QUASI-FUCHSIAN, ALMOST-FUCHSIAN AND NEARLY-FUCHSIAN MANIFOLDS

Shanghai Institute for Mathematics and Interdisciplinary Sciences

Lecture II, 01/07/2025

__Theorem__ (Nguyen - Schlenker - S. '25)

If a complete hyperbolic manifold $(M \simeq \Sigma \times \mathbb{R}, h)$ is weakly almost-Fuchsian, then it is nearly-Fuchsian.

Σ closed orientable surface

minimal surface $\sim \Sigma \times \{*\}$ with principal curvatures in $[-1,1]$

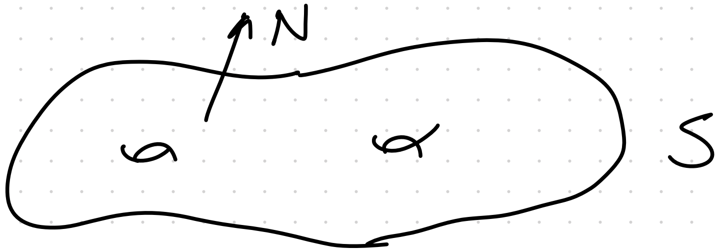
closed (not minimal) surface with principal curvatures in $(-1,1)$

Theorem (Nguyen - Schlenker - S. '25)

Let (M, h) be a hyperbolic manifold, let $S \subset M$ be an embedded, orientable, two-sided closed minimal surface with principal curvatures in $[-1, 1]$.

Then any neighbourhood U of S in M contains a (non-minimal) surface with principal curvatures in $(-1, 1)$.

Idea:



Find a "magic" function $f \in \mathcal{C}^\infty(S, \mathbb{R})$ such that

$$S_{tf} := \{ \exp_p(tfN(p)) \mid p \in S \}$$

has principal curvatures in $(-1,1)$ for small t.

embedded for small t

A little bit of differential geometry

Recall I = first fund. form

II = second fund. form

$B = -\nabla^M_{\bullet} N$ = shape operator

Weingarten equation: $\mathrm{II}(X,Y) = \mathrm{I}(B(X), Y)$

Principal curvatures are the eigenvalues of B

denoted $\lambda^+ \geq 0$, $\lambda^- \leq 0$

S minimal: $\lambda^- = -\lambda^+$

Introduce: $\|\mathrm{II}\|^2 = \operatorname{tr} B^2 = (\lambda^+)^2 + (\lambda^-)^2 = 2(\lambda^+)^2$

Principal curvatures in $[-1,1] \iff \|\mathrm{II}\|^2 \leq 2$

shape operator for S_{tf} (1,1)-Hessian

$$\frac{d}{dt}\Big|_{t=0} B_{tf} = \operatorname{Hess}^{\mathrm{I}} f + f\,\cancel{(B^2 - \mathrm{id})}$$

$$= \operatorname{Hess}^{\mathrm{I}} f$$ at $p \in S$ such that $\|\mathrm{II}(p)\|^2 = 2$

$B \sim \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $B^2 = \mathrm{id}$

(0,2)-Hessian

$$(\nabla^{\mathrm{I}}_{\cdot} df)(X, Y) = \mathrm{I}(\operatorname{Hess}^{\mathrm{I}} f(X), Y)$$

$$\frac{d}{dt}\Big|_{t=0} \lambda^{\pm}_{tf}(p) = (\nabla^S df)(e^{\pm}(p), e^{\pm}(p))$$

(e^+, e^-) orthonormal frame, $B(e^{\pm}) = \lambda^{\pm} e^{\pm}$

Denote $Z = \{ p \in S \mid \|\mathrm{II}(p)\|^2 = 2 \}$.

<u>Goal</u>: Construct $f \in \mathcal{C}^\infty(S)$ such that:

$$(\nabla^I df)(e^\pm(p), e^\pm(p)) = \mp 1$$

for every $p \in Z$

Intermediate step: understand the structure of Z

<u>Good</u> we only care about the entries on the diagonal

$$\nabla df = \begin{pmatrix} -1 & * \\ * & 1 \end{pmatrix}$$

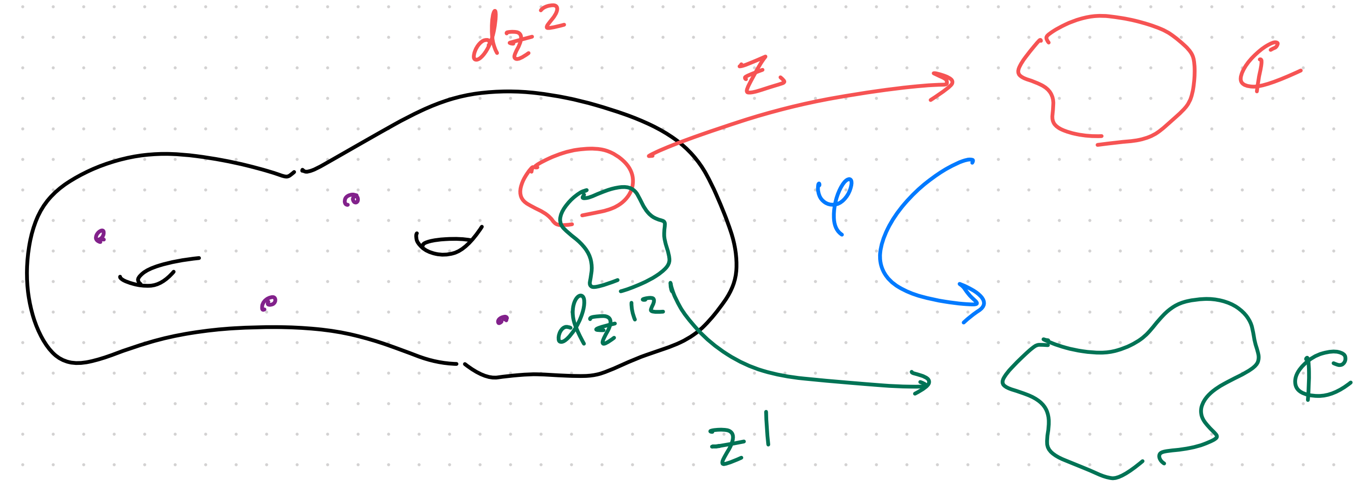
Prop Z consists of a union of (finitely many) points and simple closed curves.

Half-translation structures for minimal surfaces

Recall: given S minimal surface, $(\mathrm{I}, \mathrm{II})$,

$\mathrm{II} = \mathrm{Re}(q)$, a $\underset{\underset{[\mathrm{I}]}{\shortparallel}}{X}$-holomorphic quadratic differential

$q \overset{\text{locally}}{=} q(w)\,dw^2 \underset{\text{away from zeros}}{=} dz^2$ $\quad$ z is a determination of $\sqrt{q(w)}$



$\varphi(z) = \pm z + c$ half-translation

$\rightsquigarrow$ Well-defined Euclidean metric on $S \setminus \{q = 0\}$
$|dz|^2$

(cone singularities at zeros of q)

In such a chart $z = x + iy$,

$$\mathrm{I} = e^{2u}(dx^2 + dy^2)$$

$$\mathrm{II} = \mathrm{Re}(dz^2) = dx^2 - dy^2 \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \mathrm{I}^{-1}\,\mathrm{II} \sim \begin{pmatrix} e^{-2u} & 0 \\ 0 & -e^{-2u} \end{pmatrix}$$

<u>Rmk</u> $u \geq 0$

& $\mathcal{Z} = \{u = 0\}$

and u solves the Gauss' equation

$$K_{\mathrm{I}} = -1 + \det B$$

$$-e^{-2u}\Delta u = -1 - e^{-4u}$$

$$\Delta u = 2\cosh(2u)$$

cosh-Gordon eqn

Revisited goal:

Construct $f \in \mathcal{C}^\infty(S)$ such that, around Z and in "flat" coordinates z,

$$f_{xx} = -1 \qquad f_{yy} = 1$$

Rmk We can use the Euclidean Hessian

at $p \in Z$, $\nabla^I df = D^2 f$ $\qquad \Gamma_{ij}^k(p) = 0$

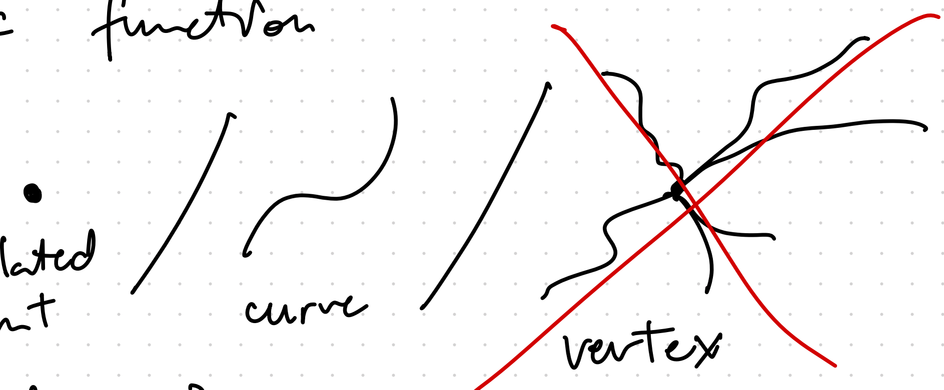
Riemannian Hessian

Euclidean Hessian

Prop $\mathcal{Z}$ consists of a union of (finitely many) points and simple closed curves.

Pf: $\mathcal{Z} = \{ \| \mathrm{II} \|^2 = 2 \}$ is the level set of an analytic function

$\underset{\text{Lojasiewicz's Theorem}}{\Longrightarrow}$ $\mathcal{Z} \underset{\text{locally}}{\simeq}$



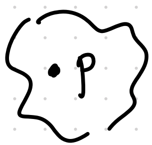
But $\mathcal{Z} = \{ u = 0 \} \subseteq \{ du = 0 \}$

and $\Delta u = 2\cosh(2u) > 0 \Rightarrow$ either $u_{xx} > 0$ or $u_{yy} > 0$

$\Rightarrow \mathcal{Z}$ is contained in a curve ($u_x = 0$ or $u_y = 0$) $\square$

Construction of f : divide into cases

0) $p \in Z$ is an isolated point



define, in the "flat" chart z

$$f := \frac{-x^2 + y^2}{2} \qquad D^2 f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Interesting situation: $\gamma \subset \mathcal{Z}$ simple closed curve

subcases according to the holonomy of γ:

$S \setminus \{q=0\}$ has a half-translation structure

$\rightsquigarrow$ $\mathrm{dev} : S \setminus \{q=0\} \longrightarrow \mathbb{C}$

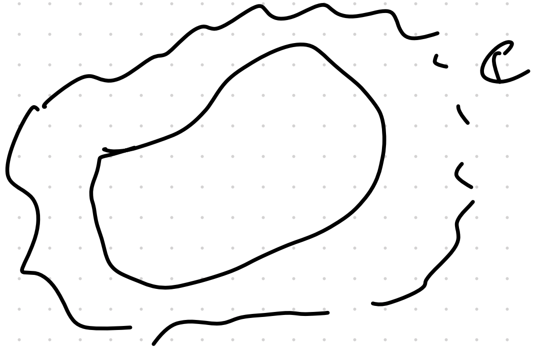
$\rho : \pi_1(S \setminus \{q=0\}) \longrightarrow \{ z \longmapsto \pm z + c \}$

holonomy

$\tilde{\gamma} : \mathbb{R} \longrightarrow \mathbb{C}$

$\tilde{\gamma}(t+1) = \rho(\gamma)\tilde{\gamma}(t)$

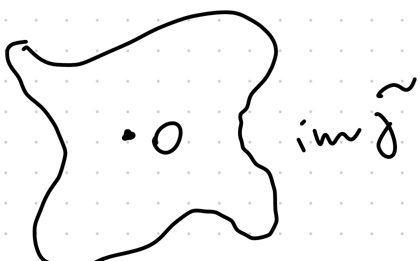
1) $\rho(\gamma) = \mathrm{id} \iff \gamma$ is covered by a single chart!



$$f := \frac{-x^2 + y^2}{2}$$

2) $\rho(\gamma)(z) = -z + c$ $\quad$ π-rotation around $\frac{c}{2}$

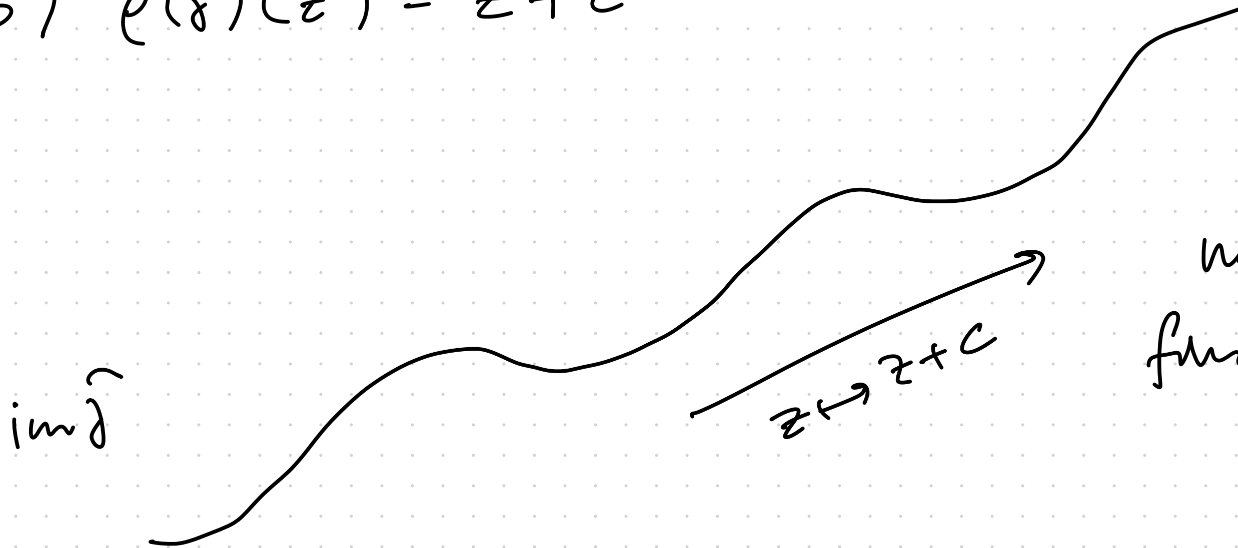
(can assume $c = 0$)



$$f := \frac{-x^2 + y^2}{2} \qquad f(-z) = f(z)$$

$$f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

3) $\rho(\gamma)(z) = z + c$



need to find f periodic

Warning: if $\tilde{\gamma}$ is a horizontal (vertical) line, then it is impossible to find f periodic with $f_{xx} < 0$ ($f_{yy} > 0$)

<u>Luckily</u>, this never happens!

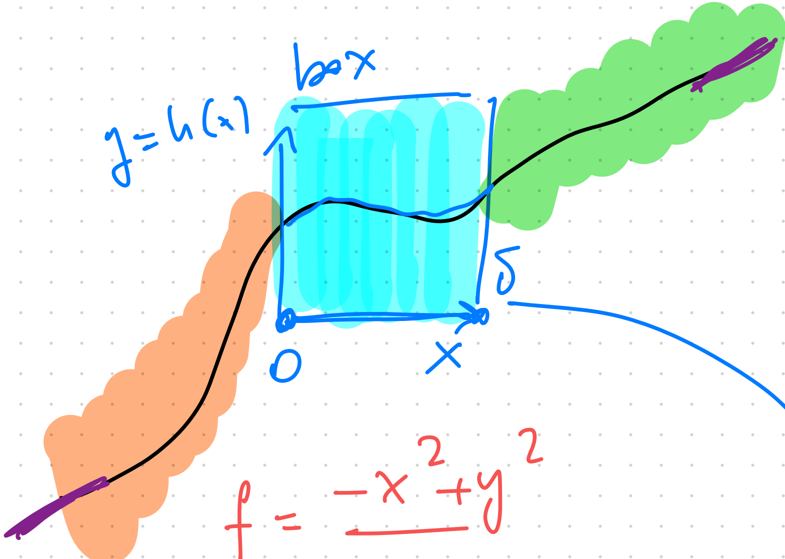
<u>Lemma</u>: A smooth curve $\gamma : (a,b) \to Z \subset S$ is <u>never</u> a geodesic for the Euclidean metric

($\Longleftrightarrow$ for the first fund. form)

at $p \in Z$, $\Gamma_{ij}^{k}(p) = 0$

See later for a sketch.

Go back to case 3. Build f as follows:



$$f = \frac{-(x-x_0)^2 + (y-y_0)^2}{2}$$

$$c = (x_0, y_0)$$

$$D^2 f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f = \frac{-x^2+y^2}{2}$$

$$D^2 f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

define f through its Hessian

Want: $D^2 f = \begin{pmatrix} -1 & \xi(x) \\ \xi(x) & 1 \end{pmatrix}$

impose $$D^2 f = \begin{pmatrix} -1 + (y - h(x))\xi'(x) & \xi(x) \\ \xi(x) & 1 \end{pmatrix}$$

$$\xi \in \mathcal{C}_0^\infty((0,\delta))$$

<u>Important</u>: gradient has to increase by $(-x_0, y_0)$

Explicity, we need:

$$\int_0^\delta \xi(x)\,dx = y_0 \qquad \& \qquad \int_0^\delta \xi(x)h'(x)\,dx = -X_0$$

Find $\xi \in \mathcal{C}_0^\infty((0,\delta))$ satisfying these conditions

The map $\mathcal{C}_0^\infty((0,\delta)) \longrightarrow \mathbb{R}^2$

$$\xi \longmapsto \left(\int_0^\delta \xi(x)\,dx, \int_0^\delta \xi(x)h'(x)\,dx \right)$$

is surjective.

By contradiction, if $\dim(\text{image}) = 1$, then $\exists \lambda$

$$\int_0^\delta \xi(x) h'(x)\, dx = \lambda \int_0^\delta \xi(x)\, dx \qquad \forall \xi \in \mathcal{C}_0^\infty((0,\delta))$$

$$\Longrightarrow \int_0^\delta \xi(x)\,(h'(x) - \lambda)\, dx = 0 \qquad \forall \xi \in \mathcal{C}_0^\infty((0,\delta))$$

$$\Longrightarrow h'(x) \equiv \lambda \Longrightarrow \tilde{\gamma} \text{ is a line.}$$

Sketch of the proof of the Lemma;

if $\tilde{\gamma}$ is a line, then $\tilde{S}$ should be a minimal surface with a 1-parameter family of symmetries

$\tilde{\gamma}$

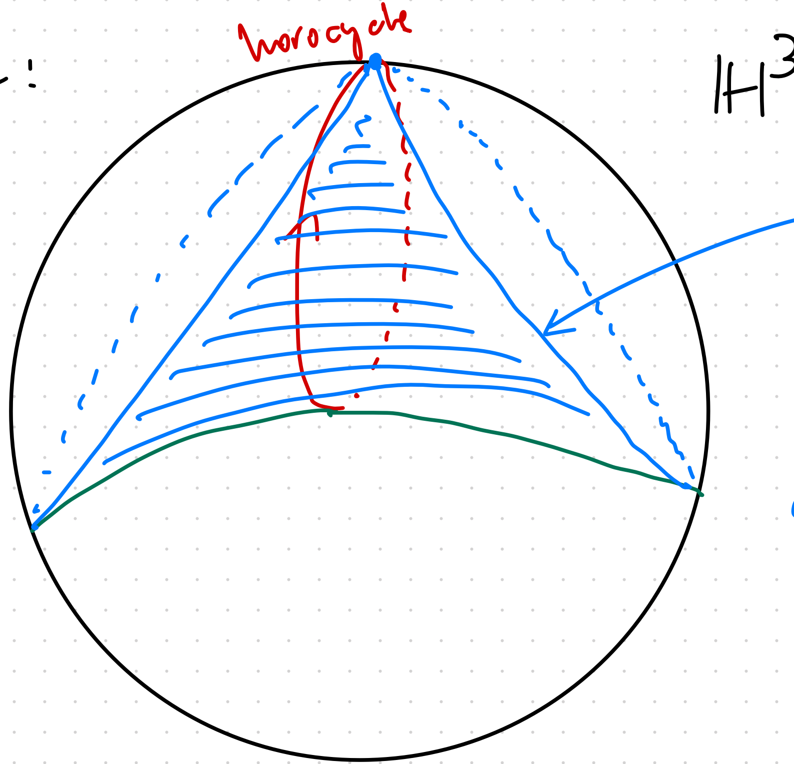
$$\Delta u = 2\cosh(2u)$$

$$u|_{\{x=0\}} = 0$$

$$du|_{\{x=0\}} = 0$$

vertical symmetry by translation (Cauchy-Kovalevskaya)

Picture:



this minimal surface is parabolic invariant and has no points where $\mathrm{II} = 0$

while $\tilde{S}$ lifted from S must have zeros of II ($\Leftrightarrow$ zeros of q) $\square$