentiable with derivative at 0 equal to 2. Hence the following holds:

**Theorem 8.D.** *There exist universal constants $\delta$ and $C_0$ such that, for any quasisymmetric homeomorphism $\phi$ of $S^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian quasiconformal extension $\Phi : \mathbb{D} \to \mathbb{D}$ has maximal dilatation $K(\Phi)$ bounded by the relation*

$$C_0 ||\phi||_{cr} \leq \log K(\Phi).$$

*The constant $C_0$ can be taken arbitrarily close to* $1/2$.

In particular, any constant $C$ satisfying the statement of Theorem 8.A cannot be smaller than $1/2$.

# Chapter 9

# Geometric transition of surfaces

The aim of this Chapter is to give a relation of the results presented in Part II, especially in Chapter 4, with those of Chapters 7 and 8. This relation will use in more detail the geometric transition we discussed in Section 1.4.

In Section 9.1 we introduce a connection for half-pipe geometry, show that it coincides with the rescaled limit of the usual connections of Anti-de Sitter and hyperbolic space, and use this connection to define a notion of second fundamental form of a spacelike surface in $\mathbb{HP}^3$. We discuss some of its natural properties, and the behaviour under the geometric transition of [Dan11]. This is stated in Proposition 9.1.13.

We then discuss the rescaling of a family of spacelike curves in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ (namely, the graphs of orientation-preserving homeomorphisms $\phi_t$) under the transition to $\mathbb{HP}^3$. Proposition 9.2.1 shows that the rescaled limit is a curve in $\partial_\infty \mathbb{HP}^3$ which can be essentially identified to the vector field obtained by differentiating $\phi_t$ at $t = 0$.

Putting together these ingredients, in Propisition 9.A we show that the statement of Theorem 4.B in Minkowski geometry (namely, a convex spacelike entire graph in $\mathbb{R}^{2,1}$ has principal curvatures bounded from below by a positive constant if and only if its support function at infinity is in the Zygmund class) is the rescaled version of the analogous statement in Anti-de Sitter space (a convex spacelike surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ has bounded principal curvatures if and only if its boundary at infinity is the graph of a quasisymmetric homeomorphism). More precisely, we show that given a family $S_t$ of surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ satisfying the second pair of equivalent conditions, converging to a totally geodesic plane (for which the principal curvatures are identically zero, and the boundary at infinity is the graph of the identity) in a reasonable way, the dual surfaces $S_t^*$, when rescaled from a point, converge to a surface in $\mathbb{R}^{2,1}$ as in Theorem 4.B.

## 9.1 Connection and curvature in $\mathbb{HP}^3$

In this section we will introduce a connection on $\mathbb{HP}^3$, and consequently a notion of curvature for immersed spacelike surfaces. We will show that this notion is the rescaled limit of the known connections and curvatures for $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and $\mathbb{H}^3$ under the transitional procedures we discussed in Section 1.4. See for instance [Lee04] for more details about the standard theory in Riemannian geometry.

**Definition 9.1.1** (Half-pipe connection)**.** Given two vector fields $V, W$ in $\mathbb{HP}^3$, we define the covariant derivative

$$\nabla_V^{\mathbb{HP}^3} W = (D_V W)^T ,$$

where $D_V W$ is the usual flat connection of the ambient $\mathbb{R}^{2,0,1}$ obtained by differentiating each component, and $(\cdot)^T$ denotes the projection on the half-pipe model determined by the splitting

$$\mathbb{R}^{2,0,1} = T_x\mathbb{HP}^3 \oplus \mathbb{R}x \,. \tag{9.1}$$

We will denote by $M$ the vector field on $\mathbb{HP}^3$ defined by $(0, 0, 1, 0)$ in $\mathbb{R}^{2,0,1}$. It is a degenerate future-directed vector field invariant for the group $\mathrm{Isom}(\mathbb{HP}^3)$ of projective isometries with unit determinant, as one can easily see from the expression of elements of $\mathrm{Isom}(\mathbb{HP}^3)$ discussed in Subsection 1.1.2.

**Proposition 9.1.2.** *The connection $\nabla^{\mathbb{HP}^3}$ defined above is the unique connection on $\mathbb{HP}^3$ which is:*

- *Symmetric;*
- *Compatible with the degenerate metric of $\mathbb{HP}^3$;*
- *Preserving every spacelike plane of $\mathbb{HP}^3$, namely for $V, W$ vector fields on a spacelike plane $P$, $\nabla_V^{\mathbb{HP}^3} W$ is tangent to $P$;*
- *Such that $\nabla^{\mathbb{HP}^3} M = 0$.*

*Proof.* It is straightforward to check that $\nabla^{\mathbb{HP}^3}$ defines a connection on $\mathbb{HP}^3$. Symmetry follows from the observation that

$$\nabla_V^{\mathbb{HP}^3} W - \nabla_W^{\mathbb{HP}^3} V = (D_V W - D_W V)^T = [V, W]^T = [V, W] \,.$$

Also compatibility is very simple: for every vector $Z$ tangent to $\mathbb{HP}^3$,

$$\begin{aligned} Z\langle V, W\rangle_{2,0,1} =& \langle D_Z V, W\rangle_{2,0,1} + \langle V, D_Z W\rangle_{2,0,1} \\ =& \langle \nabla_Z^{\mathbb{HP}^3} V, W\rangle_{2,0,1} + \langle V, \nabla_Z^{\mathbb{HP}^3} W\rangle_{2,0,1} \,. \end{aligned}$$

For the third point, let $P$ be a plane of $\mathbb{HP}^3$ obtained as intersection of $\mathbb{HP}^3$ with a linear plane $P'$ of $\mathbb{R}^{2,0,1}$. Given vector fields $V, W$ on $P$ (it suffices to define them at points of $P$), $D_V W$ is tangent to $P'$ and thus the projection to $\mathbb{HP}^3$ (using the splitting of Equation (9.1)) is still in $P$. Finally, it is clear from the construction that the derivative of $M$ in any direction vanishes.

Let us now assume the four conditions hold. In the coordinate system provided by $\mathbb{H}^2 \times \mathbb{R}$, the restriction of the half-pipe connection to every plane $\mathbb{H}^2 \times \{*\}$ preseves the plane itself (by the third point) and coincides with the Levi-Civita connection of $\mathbb{H}^2$, by the second point. Hence it is easily seen that the Christoffel symbols $\Gamma_{ij}^k$ are those of the Levi-Civita connection when $i, j, k$ correspond to coordinates of $\mathbb{H}^2$. Otherwise, using the first and fourth hypothesis, the $\Gamma_{ij}^k$ vanish. Hence the connection is uniquely determined. ◻

**Corollary 9.1.3.** *The half-pipe connection* $\nabla^{\mathbb{HP}^3}$ *is invariant for the group* $\mathrm{Isom}(\mathbb{HP}^3)$.

*Proof.* Given an isometry $\mathcal{R} \in \mathrm{Isom}(\mathbb{HP}^3)$, define

$$\nabla'_V W = (\mathcal{R}_*)^{-1} \nabla^{\mathbb{HP}^3}_{\mathcal{R}_* V}(\mathcal{R}_* W)\,.$$

It is not difficult to check that $\nabla'$ fulfills all the four conditions of Proposition 9.1.2, hence it coincides with $\nabla^{\mathbb{HP}^3}$. $\square$

**Corollary 9.1.4.** *Geodesics for the half-pipe connection* $\nabla^{\mathbb{HP}^3}$ *coincide with lines of* $\mathbb{HP}^3$.

*Proof.* Given a spacelike line $l$ of $\mathbb{HP}^3$, using the action of $\mathrm{Isom}(\mathbb{HP}^3)$ we can assume $l$ is contained in the slice $\mathbb{H}^2 \times \{0\}$. Since the connection on such slice coincides with the Levi-Civita connection, and lines of $\mathbb{HP}^3$ are geodesics for this copy of $\mathbb{H}^2$, $l$ is a geodesic for $\nabla^{\mathbb{HP}^3}$. If $l$ is not spacelike, then it is of the form $\{*\} \times \mathbb{R}$. Since $\nabla^{\mathbb{HP}^3}_M M = 0$, it is clear by construction that $l$ is geodesic, provided it is parametrized in such a way that its tangent vector is a fixed multiple of $M$ for all time.

Since there is a line of $\mathbb{HP}^3$ through every point of $\mathbb{HP}^3$ with every initial velocity, this shows that all geodesics for the connection $\nabla^{\mathbb{HP}^3}$ are lines of $\mathbb{HP}^3$. $\square$

We are now ready to define the second fundamental form of any spacelike surface in $\mathbb{HP}^3$. Recall that a spacelike surface in $\mathbb{HP}^3$ is locally the graph of a function $\bar{u} : \Omega \to \mathbb{R}$, for $\Omega \subseteq \mathbb{H}^2$, and the first fundamental form is just the hyperbolic metric on the base $\mathbb{H}^2$.

Given a spacelike immersion $\sigma : S \to \mathbb{HP}^3$ and two vector fields $\hat{v}, \hat{w}$ on $S$, using symmetry and compatibility with the metric it is easy to prove that the tangential component of $\nabla^{\mathbb{HP}^3}_{\sigma_* v}(\sigma_* \hat{w})$ in the splitting (9.1), which we denote again by $(\nabla^{\mathbb{HP}^3}_{\sigma_* v}(\sigma_* \hat{w}))^T$, coincides with the Levi-Civita connection of the first fundamental form.

**Definition 9.1.5.** Given a spacelike immersion $\sigma : S \to \mathbb{HP}^3$, the second fundamental form of $S$ is defined by

$$\nabla^{\mathbb{HP}^3}_{\sigma_* v}(\sigma_* \hat{w}) = (\nabla^{\mathbb{HP}^3}_{\sigma_* v}(\sigma_* \hat{w}))^T + I\!I(v, w) M\,,$$

for every pair of vectors $v, w \in T_x S$, where $\hat{w}$ is any extension on $S$ of the vector $w \in T_x S$.

It is easy to check, as in the classical Riemannian case, that $I\!I$ is linear in both arguments, and thus defines a 2-tensor.

**Definition 9.1.6.** The shape operator of $\sigma : S \to \mathbb{HP}^3$ is the $(1,1)$-tensor such that $I\!I(v, w) = I(B(v), w)$ for every $v, w \in T_x S$, where $I$ is the first fundamental form of $S$. The extrinsic curvature of $S$ is the determinant of the shape operator.

As usual the second fundamental form is symmetric and thus the definition does not depend on the extension of any of the vectors $v$ and $w$.

**Lemma 9.1.7.** *Given a spacelike embedded graph $S$ in $\mathbb{HP}^3$, consider the embedding $\sigma : \Omega \to \mathbb{HP}^3 \cong \mathbb{H}^2 \times \mathbb{R}$ defining $S$ as a graph:*

$$\sigma(x) = (x, \bar{u}(x))$$

*for $\bar{u} : \Omega \to \mathbb{R}$ and $\Omega \subseteq \mathbb{H}^2$. Then the shape operator of $S$ for the embedding $\sigma$ is*

$$B = \operatorname{Hess} \bar{u} - \bar{u}\, E\,, \tag{9.2}$$

*where* $\operatorname{Hess} \bar{u} = \nabla^{\mathbb{H}^2} \operatorname{grad} \bar{u}$ *denotes the hyperbolic Hessian of $\bar{u}$.*

*Proof.* Fix a point $x_0 \in \Omega$. By composing with an element in $\operatorname{Isom}(\mathbb{HP}^3)$ of the form $(x,t) \mapsto (x, t + f(x))$, where $f(x) = \langle x, t_0 \rangle_{2,1}$, we can assume $S$ is tangent to the horizontal plane $\mathbb{H}^2 \times \{0\}$ at $x_0$ (i.e. obtained as $\{x^4 = 0\}$ in $\mathbb{R}^{2,0,1}$). Indeed, since $\operatorname{Hess} f - fE = 0$, it suffices to prove the statement in this case.

We consider $\mathbb{H}^2 \times \{0\}$ inside the copy of $\mathbb{R}^{2,1}$ obtained as $\{x^4 = 0\}$ in $\mathbb{R}^{2,0,1}$. Let $\hat{v}$, $\hat{w}$ two vector fields on $\Omega \subseteq \mathbb{H}^2$. Then at $x_0 \in \mathbb{H}^2$ one has

$$D_{\hat{v}} \hat{w} = \nabla^{\mathbb{H}^2}_{\hat{v}} \hat{w} + I(\hat{v}, \hat{w}) x_0\,,$$

since $x_0$ is the normal vector to $\mathbb{H}^2$ at $x_0$ itself and the first and second fundamental form of $\mathbb{H}^2$ coincide.

Consider now the vector fields $\sigma_*(\hat{v}) = \hat{v} + d\bar{u}(\hat{v})M$ and $\sigma_*(\hat{w}) = \hat{w} + d\bar{u}(\hat{w})M$ on $S$. We choose extensions $V$ and $W$ in a neighborhood of $S$ which are invariant for translations $t \mapsto t + t_0$ in the degenerate direction of $\mathbb{H}^2 \times \mathbb{R}$. We can now compute

$$\begin{aligned} D_V W &= D_{\hat{v}} \hat{w} + D_{\hat{v}}(d\bar{u}(\hat{w})M) + d\bar{u}(\hat{v}) D_M W \\ &= \nabla^{\mathbb{H}^2}_{\hat{v}} \hat{w} + I(\hat{v}, \hat{w}) x_0 + \hat{v} \langle \operatorname{grad} \bar{u}, \hat{w} \rangle_{\mathbb{H}^2} M \\ &= \nabla^{\mathbb{H}^2}_{\hat{v}} \hat{w} + I(\hat{v}, \hat{w}) x_0 + \Big( \langle \operatorname{Hess}\bar{u}(\hat{v}), \hat{w} \rangle_{\mathbb{H}^2} + \langle \operatorname{grad} \bar{u}, \nabla^{\mathbb{H}^2}_{\hat{v}} \hat{w} \rangle_{\mathbb{H}^2} \Big) M\,, \end{aligned}$$

where in the first equality we have substituted the expressions for $V$ and $W$, and in the second equality we have used that $DM = 0$ and that the chosen extension $W$ is invariant along the direction of $M$. Since $\operatorname{grad} \bar{u}$ vanishes at $x_0$ by construction, we get at $x_0$:

$$\nabla^{\mathbb{HP}^3}_V W = \nabla^{\mathbb{H}^2}_{\hat{v}} \hat{w} + I(\operatorname{Hess}\bar{u}(v), w)\,.$$

By Proposition 9.1.2, $\nabla^{\mathbb{H}^2}_{\hat{v}} \hat{w}$ is tangent to the slice $\mathbb{H}^2 \times \{0\}$ itself, and thus the second fundamental form is:

$$I\!I(v, w) = I(\operatorname{Hess}\bar{u}(v), w)\,.$$

Since $\bar{u}(x_0) = 0$, this concludes the proof. □

**Corollary 9.1.8.** *A spacelike surface in $\mathbb{HP}^3$ is totally geodesic if and only if $B \equiv 0$.*

*Proof.* Totally geodesic planes are graphs of functions of the form $\bar{u}(x) = \langle x, p \rangle_{2,1}$, which (as showed in Proposition 5.1.3) are exactly the functions such that $\operatorname{Hess} \bar{u} - \bar{u}\, E$ vanishes. □

*Remark* 9.1.9. We have already showed that, given a surface endowed with a hyperbolic metric $h$, namely

$$K_h = -1 \tag{G-$\mathbb{HP}^3$}$$

the tensor $B = \mathrm{Hess}\bar{u} - \bar{u}E$, where $\bar{u} : S \to \mathbb{R}$, satisfies Codazzi equation

$$d^{\nabla_h} B = 0 \tag{Cod}$$

for the hyperbolic metric $h$.

The above two equations (G-$\mathbb{HP}^3$) and (Cod) can be interpreted as a *baby-version* of the Gauss-Codazzi equations for half-pipe geometry. Of course this is a very simple version, since the two equations are not really coupled: the first equation is independent on $B$. Indeed, the proof of the fundamental theorem of immersions is going to be very simple.

**Proposition 9.1.10** (Fundamental property of immersed surfaces in half-pipe geometry)**.** *Let $\tilde{S}$ be a simply connected surface. Given any pair $(h, B)$, where $h$ is a Riemannian metric on $\tilde{S}$ and $B$ is a $(1,1)$-tensor self-adjoint for $h$, such that the equations*

$$\begin{cases} K_h = -1 \\ d^{\nabla^h} B = 0 \end{cases} \tag{GC-$\mathbb{HP}^3$}$$

*are satisfied, there exists a smooth immersion $\sigma : \tilde{S} \to \mathbb{HP}^3$ such that the first fundamental form is $h$ and the shape operator is $B$. Moreover, given any two such immersions $\sigma$ and $\sigma'$, there exists $\mathcal{R} \in \mathrm{Isom}(\mathbb{HP}^3)$ such that $\sigma' = \mathcal{R} \circ \sigma$.*

*Proof.* As a consequence of [OS83], there exists a function $\bar{u} : \tilde{S} \to \mathbb{R}$ such that $B = \mathrm{Hess}\, \bar{u} - \bar{u}\, E$. Let $\mathrm{dev} : \tilde{S} \to \mathbb{H}^2$ be a developing map for the hyperbolic metric $h$ on $\tilde{S}$. Then we define

$$\sigma : \tilde{S} \to \mathbb{HP}^3 \cong \mathbb{H}^2 \times \mathbb{R}$$

by means of

$$\sigma(x) = (\mathrm{dev}(x), \bar{u}(x))\,.$$

Since dev is a local isometry, and the metric of $\mathbb{HP}^3$ has the degenerate form $g_{\mathbb{H}^2} + 0 \cdot dt^2$, the first fundamental form of $\sigma$ is $h$. By Lemma 9.1.7, the shape operator is $B$.

Given any other immersion $\sigma'$ with embedding data $(h, B)$, the projection to the first component is a local isometry, hence it differs from dev by postcomposition by an isometry $A$ of $\mathbb{H}^2$. By composing with an isometry of $\mathbb{HP}^3$ which acts on $\mathbb{H}^2$ by means of $A$ and leaves the coordinate $t$ invariant, we can assume the first component of $\sigma$ and $\sigma'$ in $\mathbb{H}^2 \times \mathbb{R}$ coincide.

Hence we have $\sigma'(x) = (\mathrm{dev}(x), \bar{v}(x))$ where $\bar{v}$ is such that $B = \mathrm{Hess}\, \bar{v} - \bar{v}\, E$. Therefore $\mathrm{Hess}(\bar{u} - \bar{v}) - (\bar{u} - \bar{v})E = 0$, which implies (again by Proposition 5.1.3)

$$\bar{u}(x) - \bar{v}(x) = \langle \mathrm{dev}(x), t_0 \rangle_{2,1}$$

for some vector $t_0 \in \mathbb{R}^{2,1}$. This shows that $\sigma$ and $\sigma'$ differ by the isometry

$$(x, t) \mapsto (x, t + \langle x, t_0 \rangle_{2,1})$$

which is an element of $\mathrm{Isom}(\mathbb{HP}^3)$. This concludes the proof. $\square$

We now compute the embedding data of the dual surface to a strictly convex surface. The formulae we will obtain are exactly the same as in the $\mathbb{A}\mathrm{d}\mathbb{S}^3$-$\mathbb{A}\mathrm{d}\mathbb{S}^3$ duality and in the $\mathbb{H}^3$-$\mathrm{d}\mathbb{S}^3$ duality.

**Corollary 9.1.11.** *Given a spacelike strictly convex immersion in $\mathbb{R}^{2,1}$ (resp. $\mathbb{HP}^3$) with embedding data $(I, B)$, the dual immersion in $\mathbb{HP}^3$ (resp. $\mathbb{R}^{2,1}$) has embedding data $(I\!I\!I, B^{-1})$.*

*Proof.* Clearly it suffices to show the statement for the dual of an immersion $\sigma$ in $\mathbb{R}^{2,1}$. Moreover, the statement is local, hence we can assume $S$ is embedded. Therefore we assume $S$ is a graph over an open subset $\Omega$ of $\mathbb{H}^2$. We have already showed (see Subsection 1.1.3) that the first fundamental form of the dual embedding $\sigma^* = d \circ \sigma$ is the third fundamental form. Moreover, we have showed that the inverse of the shape operator of $S$, by means of the inverse of the Gauss map $G^{-1} : \mathbb{H}^2 \to S \subset \mathbb{R}^{2,1}$, is $B^{-1} = \mathrm{Hess}\, \bar{u} - \bar{u}\, E$, where $\bar{u} : \mathbb{H}^2 \to \mathbb{R}$ is the support function. Since the dual surface of $S$ is precisely the graph of $\bar{u}$, this concludes the proof. □

Pushing the analogy with the Anti-de Sitter and hyperbolic case further, we can prove the following result concerning the existence of surfaces of zero mean curvature in half-pipe geometry.

**Proposition 9.1.12.** *Given any continuous function $\varphi : S^1 \to \mathbb{R}$, there exists a unique complete zero mean curvature smooth spacelike surface $S$ in $\mathbb{HP}^3$ such that $\partial_\infty S = gr(\varphi)$.*

*Proof.* Observe that taking the trace in the expression

$$B = \mathrm{Hess}\, u - u\, E$$

for the shape operator of a spacelike surface, one obtains that a surface has zero mean curvature if and only if

$$\Delta u - 2u = 0\,, \tag{9.3}$$

where $\Delta$ denotes the hyperbolic Laplacian. Hence the proof follows straightforwardly from the existence and uniqueness of solutions to the linear PDE (9.3). □

Let us remark that the results of Chapter 4 can be interpreted as results of existence and uniqueness of surfaces of constant curvature in $\mathbb{HP}^3$ with a given asymptotic boundary.

We will now show that the notion of curvature in half-pipe geometry is the rescaled limit of the usual notions in hyperbolic space and Anti-de Sitter space.

**Proposition 9.1.13.** *Suppose $\sigma_t$ is a $C^2$ family of smooth immersions of a simply connected surface $\tilde{S}$ into $\mathbb{A}\mathrm{d}\mathbb{S}^3$ or $\mathbb{H}^3$, such that $\sigma_0$ is contained in the totally geodesic plane $\{x^3 = 0\}$. Let*

$$\sigma = \lim_{t \to 0} (\mathfrak{r}_t \circ \sigma_t)$$

*be the rescaled immersion in $\mathbb{HP}^3$. Then:*

- *The first fundamental form of $\sigma$ coincides with the first fundamental form of $\sigma_0$:*
$$I(v,w) = \lim_{t\to 0} I_t(v,w)\,;$$
- *The second fundamental form of $\sigma$ is the first derivative of the second fundamental form of $\sigma_t$:*
$$I\!I(v,w) = \lim_{t\to 0} \frac{I\!I_t(v,w)}{t}\,;$$
- *The shape operator $B$ of $\sigma$ is the first derivative of the shape operator $B_t$ of $\sigma_t$:*
$$B(v) = \lim_{t\to 0} \frac{B_t(v)}{t}\,;$$
- *The extrinsic curvature $K^{ext}$ of $\sigma$ is the second derivative of the Gauss-Kronecker curvature $K^{ext}_t = \det B_t$ of $\sigma_t$:*
$$K^{ext}(x) = \lim_{t\to 0} \frac{K^{ext}_t(x)}{t^2}\,.$$

*Proof.* The first point is clear, since we have already showed in Subsection 1.4.1 that $\sigma(x) = (\sigma_0(x), \bar{u}(x))$ for some function $\bar{u}$ which encodes the derivative of the $x^3$-component of $\sigma_t$.

For the second point, we will focus on the case of $\mathbb{H}^3$ for definiteness. Consider the second fundamental form of $\sigma_t$. Given two vectors $v, w \in T_x\tilde{S}$, consider extensions $\hat{v}$ and $\hat{w}$ on a neighborhood of $x$. As $t \to 0$, the connection of $\mathbb{H}^3$ converges to the connection of $\mathbb{HP}^3$. Indeed, recall $\mathfrak{r}_t$ maps $\mathbb{H}^3$ isometrically to
$$\mathbb{H}^3_t := \mathfrak{r}_t(\mathbb{H}^3) = \{x : \langle x,x\rangle^t_{3,1} = -1\}$$
and the connection of $\mathfrak{r}_t(\mathbb{H}^3)$ can be obtained by projecting the flat connection $D_v\hat{w}$ of the ambient space tangentially to $\mathfrak{r}_t(\mathbb{H}^3)$. As $t \to 0$, $\mathfrak{r}_t(\mathbb{H}^3)$ converges to the half-pipe model, hence the tangent projection converges exactly to that of $\mathbb{HP}^3$.

Now, consider the unit normal vector fields $N_t$ to $\sigma_t(\tilde{S})$, chosen so that at time $t = 0$ the vector field is $(0,0,1,0)$ and $N_t$ varies continuously with $t$. Again the computation is local, so we may assume the $\sigma_t$ are embeddings on a subset of $\tilde{S}$. Let us write, using that $\mathfrak{r}_t$ acts as a projective transformation which is an isometry between $\mathbb{H}^3$ and $\mathbb{H}^3_t$:
$$\nabla^{\mathbb{H}^3_t}_{(\mathfrak{r}_t\sigma_t)_*v}((\mathfrak{r}_t\sigma_t)_*\hat{w}) - (\nabla^{\mathbb{H}^3_t}_{(\mathfrak{r}_t\sigma_t)_*v}((\mathfrak{r}_t\sigma_t)_*\hat{w}))^T = I\!I_t(v,w)\mathfrak{r}_t(N_t) = \frac{I\!I_t(v,w)}{t}\mathfrak{r}_t(N_t)t\,.$$
Where in the LHS $(\cdot)^T$ denotes the projection to the tangent plane $T_{\sigma_t(x)}\sigma_t(\tilde{S})$. Thus by the above claim, the LHS converges to $(\nabla^{\mathbb{HP}^3}_{\sigma_* v}(\sigma_*\hat{w}))^T$. On the other hand, if
$$N_t = (N^1_t, N^2_t, N^3_t, N^4_t)\,,$$
with $N_0 = (0,0,1,0)$, we have
$$\mathfrak{r}_t(N_t) = (N^1_t, N^2_t, \frac{N^3_t}{t}, N^4_t)\,,$$

and therefore

$$\mathfrak{r}_t(N_t)t = (tN_t^1, tN_t^2, N_t^3, tN_t^4) \xrightarrow{t\to 0} (0,0,1,0) = M\,.$$

This shows that $\mathit{II}_t(v,w)/t$ converges to the second fundamental form of $\sigma$ in $\mathbb{HP}^3$. The case of rescaling from Anti-de Sitter space is analogous of course.

Since $\mathit{II}_t(v,w) = I_t(B_t(v),w)$ and $\mathit{II}(v,w) = I(B(v),w)$, the third point follows from the first two statements. The last point is a consequence of the third point and the fact that $K_t^{ext} = \det B_t$ and $K^{ext} = \det B$. □

## 9.2 The rescaling of the boundary at infinity

In this section we describe the behaviour of the asymptotic boundary of embedded surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ under the rescaling to $\mathbb{HP}^3$. In this work, we have always considered spacelike surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ whose boundary at infinity is a weakly spacelike curve. Equivalently, the boundary at infinity can be regarded as the graph of an orientation-preserving homeomorphism $\phi : S^1 \to S^1$.

Recall in Subsection 1.1.2 we introduced the boundary at infinity $\partial_\infty\mathbb{HP}^3$ of half-pipe geometry.

**Proposition 9.2.1.** *Let $\phi_t : S^1 \to S^1$ be a differentiable family of orientation-preserving homeomorphisms of $S^1 \cong \mathbb{R}\mathrm{P}^1$, such that $\phi_0 = \mathrm{id}$. Then the rescaled limit of $gr(\phi_t) \subset \partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3$ in $\partial_\infty\mathbb{HP}^3 \cong \partial\mathbb{D}\times\mathbb{R}$ is the graph of the function $\varphi/2$, where $\varphi : S^1 \to \mathbb{R}$ is corresponds to the vector field*

$$\dot\phi(z) = \left.\frac{d}{dt}\right|_{t=0} \phi_t(z)$$

*under the standard trivialization of $S^1$, namely $\dot\phi(z) = iz\varphi(z)$ for $z \in \partial\mathbb{D}$.*

*Proof.* It suffices to prove the statement in the affine charts of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and $\mathbb{HP}^3$ given by $\{x^4 \neq 0\}$. We will identify $S^1$ with the boundary at infinity of the totally geodesic plane $\{x^3 = 0\}$ by means of $\theta \mapsto (\cos\theta, \sin\theta, 0)$. We claim that $gr(\phi_t)$ is composed of points of the form

$$(\cos\theta - h_t(\theta)\sin\theta, \sin\theta + h_t(\theta)\cos\theta, h_t(\theta))\,, \tag{9.4}$$

where

$$h_t(\theta) = \tan\left(\frac{\phi_t(\theta)-\theta}{2}\right)\,. \tag{9.5}$$

Indeed (compare Lemma 8.1.13) the point corresponding to $(\theta, \phi_t(\theta))$ in $\partial_\infty\mathbb{A}\mathrm{d}\mathbb{S}^3 \cong S^1 \times S^1$ is obtained as the intersection of the left ruling through the point $\theta \in S^1$, and the right ruling through $\phi_t(\theta)$. The left ruling is parametrized by

$$(\cos\theta - x\sin\theta, \sin\theta + x\cos\theta, x) = (\sqrt{1+x^2}\cos(\theta+\beta), \sqrt{1+x^2}\sin(\theta+\beta), x)\,,$$

where $\tan\beta = x$, while the right ruling is parametrized by

$$\begin{aligned}(\cos\phi_t(\theta) &+ x\sin\phi_t(\theta), \sin\phi_t(\theta) - x\cos\phi_t(\theta), x)\\ &= (\sqrt{1+x^2}\cos(\phi_t(\theta)-\beta), \sqrt{1+x^2}\sin(\phi_t(\theta)-\beta), x)\,.\end{aligned}$$

Of course at the intersection point the value of $x$ - which will provide the desired value of $h_t(\theta)$) - has to be the same. Hence one obtains

$$\phi_t(\theta) = \theta + 2\beta$$

and thus the expression (9.5).

We are finally ready to compute the rescaled limit of $gr(\phi_t)$. In the affine chart we consider, the rescaling $\mathfrak{r}_t$ acts as $(x, y, z) \mapsto (x, y, z/t)$. Hence we have, for the point at infinity expressed by (9.4)

$$\lim_{t \to 0} \mathfrak{r}_t(\cos\theta - h_t(\theta)\sin\theta, \sin\theta + h_t(\theta)\cos\theta, h_t(\theta)) = \left(\cos\theta, \sin\theta, \left.\frac{d}{dt}\right|_{t=0} h_t(\theta)\right).$$

By computing (under the standard identification of $\mathbb{R}$ to the tangent line of $S^1$)

$$\left.\frac{d}{dt}\right|_{t=0} h_t(\theta) = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \phi_t(\theta)\,,$$

the proof is concluded. □

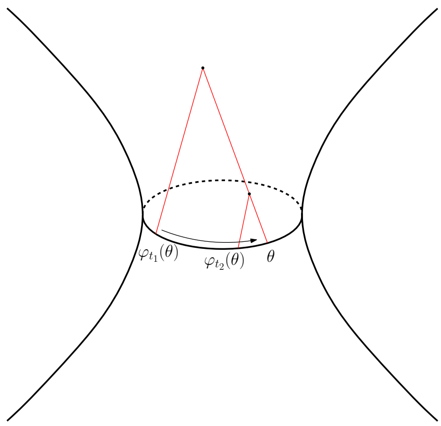



**Figure 9.1:** As $t \to 0$, the curve $gr(\phi_t)$ tends to the boundary of the totally geodesic plane $\{x^3 = 0\}$.

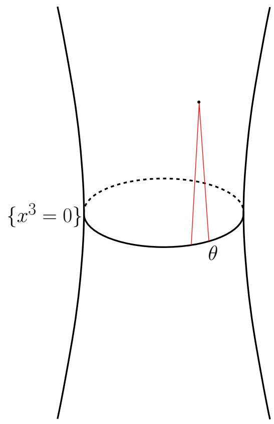



**Figure 9.2:** The rescaling towards half-pipe geometry.

## 9.3 Degeneration of convex surfaces

Already at the beginning of this thesis, we stated that Theorem 4.B is an infinitesimal version of the analogous phenomenon in Anti-de Sitter space. Recall that Theorem 4.B shows that a continuous function $\varphi : S^1 \to \mathbb{R}$ is in the Zygmund class if and only if it is the support function at infinity of a convex surface in Minkowski space with principal curvatures bounded from below by a positive constant. We discussed that this condition is also equivalent to the fact that the dual measured geodesic lamination associated to the domain of dependence defined by $\varphi$ has finite Thurston norm.

The corresponding statement in Anti-de Sitter space is the following.

**Proposition 9.3.1.** *Let $\phi : S^1 \to S^1$ an orientation-preserving homeomorphism of the circle. Then the following are equivalent:*

*i) The homeomorphism $\phi$ is quasisymmetric;*

*ii) The curve $gr(\phi)$ is the boundary at infinity of a convex surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with principal curvatures bounded from above;*

*iii) The curve $gr(\phi)$ is the boundary at infinity of a convex surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ of constant curvature with principal curvatures bounded from above;*

*iv) The bending laminations of the boundary of the convex hull of $gr(\phi)$ are bounded.*

We provide here a sketch of the proof, for convenience of the reader.

*Sketch of proof.* It is obvious that *iii*) implies *ii*). The equivalence between *i*) and *iv*) follows from the construction of Mess in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, which associates an earthquake map extending $\phi$ to the pleated surface having boundary at infinity $gr(\phi)$. Hence, using Theorem 2.3.9, the bending lamination is bounded if and only if $\phi$ is quasisymmetric.

The fact that *ii*) implies *i*) follows from a direct computation of the quasiconformal dilatation of the map $\Phi = (\Phi_l)^{-1} \circ (\Phi_r)$, which is an extension of $\phi$. Using the expression (8.22) and (8.22), one sees easily that if the eigenvalues of $B$ are bounded from above, then the maximal dilatation of $\Phi$ is bounded.

Finally, we show that, if $\phi$ is quasisymmetric, then there exists a surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with bounded principal curvatures and with boundary at infinity $gr(\phi)$. Let us consider the surface at distance $\pi/4$ from the maximal surface $S_0$ bounded by $gr(\phi)$. In [BS10] it was proved that the principal curvatures of $S_0$ are in $[-1+\epsilon, 1-\epsilon]$ since $\phi$ is quasisymmetric. Then using equation (8.21), by a simple computation one notices that the principal curvatures of $S$ are bounded from above, and moreover that $S$ has constant curvature $-2$. This shows *i*) $\Rightarrow$ *iii*) and concludes the proof. □

*Remark* 9.3.2. If $S$ is a strictly convex surface in Anti-de Sitter space, the dual surface $S^*$ has the same boundary at infinity of $S$ and the shape operator of $S^*$ is the inverse of the shape operator of $S$. Hence, if the principal curvatures of $S$ are $\lambda_1, \lambda_2$, the principal curvatures of $S^*$ are $1/\lambda_1, 1/\lambda_2$. Hence the existence of a convex surface with principal curvatures bounded from above is equivalent to the existence of a concave surface with principal curvatures bounded from below.

The following Proposition shows that the situation of Theorem 4.B is the rescaled limit of Proposition 9.3.1.

**Proposition 9.A.** *Let $\sigma_t : \mathbb{H}^2 \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ be a $C^2$ family of smooth embeddings with image surface $S_t = \sigma_t(\mathbb{H}^2)$. Suppose the boundary at infinity of $S_t$ is the graph of the quasisymmetric homeomorphism $\phi_t : S^1 \to S^1$, satisfying the following:*

- *For $t = 0$, $\sigma_t$ is an isometric embedding of the totally geodesic plane $\{x^3 = 0\}$;*
- *The principal curvatures of $S_t$ are $\lambda_i(x) = O(t)$, for $i = 1, 2$, i.e. are uniformly bounded for small $t$ by some function $Ct$ independently of the point $x$;*
- *The path $\phi_t$ is tangent at $\phi_0 = \mathrm{id}$ to a Zygmund field $\dot{\phi}$ on $S^1$.*

*Then the rescaled limit in $\mathbb{R}^{2,1}$ of the surfaces $S_t^*$ dual to $S_t$ is a spacelike entire graph in $\mathbb{R}^{2,1}$, with principal curvatures bounded from below by a positive constant and with support function at infinity the function $\varphi$ (in the Zygmund class) which corresponds to $\dot{\phi}/2$ under the standard trivialization of $TS^1$.*

*Proof.* The proof follows from the previous results. From Propositon 9.2.1 we know that $S_t$ converges to a surface in half-pipe geometry whose boundary at infinity is the graph of $(1/2)\dot{\phi}$, under the usual correspondence between vector fields and functions on $S^1$. Moreover, we know from Propositon 9.2.1 that the rescaled limit of $S_t$ is a surface $S$ in $\mathbb{HP}^3$ with shape operator

$$B = \lim_{t \to 0} \frac{B_t}{t}\,.$$

Hence the principal curvatures of $S$ - in the sense of half-pipe geometry - are bounded from above by a constant $C$.

On the other hand, in Proposition 1.4.2 we showed that the surfaces $S_t^*$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ converge to the surface $S^*$ in $\mathbb{R}^{2,1}$ which is dual - in the $\mathbb{R}^{2,1}$-$\mathbb{HP}^3$ duality - to $S$. Since the shape operator of $S^*$ is the inverse of the shape operator of $S$, $S^*$ has principal curvatures bounded from below by $1/C$ and has support function at infinity the Zygmund function associated to the Zygmund field $\dot{\phi}/2$. □

*Remark* 9.3.3. An analogous statement could be considered by replacing the smooth surfaces $S_t$ with pleated surfaces. The rescaled convergence of pleated surfaces $P_t$ and their bending lamination, with limit in half-pipe geometry, was discussed in [DMS14]. The role of the shape operator is replaced by the bending lamination.

The dual object to a pleated surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ turns out to be a real tree, which we denote by $P_t^*$. Assume for instance the bending lamination of $P_t$ is $t\mu$, where $\mu$ is a measured geodesic lamination. It follows from the arguments in [DMS14] that the rescaled limit of $P_t$ in $\mathbb{HP}^3$ is a pleated surface $P$ with bending lamination $\mu$. When considering the rescaled convergence of $P_t^*$ under the blow-up of the point $(P_0)^{\perp}$ dual to the totally geodesic plane $P_0$, the limit is a real tree in $\mathbb{R}^{2,1}$ which is precisely the dual object to the pleated surface $P$ of $\mathbb{HP}^3$.

# Possible developments

In this thesis, several instances of the link between the infinitesimal theory of Teichmüller spaces and the geometry of surfaces in Minkowski spaces have been studied. Through the idea of geometric transition, results expressed in terms of Minkowski geometry have been related to other results in hyperbolic geometry and Anti-de Sitter geometry.

To the opinion of the author, there is still a number of interesting questions which are left for future developments.

## Convex surfaces in Minkowski space

Theorem 4.B provides a characterization of convex spacelike entire graphs in $\mathbb{R}^{2,1}$ with principal curvatures bounded from below in terms of the regularity of the support function at infinity. An interesting open problem is to characterize complete spacelike entire graphs. It seems a challenging problem to express the condition of completeness in terms of the regularity of the boundary value of the support function.

From Theorem 4.C we know that the surface is complete if this regularity is Zygmund. For the case of surfaces of constant curvature, there exist noncomplete entire graphs, as constructed in Section 4.2. The principal curvatures will obviously be unbounded in this case, and the support function at infinity is not even continuous. This is a striking difference with the case of constant mean curvature surfaces: it is indeed known that a CMC entire graph in $\mathbb{R}^{2,1}$ is complete.

Another interesting question is to solve Minkowski problem for domains of dependence which are not contained in the future of a point. This would include the case of domains of dependence whose support function is finite only on some subset of $\overline{\mathbb{D}}$ which is obtained as the convex hull of a subset $E$ of $\partial\mathbb{D}$. It is simple to check that there is no solution for the constant curvature problem when $E$ contains 0, 1 or 2 points (the corresponding domains are the whole space, the future of a lightlike plane, or the future of a spacelike line). An existence result in case where $E$ is an interval is given in [GJS06] with some assumption on the smoothness of the support function on $E$. It seems that the construction of the support function solving the relevant Monge-Ampère equation is not difficult to generalize in this setting, but the barriers used to prove that the corresponding surfaces are entire graph seem to be ineffective.

## Surfaces in Anti-de Sitter space

In Minkowski space, Theorem 4.C characterized spacelike surfaces with constant curvature and bounded second fundamental form, which are essentially parametrized by the tangent space of universal Teichmüller space. In light of the degeneration of convex surfaces discussed in Chapter 9, it seems a natural question to ask if for any orientation-preserving homeomorphism $\phi$ of $S^1$ there exists a spacelike convex surface $S$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ of constant curvature $K$, with boundary at infinity the graph of $\phi$, for every $K < -1$. For $K = -2$, this was already observed by noticing that the surface at distance $\pi/4$ from the maximal surface satisfies the requirement.

If the above is true, it would be natural to show that the principal curvatures of $S$ are bounded exactly when $\phi$ is quasisymmetric, and that such $K$-surfaces provide a foliation of the complement of the convex hull of the domain of dependence determined by $gr(\phi)$, as $K$ varies in $(-\infty, -1)$. When the curve at infinity $gr(\phi)$ is invariant for the action of the fundamental group of a closed surface, acting by two Fuchsian representations in the two copies of $\mathrm{PSL}(2, \mathbb{R})$, these results were proved in [BBZ11].

This would be an interesting step towards results of *regeneration* of convex surfaces in Minkowski space. Indeed, in Chapter 9 the degeneration of surfaces of constant curvature in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ to surfaces of constant curvature in $\mathbb{R}^{2,1}$ was studied under some assumptions. It would be interesting to show that every $K$-surface which is an entire graph in $\mathbb{R}^{2,1}$ is the rescaled limit of surfaces of constant curvature in $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Regarding maximal surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, we have already remarked that Theorem 8.A is interesting only for quasisymmetric homeomorphisms with small cross-ratio norm ("close" to being a totally geodesic spacelike plane). The author believes an interesting problem is showing that there is an estimate holding for all quasisymmetric homeomorphisms, even with large cross-ratio norm, in which case the width approaches $\pi/2$. In the special case of maximal globally hyperbolic Anti-de Sitter manifolds, such estimate would provide a direct comparison between the Teichmüller distance between the left and right hyperbolic metrics which determine the globally hyperbolic manifold $M$, and the width of the convex hull of $M$.

# Bibliography


[ABB$^+$07] Lars Andersson, Thierry Barbot, Riccardo Benedetti, Francesco Bonsante, William M. Goldman, François Labourie, Kevin P. Scannell, and Jean-Marc Schlenker. Notes on: "Lorentz spacetimes of constant curvature" [Geom. Dedicata **126** (2007), 3–45; mr2328921] by G. Mess. *Geom. Dedicata*, 126:47–70, 2007.

[Ahl38] L. V. Ahlfors. An extension of Schwarz's lemma. *Trans. Amer. Math. Soc.*, 43:359–364, 1938.

[Ahl06] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard.

[And83] Michael T. Anderson. Complete minimal hypersurfaces in hyperbolic $n$-manifolds. *Comment. Math. Helv.*, 58(2):264–290, 1983.

[BA56] A. Beurling and L. Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96:125–142, 1956.

[Bar05] Thierry Barbot. Globally hyperbolic flat space-times. *J. Geom. Phys.*, 53(2):123–165, 2005.

[BBS11] Thierry Barbot, Francesco Bonsante, and Jean-Marc Schlenker. Collisions of particles in locally AdS spacetimes I. Local description and global examples. *Comm. Math. Phys.*, 308(1):147–200, 2011.

[BBS14] Thierry Barbot, Francesco Bonsante, and Jean-Marc Schlenker. Collisions of particles in locally AdS spacetimes II. Moduli of globally hyperbolic spaces. *Comm. Math. Phys.*, 327(3):691–735, 2014.

[BBZ11] Thierry Barbot, François Béguin, and Abdelghani Zeghib. Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes: application to the Minkowski problem in the Minkowski space. *Ann. Inst. Fourier (Grenoble)*, 61(2):511–591, 2011.

[Bel14] Mehdi Belraouti. Sur la géométrie de la singularité initiale des espaces-temps plats globalement hyperboliques. *Annales de l'Institut Fourier*, 64(2):457–466, 2014.

[Ber60] Lipman Bers. Simultaneous uniformization. *Bull. Amer. Math. Soc.*, 66:94–97, 1960.

[BF14] Francesco Bonsante and François Fillastre. The equivariant Minkowski problem in Minkowski space. *ArXiv: 1405.4376*, 2014.

[BG00] Riccardo Benedetti and Enore Guadagnini. Geometric cone surfaces and $(2+1)$-gravity coupled to particles. *Nuclear Phys. B*, 588(1-2):436–450, 2000.

[Bon05] Francesco Bonsante. Flat spacetimes with compact hyperbolic Cauchy surfaces. *J. Differential Geom.*, 69(3):441–521, 2005.

[BP92] Riccardo Benedetti and Carlo Petronio. *Lectures on Hyperbolic Geometry.* Universitext. Springer-Verlag, New York, 1992.

[BS09] Francesco Bonsante and Jean-Marc Schlenker. AdS manifolds with particles and earthquakes on singular surfaces. *Geom. Funct. Anal.*, 19(1):41–82, 2009.

[BS10] Francesco Bonsante and Jean-Marc Schlenker. Maximal surfaces and the universal Teichmüller space. *Invent. Math.*, 182(2):279–333, 2010.

[BS12] Francesco Bonsante and Jean-Marc Schlenker. Fixed points of compositions of earthquakes. *Duke Math. J.*, 161(6):1011–1054, 2012.

[BS15a] Francesco Bonsante and Andrea Seppi. On Codazzi Tensors on a Hyperbolic Surface and Flat Lorentzian Geometry. *To appear on International Mathematics Research Notices*, 2015. ArXiv: 1501.04922.

[BS15b] Francesco Bonsante and Andrea Seppi. Spacelike convex surfaces with prescribed curvature in (2+1)-Minkowski space. *ArXiv: 1505.06748v1*, 2015.

[CB68] Yvonne Choquet-Bruhat. Théorème global d'unicité pour les solutions des équations de'Einstein. *Bull. Soc. Math. France*, 96:181–192, 1968.

[CFKP97] James W. Cannon, William J. Floyd, Richard Kenyon, and Walter R. Parry. Hyperbolic geometry. In *Flavors of geometry*, volume 31 of *Math. Sci. Res. Inst. Publ.*, pages 59–115. Cambridge Univ. Press, Cambridge, 1997.

[Che55a] Shiing-shen Chern. An elementary proof of the existence of isothermal parameters on a surface. *Proceedings of the American Mathematical Society*, 6(5):771–782, 1955.

[Che55b] Shiing-shen Chern. An elementary proof of the existence of isothermal parameters on a surface. *Proc. Amer. Math. Soc.*, 6:771–782, 1955.

[Cos13] Baris Coskunuzer. Asymptotic plateau problem. In *Proc. Gokova Geom. Top. Conf.*, pages 120–146, 2013.

[CT90] Hyeong In Choi and Andrejs Treibergs. Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space. *J. Differential Geom.*, 32(3):775–817, 1990.

[Cus09] Thomas Cuschieri. *Complete Noncompact CMC Surfaces in Hyperbolic 3-Space*. PhD thesis, University of Warwick, 2009.

[CY77] Shiu Yuen Cheng and Shing Tung Yau. On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial s x_j) = F(x, u)$. *Comm. Pure Appl. Math.*, 30(1):41–68, 1977.

[Dan11] Jeffrey Danciger. *Geometric transition: from hyperbolic to AdS geometry*. PhD thesis, Stanford University, 2011.

[Dan13] Jeffrey Danciger. A geometric transition from hyperbolic to anti-de Sitter geometry. *Geom. Topol.*, 17(5):3077–3134, 2013.

[Dan14] Jeffrey Danciger. Ideal triangulations and geometric transitions. *J. Topol.*, 7(4):1118–1154, 2014.

[DE86] Adrien Douady and Clifford J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157(1-2):23–48, 1986.

[DGK13] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Geometry and topology of complete lorentz spacetimes of constant curvature. *To appear on Ann. Sci. Éc. Norm. Supér*, 2013. ArXiv: 1306.2240.

[DMS14] Jeffrey Danciger, Sara Maloni, and Jean-Marc Schlenker. Polyhedra inscribed in a quadric. *ArXiv: 1410.3774*, 2014.

[EM06] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces [mr0903852]. In *Fundamentals of hyperbolic geometry: selected expositions*, volume 328 of *London Math. Soc. Lecture Note Ser.*, pages 117–266. Cambridge Univ. Press, Cambridge, 2006.

[Eps84] Charles L. Epstein. Envelopes of horospheres and Weingarten surfaces in hyperbolic 3-space. Princeton Univ, 1984.

[Eps86] Charles L. Epstein. The hyperbolic Gauss map and quasiconformal reflections. *J. Reine Angew. Math.*, 372:96–135, 1986.

[Eps87] C. L. Epstein. Univalence criteria and surfaces in hyperbolic space. *J. Reine Angew. Math.*, 380:196–214, 1987.

[ES64] J. Eells and J.H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math*, 86:109–159, 1964.

[FKM13] Alastair Fletcher, Jeremy Kahn, and Vladimir Markovic. The moduli space of Riemann surfaces of large genus. *Geom. Funct. Anal.*, 23(3):867–887, 2013.

[FM07] A. Fletcher and V. Markovic. *Quasiconformal maps and Teichmüller theory*, volume 11 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2007.

[FT84a] Arthur E. Fischer and Anthony J. Tromba. On a purely "Riemannian" proof of the structure and dimension of the unramified moduli space of a compact Riemann surface. *Math. Ann.*, 267(3):311–345, 1984.

[FT84b] Arthur E. Fischer and Anthony J. Tromba. On the Weil-Petersson metric on Teichmüller space. *Trans. Amer. Math. Soc.*, 284(1):319–335, 1984.

[FV13] François Fillastre and Giona Veronelli. Lorentzian area measures and the Christoffel problem. *To appear on Annali della Scuola Normale Superiore*, 2013. ArXiv: 1302.6169.

[Gar87] F. P. Gardiner. *Teichmüller theory and quadratic differentials*. Wiley, 1987.

[GHL02] F. P. Gardiner, J. Hu, and N. Lakic. Earthquake curves. In *Complex manifolds and hyperbolic geometry (Guanajuato, 2001)*, volume 311 of *Contemp. Math.*, pages 141–195. Amer. Math. Soc., Providence, RI, 2002.

[GHW10] Ren Guo, Zheng Huang, and Biao Wang. Quasi-Fuchsian 3-manifolds and metrics on Teichmüller space. *Asian J. Math.*, 14(2):243–256, 2010.

[GJS06] Bo Guan, Huai-Yu Jian, and Richard M. Schoen. Entire spacelike hypersurfaces of prescribed Gauss curvature in Minkowski space. *J. Reine Angew. Math.*, 595:167–188, 2006.

[GL00] F. P. Gardiner and N. Lakic. *Quasiconformal Teichmüller Theory*. American Mathematical Soc., 2000.

[Gol80] William M. Goldman. *Discontinuous Groups and the Euler Class*. PhD thesis, University of California, Berkeley, 1980.

[Gol82] William M. Goldman. Characteristic classes and representations of discrete subgroups of Lie groups. *Bull. Amer. Math. Soc. (N.S.)*, 6(1):91–94, 1982.

[Gol84] William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[Gut01] Cristian E. Gutiérrez. *The Monge-Ampère equation*. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston, Inc., Boston, MA, 2001.

[HM12] Jun Hu and Oleg Muzician. Cross-ratio distortion and Douady-Earle extension: I. A new upper bound on quasiconformality. *J. Lond. Math. Soc. (2)*, 86(2):387–406, 2012.

[HN83] Jun-ichi Hano and Katsumi Nomizu. On isometric immersions of the hyperbolic plane into the Lorentz-Minkowski space and the Monge-Ampère equation of a certain type. *Math. Ann.*, 262(2):245–253, 1983.

[HR93] Craig D. Hodgson and Igor Rivin. A characterization of compact convex polyhedra in hyperbolic 3-space. *Invent. Math.*, 111(1):77–111, 1993.

[Huy05] D. Huybrechts. *Complex Geometry: An Introduction.* Universitext (1979). Springer, 2005.

[HW13a] Zheng Huang and Biao Wang. Counting minimal surfaces in quasifuchsian manifolds. *To appear in Trans. Amer. Math. Soc.*, 2013.

[HW13b] Zheng Huang and Biao Wang. On almost-Fuchsian manifolds. *Trans. Amer. Math. Soc.*, 365(9):4679–4698, 2013.

[JMR11] T. Jeffres, R. Mazzeo, and Y.A. Rubinstein. Kähler-Einstein metrics with edge singularities. *ArXiv: 1105.5216*, 2011.

[Jos06] Jürgen Jost. *Compact Riemann surfaces.* Universitext. Springer-Verlag, Berlin, third edition, 2006. An introduction to contemporary mathematics.

[Koe09] Paul Koebe. Über die Uniformisierung der algebraischen Kurven. I. *Math. Ann.*, 67(2):145–224, 1909.

[KS07] Kirill Krasnov and Jean-Marc Schlenker. Minimal surfaces and particles in 3-manifolds. *Geom. Dedicata*, 126:187–254, 2007.

[KS08] Kirill Krasnov and Jean-Marc Schlenker. On the renormalized volume of hyperbolic 3-manifolds. *Comm. Math. Phys.*, 279(3):637–668, 2008.

[Lab92] François Labourie. Surfaces convexes dans l'espace hyperbolique et $\mathbf{C}\mathrm{P}^1$-structures. *J. London Math. Soc. (2)*, 45(3):549–565, 1992.

[Lee04] John M Lee. *Riemannian Manifolds: An Introduction to Curvature.* September 2004.

[Leh83] Matti Lehtinen. The dilatation of Beurling-Ahlfors extensions of quasisymmetric functions. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 8(1):187–191, 1983.

[Leh87] O. Lehto. *Univalent functions and Teichmüller spaces.* Springer Verlag, Berlin 1987.

[Li95] An-Min Li. Spacelike hypersurfaces with constant Gauss-Kronecker curvature in the Minkowski space. *Arch. Math.*, 64:534–551, 1995.

[Lic16] Leon Lichtenstein. Zur theorie der konformen abbildung. *Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Sér. A.*, pages 192–217, 1916.

[LS14] Cyril Lecuire and Jean-Marc Schlenker. The convex core of quasifuchsian manifolds with particles. *Geom. Topol.*, 18(4):2309–2373, 2014.

[Mes07] Geoffrey Mess. Lorentz spacetimes of constant curvature. *Geom. Dedicata*, 126:3–45, 2007.

[Mon10] Gabriele Mondello. Poisson structures on the Teichmüller space of hyperbolic surfaces with conical points. In *In the tradition of Ahlfors-Bers. V*, volume 510 of *Contemp. Math.*, pages 307–329. Amer. Math. Soc., Providence, RI, 2010.

[MS09] Sergiu Moroianu and Jean-Marc Schlenker. Quasi-Fuchsian manifolds with particles. *J. Differential Geom.*, 83(1):75–129, 2009.

[MŠ12] Hideki Miyachi and Dragomir Šarić. Uniform weak* topology and earthquakes in the hyperbolic plane. *Proc. Lond. Math. Soc. (3)*, 105(6):1123–1148, 2012.

[NN57] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. of Math. (2)*, 65:391–404, 1957.

[OS83] V. I. Oliker and U. Simon. Codazzi tensors and equations of Monge-Ampère type on compact manifolds of constant sectional curvature. *J. Reine Angew. Math.*, 342:35–65, 1983.

[Pet06] Peter Petersen. *Riemannian Geometry.* Number 171 in Graduate Text in Mathematics. Springer-Verlag, 2006.

[Pog71] A. V. Pogorelov. A regular solution of the $n$-dimensional Minkowski problem. *Soviet Math. Dokl.*, 12:1192–1196, 1971.

[Poi08] H. Poincaré. Sur l'uniformisation des fonctions analytiques. *Acta Math.*, 31(1):1–63, 1908.

[Rat48] John G. Ratcliffe. *Foundations of hyperbolic manifolds.* Number 149 in Graduate Text in Mathematics. Springer-Verlag, 1948.

[Roc70] R. Tyrrell Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[San14] Andrew Sanders. Entropy, minimal surfaces, and negatively curved manifolds. *Arxiv: 1404.1105*, 2014.

[Šar06] Dragomir Šarić. Real and complex earthquakes. *Transactions of the American Mathematical Society*, 358(1):233–249 (electronic), 2006.

[Šar08] Dragomir Šarić. Bounded earthquakes. *Proceedings of the American Mathematical Society*, 136(3):889–897 (electronic), 2008.

[Sca96] Kevin P. Scannell. *Flat conformal structures and causality in de Sitter manifolds*. PhD thesis, University of California, Los Angeles, 1996.

[Sca12] Carlos Scarinci. *The phase space of 2+1 AdS gravity*. PhD thesis, University of Nottingham, 2012.

[Sch98] Jean-Marc Schlenker. Métriques sur les polyèdres hyperboliques convexes. *J. Differential Geom.*, 48(2):323–405, 1998.

[Sep14] Andrea Seppi. Minimal surfaces in Hyperbolic space and maximal surfaces in Anti-de Sitter space. *ArXiv: 1411.3412v1*, 2014.

[SK13] Carlos Scarinci and Kirill Krasnov. The universal phase space of $AdS_3$ gravity. *Comm. Math. Phys.*, 322(1):167–205, 2013.

[ST11] Georg Schumacher and Stefano Trapani. Weil-Petersson geometry for families of hyperbolic conical Riemann surfaces. *Michigan Math. J.*, 60(1):3–33, 2011.

[Sug07] Toshiyuki Sugawa. The universal Teichmüller space and related topics. In *Quasiconformal mappings and their applications*, pages 261–289. Narosa, New Delhi, 2007.

[Thu86] William P. Thurston. Earthquakes in two-dimensional hyperbolic geometry. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 91–112. Cambridge Univ. Press, Cambridge, 1986.

[Thu97a] William P. Thurston. *Three-Dimensional Geometry and Topology*. Princeton University Press, 1997.

[Thu97b] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[Tou15] Jérémy Toulisse. Maximal surfaces in anti-de sitter 3-manifolds with particles. *ArXiv: 1312.2724*, 2015.

[Tre82] Andrejs E. Treibergs. Entire spacelike hypersurfaces of constant mean curvature in Minkowski space. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 229–238. Princeton Univ. Press, Princeton, N.J., 1982.

[Tro91] Marc Troyanov. Prescribing curvature on compact surfaces with conical singularities. *Trans. Amer. Math. Soc.*, 324(2):793–821, 1991.

[TW08] N. Trudinger and X.-J. Wang. The Monge-Ampére equation and its geometric applications. In *Handbook of geometric analysis*, volume 7 of *Adv. Lect. Math. (ALM)*, pages 467–524. Int. Press, Somerville, MA, 2008.

[Uhl83] Karen K. Uhlenbeck. Closed minimal surfaces in hyperbolic 3-manifolds. In *Seminar on minimal submanifolds*, volume 103 of *Ann. of Math. Stud.*, pages 147–168. Princeton Univ. Press, Princeton, NJ, 1983.

[Wit89] Edward Witten. Quantum field theory and the Jones polynomial. In *Braid group, knot theory and statistical mechanics*, volume 9 of *Adv. Ser. Math. Phys.*, pages 239–329. World Sci. Publ., Teaneck, NJ, 1989.

[ZT87] P. G. Zograf and L. A. Takhtadzhyan. On the uniformization of Riemann surfaces and on the Weil-Petersson metric on the Teichmüller and Schottky spaces. *Mat. Sb. (N.S.)*, 132(174)(3):304–321, 444, 1987.