<u>Théorème</u> Si $f : \Omega \to \mathbb{R}$ est Gâteaux-différentiable dans un voisinage de $x_0 \in \Omega$ et ses dérivées partielles sont continues dans x_0, alors f est Fréchet-différentiable.

Dans l'exemple, calculons $\partial_x f$: $v = (u, w) = (1, 0)$

$$\partial_x f(0,0) = 1$$

$$\partial_x f(x,y) = \frac{3x^2(x^2+y^2) - x^3 \cdot 2x}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}$$

donc si $y = 0$, $\partial_x f(x,0) = 0$

$$\partial_x f(x,0) \underset{x \to 0}{\not\to} \partial_x f(0,0).$$

Une fonction $f : \Omega \to \mathbb{R}$ est de classe C^k si toutes les dérivées partielles $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}$ — existent jusqu'au ordre k et ils sont continues dans Ω.

<u>Théorème de Schwarz</u>

Si les dérivées deuxièmes $\frac{\partial^2 f}{\partial x_i \partial x_j}$ et $\frac{\partial^2 f}{\partial x_j \partial x_i}$ existent dans un voisinage de x_0 et ils sont continues en x_0, alors $\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_0)$.

Fonctions à valeurs vectorielles

<u>Déf</u> Une fonction $f : \Omega \to \mathbb{R}^m$, $\Omega \subseteq \mathbb{R}^n$ ouvert, est Fréchet-différentiable s'il existe $l : \mathbb{R}^n \to \mathbb{R}^m$ linéaire telle que

$$f(x) = f(x_0) + l(x - x_0) + r(x)$$

où $$\lim_{x \to x_0} \frac{r(x)}{\|x - x_0\|} = 0$$

Dans ce cas, $l = df(x_0)$

$f : \Omega \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$, est Fréchet-différentiable

$\Longleftrightarrow$ $f = (f_1, \dots, f_m)$ $\quad f_i : \Omega \to \mathbb{R}$

f_i Fréchet différentiable

En plus, $df = (df_1, \dots, df_m)$

La matrice associée à l'application $l = df$ est appellée <u>matrice Jacobienne</u>

$$Jf(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}$$

Avec cette définition, $df(x_0)(v) = Jf(x_0) \cdot v$

Théorème Si $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ est différentiable dans $x_0 \in \Omega$

et $g : \Omega' \subseteq \mathbb{R}^m \to \mathbb{R}^k$ est différentiable dans $y_0 = f(x_0) \in \Omega'$,

alors $g \circ f$ est différentiable dans $g(f(x_0))$

et $$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

$$\frac{\partial (g \circ f)}{\partial x_i}(x_0) = \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(f(x_0)) \frac{\partial f}{\partial x_i}(x_0)$$

$$\Rightarrow \quad \underbrace{J_{g \circ f}(x_0)}_{k \times n} = \underbrace{J_g(f(x_0))}_{k \times m} \cdot \underbrace{J_f(x_0)}_{m \times n}$$

Chain rule

Règle de la chaîne

Exemple si $f : \mathbb{R}^n \to \mathbb{R}^n$ est l'application identité

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

alors $$J_f(x_0) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$ i.e. $df = id : \mathbb{R}^n \to \mathbb{R}^n$

Exemple $f : (0, \infty) \times (-\pi, \pi) \to \mathbb{R}^2 \setminus \{(t, 0) \mid t \leq 0\}$

$f(r, \theta) = (r\cos\theta, r\sin\theta)$

$$J_f(r_0, \theta_0) = \begin{pmatrix} \frac{\partial f_1}{\partial r}(r_0, \theta_0) & \frac{\partial f_1}{\partial \theta}(r_0, \theta_0) \\ \frac{\partial f_2}{\partial r}(r_0, \theta_0) & \frac{\partial f_2}{\partial \theta}(r_0, \theta_0) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_0 & -r_0\sin\theta_0 \\ \sin\theta_0 & r_0\cos\theta_0 \end{pmatrix}$$

L'inverse de $f : (0,\infty) \times (-\pi,\pi) \to \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\}$

est
$$g : \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\} \to (0,\infty) \times (-\pi,\pi)$$
$$g(x,y) = \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right)$$

$$J_g(x_0,y_0) = \begin{pmatrix} \dfrac{x_0}{(x_0^2+y_0^2)^{1/2}} & \dfrac{y_0}{(x_0^2+y_0^2)^{1/2}} \\[2ex] \dfrac{-y_0/x_0^2}{1+y_0^2/x_0^2} & \dfrac{1/x_0^2}{1+y_0^2/x_0^2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{x_0}{(x_0^2+y_0^2)^{1/2}} & \dfrac{y_0}{(x_0^2+y_0^2)^{1/2}} \\[2ex] -\dfrac{y_0}{x_0^2+y_0^2} & \dfrac{x_0}{x_0^2+y_0^2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{r_0 \cos\theta_0}{r_0} & \dfrac{r_0 \sin\theta_0}{r_0} \\[2ex] -\dfrac{r_0 \sin\theta_0}{r_0^2} & \dfrac{r_0\cos\theta_0}{r_0^2} \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\[1ex] -\dfrac{\sin\theta_0}{r_0} & \dfrac{\cos\theta_0}{r_0} \end{pmatrix}$$

En effet, $\quad g \circ f = id \Rightarrow dg(x_0,y_0) \circ df(r_0,\theta_0) = id$

$$\Rightarrow J_g(x_0,y_0) \cdot J_f(r_0,\theta_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(x_0,\theta_0) = f(r_0,\theta_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$$

$$J_g(x_0,y_0) = J_f(r_0,\theta_0)^{-1}.$$

<u>Vérifier</u> !

Laplacien en coordonnées polaires

Si $u : \mathbb{R}^n \to \mathbb{R}$ est $\mathcal{C}^2$, le Laplacien de u est défini :

$$\Delta u\,(x_1^0, \dots, x_n^0) = \frac{\partial^2 u}{\partial x_1^2}(x_1^0, \dots, x_n^0) + \dots + \frac{\partial^2 u}{\partial x_n^2}(x_1^0, \dots, x_n^0)$$

si $n = 2$, $\quad \Delta u\,(x_0, y_0) = \frac{\partial^2 u}{\partial x^2}(x_0, y_0) + \frac{\partial^2 u}{\partial y^2}(x_0, y_0)$.

Calculons Δu en coordonnées polaires, c'est-à-dire :

$$\begin{tikzcd}
\mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\} \arrow[rr, "u"] \arrow[dr, bend right, "g"'] & & \mathbb{R} \\
& \mathbb{R}_{>0} \times (-\pi, \pi) \arrow[ul, "f"'] \arrow[ur, "\hat{u}"'] &
\end{tikzcd}$$

$\hat{u}(r_0, \theta_0) = u(x_0, y_0)$

où $(x_0, y_0) = f(r_0, \theta_0)$

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial r}{\partial x}(x_0, y_0)\,\frac{\partial \hat{u}}{\partial r}(r_0, \theta_0) + \frac{\partial \theta}{\partial x}(x_0, y_0)\,\frac{\partial \hat{u}}{\partial \theta}(r_0, \theta_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial r}{\partial y}(x_0, y_0)\,\frac{\partial \hat{u}}{\partial r}(r_0, \theta_0) + \frac{\partial \theta}{\partial y}(x_0, y_0)\,\frac{\partial \hat{u}}{\partial \theta}(r_0, \theta_0)$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{u}}{\partial r} \\ \frac{\partial \hat{u}}{\partial \theta} \end{pmatrix}$$

$$du(x_0, y_0) = d\hat{u}(r_0, \theta_0) \circ dg(x_0, y_0)$$

$$\nabla u = \nabla \hat{u} \cdot J_g$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{u}}{\partial r} & \frac{\partial \hat{u}}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix}.$$

$$\frac{\partial u}{\partial x} = \cos\theta \, \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \, \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \sin\theta \, \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \, \frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta \, \frac{\partial}{\partial r}\left(\cos\theta \, \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \, \frac{\partial u}{\partial \theta}\right)$$

$$- \frac{\sin\theta}{r} \, \frac{\partial}{\partial \theta}\left(\cos\theta \, \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \, \frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta \, \frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta\cos\theta}{r} \, \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin\theta\cos\theta}{r^2} \, \frac{\partial u}{\partial \theta}$$

$$+ \frac{\sin^2\theta}{r} \, \frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r} \, \frac{\partial^2 u}{\partial \theta \partial r} + \frac{\sin^2\theta}{r^2} \, \frac{\partial^2 u}{\partial \theta^2} + \frac{\sin\theta\cos\theta}{r^2} \, \frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \sin\theta \, \frac{\partial}{\partial r}\left(\sin\theta \, \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \, \frac{\partial u}{\partial \theta}\right)$$

$$+ \frac{\cos\theta}{r} \, \frac{\partial}{\partial \theta}\left(\sin\theta \, \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \, \frac{\partial u}{\partial \theta}\right)$$

$$= \sin^2\theta \, \frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta\cos\theta}{r} \, \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r^2} \, \frac{\partial u}{\partial \theta}$$

$$+ \frac{\sin\theta\cos\theta}{r} \, \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r} \, \frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2} \, \frac{\partial^2 u}{\partial \theta^2} - \frac{\cos\theta\sin\theta}{r^2} \, \frac{\partial u}{\partial \theta}$$

Du coup,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2} \, \frac{\partial^2 u}{\partial \theta^2}$$

<u>Laplacien en coordonnées polaires.</u>