Exercice Calculer la matrice jacobienne des fonctions $f : \mathbb{R}^2 \to \mathbb{R}$;

- $f(x,y) = (x + 3y^2,\ 6x + 7y^2)$

$$J_f(x,y) = \begin{pmatrix} 1 & 6y \\ 6 & 14y \end{pmatrix}$$

- $f(x,y) = (2xy,\ 1 - x^3y - 4xy^2)$

$$J_f(x,y) = \begin{pmatrix} 2y & 2x \\ -3x^2y - 4y^2 & -x^3 - 8xy \end{pmatrix}$$

- $f(x,y) = (e^x \sin y,\ -e^x \cos y)$

$$J_f(x,y) = \begin{pmatrix} e^x \sin y & e^x \cos y \\ -e^x \cos y & e^x \sin y \end{pmatrix}$$

<u>Exercice</u> Calculer la matrice hessienne et le laplacien des fonctions :

- $f_1 : \mathbb{R}^2 \to \mathbb{R}$ $\quad f_1(x,y) = \frac{1}{2}(ax^2 + 2bxy + cy^2)$

$\nabla f_1(x,y) = (ax+by,\ bx+cy)$

$D^2 f_1(x,y) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\quad \Delta f(x,y) = a+c$

- $f_2 : \mathbb{R}^2 \to \mathbb{R}$ $\quad f_2(x,y) = \sqrt{x^2+y^2}$

$\nabla f_2(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}} \right)$

$$D^2 f_2(x,y) = \begin{pmatrix} \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} & -\frac{xy}{(x^2+y^2)^{3/2}} \\ -\frac{xy}{(x^2+y^2)^{3/2}} & \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)^{3/2}} \end{pmatrix}$$

$$= \frac{1}{(x^2+y^2)^{3/2}} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

$$\Delta f_2(x,y) = \frac{1}{(x^2+y^2)^{3/2}}(x^2+y^2) = \frac{1}{\sqrt{x^2+y^2}}$$

Remarque : le calcul est plus simple en coordonnées polaires :

$$f_2(r,\theta) = r$$

$$\Delta f_2(r,\theta) = \frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_2}{\partial \theta^2} = \frac{1}{r}$$

car $\frac{\partial f_2}{\partial r} = 1$, $\frac{\partial^2 f_1}{\partial r^2} = 0$, $\frac{\partial f_2}{\partial \theta} = 0$, $\frac{\partial^2 f_2}{\partial \theta^2} = 0$.

- $f_3 : \mathbb{R}^n \to \mathbb{R}$ $\quad f_3(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$

$$\nabla f_3(x_1, \dots, x_n) = (2x_1, \dots, 2x_n)$$

$$D^2 f_3(x_1, \dots, x_n) = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 2 \end{pmatrix}$$

$$\Delta f_3(x_1, \dots, x_n) = 2n$$

Remarque : en fait, en utilisant la formule

$\Delta f_3 = h''(r) + \frac{n-1}{r} h'(r)$, où $f_3(x_1, \dots, x_n) = h(r) = r^2$

est une fonction radiale, on obtient :

$$\Delta f_3(x_1, \dots, x_n) = 2 + \frac{n-1}{r} \cdot 2r = 2 + (2n-2) = 2n.$$

Exercice Vérifier que la fonction $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$

$$u(x,y) = \frac{x}{x^2+y^2}$$

est harmonique (i.e. $\Delta u = 0$).

1) En coordonnées polaires,

$$u(r,\theta) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial u}{\partial r}(r,\theta) = -\frac{\cos\theta}{r^2} \qquad \frac{\partial^2 u}{\partial r^2}(r,\theta) = \frac{2\cos\theta}{r^3}$$

$$\frac{\partial u}{\partial \theta}(r,\theta) = -\frac{\sin\theta}{r} \qquad \frac{\partial^2 u}{\partial \theta^2}(r,\theta) = -\frac{\cos\theta}{r}$$

$$\Delta u(r,\theta) = \frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(r,\theta) =$$

$$= \frac{2\cos\theta}{r^3} - \frac{\cos\theta}{r^3} - \frac{\cos\theta}{r^3} = 0.$$

2) En coordonnées cartesiennes,

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x(x^2+y^2)^2 - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} =$$

$$= \frac{(x^2+y^2)(-2x(x^2+y^2) - 4x(y^2-x^2))}{(x^2+y^2)^4} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2x(x^2+y^2)^2 + 2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} =$$

$$= \frac{(x^2+y^2)(-2x(x^2+y^2)+8xy^2)}{(x^2+y^2)^4} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

Du coup $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

Exercice Trouver les solutions radiales $u = u(r)$ sur $\mathbb{R}^2$ de l'équation $\Delta u = 0$.

En coordonnées polaires, $u = u(r, \theta) = f(r)$ ne dépend pas de θ.

On comp,

$$\Delta u(r_0, \theta_0) = \frac{\partial^2 u}{\partial r^2}(r_0, \theta_0) + \frac{1}{r_0}\frac{\partial u}{\partial r}(r_0, \theta_0) + \frac{1}{r_0^2}\frac{\partial^2 u}{\partial \theta^2}(r_0, \theta_0)$$

$$= f''(r_0) + \frac{1}{r_0} f'(r_0) = 0$$

Soit $h = f'$. Alors on a

$$h'(t) + \frac{1}{t} h(t) = 0$$

$$\frac{h'(t)}{h(t)} = -\frac{1}{t} \quad (\text{si } h(t) \neq 0) \quad t \geq 0$$

$$\Rightarrow \quad \log h(t) = -\log t + c \quad (\text{si } h(t) > 0)$$
$$\log(-h(t)) = -\log t + c \quad (\text{si } h(t) < 0)$$

$$\Rightarrow \quad h(t) = \frac{e^c}{t} \quad (\text{si } h(t) \geq 0)$$
$$h(t) = -\frac{e^c}{t} \quad (\text{si } h(t) \geq 0)$$

en récupérant $h(t) \equiv 0$, on obtient la solution générale $h(t) = \frac{\hat{c}}{t}$, $\hat{c} \in \mathbb{R}$.

En intégrant, $f(t) = \int h(t)\,dt = \hat{c} \log t + \hat{D}$

$$u(r, \theta) = f(r) = A \log r + B \qquad A, B \in \mathbb{R}.$$

Exercice Soit $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ une fonction radiale, c'est-à-dire, $u = f(r)$ où $f : (0, +\infty) \to \mathbb{R}$

$r = \sqrt{x_1^2 + \dots + x_n^2}$.

1) Calculer Δu.

Pour tout $i = 1, \dots, n$, car $u = f \circ r$, $r : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$

$$\frac{\partial u}{\partial x_i}(x_1, \dots, x_n) = \frac{\partial r}{\partial x_i}(x_1, \dots, x_n) \cdot f'(r)$$

$$= \frac{x_i}{r} f'(r).$$

$$\frac{\partial^2 u}{\partial x_i}(x_1, \dots, x_n) = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r} f'(r)\right) =$$

$$= \frac{x_i}{r} \frac{\partial}{\partial x_i}(f'(r)) + \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) \cdot f'(r) =$$

$$= \frac{x_i}{r} \cdot \frac{\partial r}{\partial x_i} \cdot f''(r) + \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) f'(r)$$

$$= \frac{x_i^2}{r^2} f''(r) + \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) f'(r).$$

Donc (rappel $r^2 = x_1^2 + \dots + x_n^2$)

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \left(\sum_{i=1}^n \frac{x_i^2}{r^2}\right) f''(r) + \left(\sum_{i=1}^n \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right) f'(r)$$

$$= f''(r) + \left(\frac{n}{r} - \frac{r^2}{r^3}\right) f'(r) = f''(r) + \frac{n-1}{r} f'(r).$$

2) Trouver les solutions radiales de $\Delta u = 0$ pour $n > 2$ (on a déjà fait ça si $n=2$).

On cherche $u(x_1, \dots, x_n) = f(r)$ telle que

$$\Delta u = f'' + \frac{n-1}{r} f' = 0$$

Soit $h = f'$. On a $h'(r) + \frac{n-1}{r} h(r) = 0$

$$\frac{h'(r)}{h(r)} = \frac{1-n}{r} \qquad (\text{si } h(r) \neq 0)$$

$$\Rightarrow \begin{array}{ll} \log h(r) = (1-n) \log r + C & \text{si } h(r) > 0 \\ \log(-h(r)) = (1-n) \log r + C & \text{si } h(r) < 0 \end{array}$$

$$\Rightarrow h(r) = \pm e^C r^{1-n}$$

En récupérant $h(r) = 0$, on obtient la solution générale $h(r) = \frac{\hat{C}}{r^{n-1}}$, $\hat{C} \in \mathbb{R}$, $n \geq 3$

Du coup,

$$f(r) = \int h(r) dr = \hat{C} \int \frac{dr}{r^{n-1}} = \frac{\hat{C}}{(n-2) r^{n-2}} + \hat{D}$$

car $n-2 \geq 0$.

En conclusion, la solution générale est

$u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$

$$u = \frac{A}{r^{n-2}} + B, \qquad A, B \in \mathbb{R}.$$