QUASI-FUCHSIAN, ALMOST-FUCHSIAN AND NEARLY-FUCHSIAN MANIFOLDS

Shanghai Institute for Mathematics and Interdisciplinary Sciences

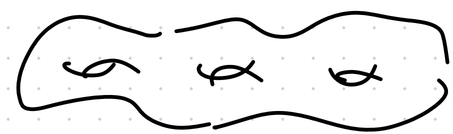
Lecture I, 30/06/2025

A Riemannian manifold (M^n, h) is **hyperbolic** if h has sectional curvature $\equiv -1$.

If h is moreover **complete**, then $M \cong \mathbb{H}^n / \Gamma$

$\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ torsion-free, discrete

$$\Gamma \cong \pi_1 M$$



We will focus on hyp. manifolds $M \underset{\text{homeo.}}{\cong} \Sigma \times \mathbb{R}$,

Σ = closed oriented surface of genus ≥ 2

__Theorem__ (Kahn-Markovic 2012)

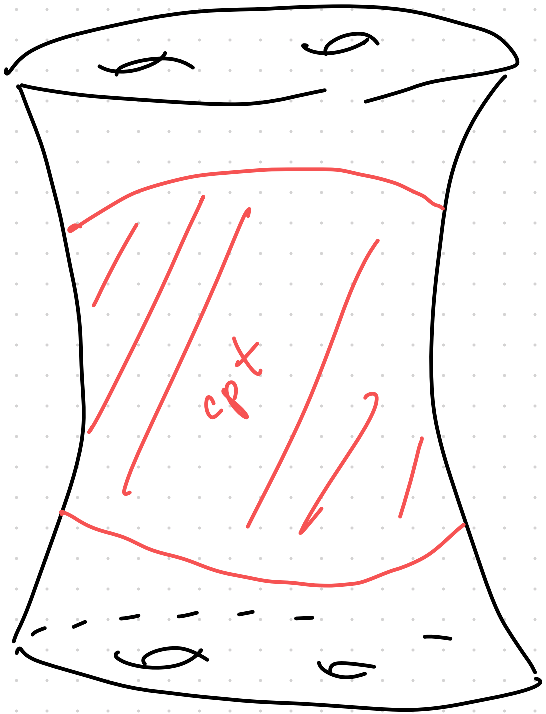
If M^3 is closed and hyperbolic, then $\pi_1 M$ contains $\pi_1 \Sigma$, Σ = closed oriented surface of genus ≥ 2

many !

$\sim g^{2g}$

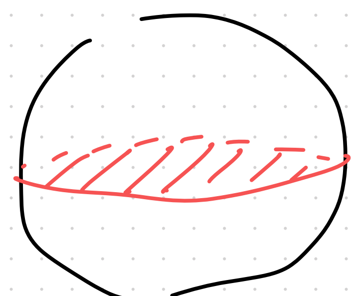
$\Longrightarrow$ the covering $\hat{M} \xrightarrow{p} M$ such that $p_* \pi_1 \hat{M} = G \cong \pi_1 \Sigma$, then $\hat{M} \underset{\text{diffeo}}{\simeq} \Sigma \times \mathbb{R}$

Def $(M^3 \simeq \Sigma \times \mathbb{R}, h)$ is quasi-Fuchsian if M contains a compact, geodesically convex subset.



Ex "Fuchsian" manifolds

$$\Gamma < \underset{\mathrm{Isom}\,\mathbb{H}^2}{\underset{\shortparallel}{PSL_2\mathbb{R}}} < \underset{\mathrm{Isom}\,\mathbb{H}^3}{\underset{\shortparallel}{PSL_2\mathbb{C}}}$$



$$\operatorname{int}\left\{ \begin{array}{c} \rho : \pi_1 \Sigma \to \operatorname{Isom} \mathbb{H}^3 \\ \text{discrete faithful} \end{array} \right\} \Big/ \text{conj.} = \{\rho \text{ quasi-Fuchsian}\} \Big/ \text{conj.}$$

i.e. $\mathbb{H}^3 / \rho(\pi_1 \Sigma)$ is QF.

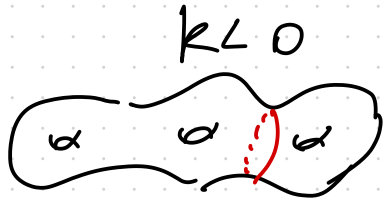
$$\text{IS}$$

$$\mathbb{R}^{12g-12}$$

Use <u>minimal surfaces</u>

Idea:

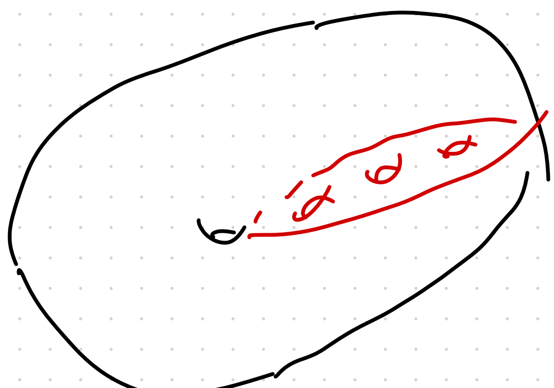
dim = 2



locally length minimizing

find a (unique) geodesic representative of a closed curve in the homotopy class

dim = 3

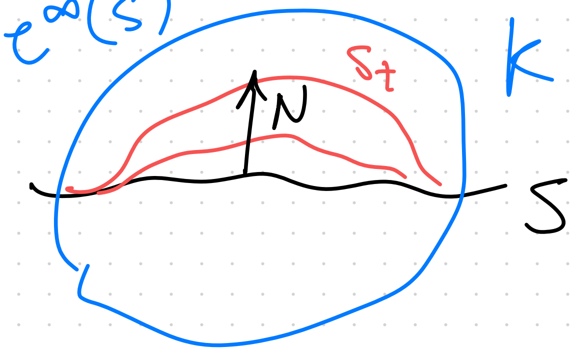


locally area minimizing

look for a minimal representative of a closed surface in the homotopy class

Def An embedded surface $S \subset (M^3, h)$ is minimal if for every smooth variation S_t, $S_0 = S$ (compactly supported in $K \subset\subset M$)

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(S_t \cap K) = 0$$



$$\Longleftrightarrow \quad H := \mathrm{tr}_{\mathrm{I}} \mathrm{II} = 0$$

mean curvature

(e_1, e_2) I-orthonormal frame

$$H = \mathrm{II}(e_1, e_1) + \mathrm{II}(e_2, e_2)$$

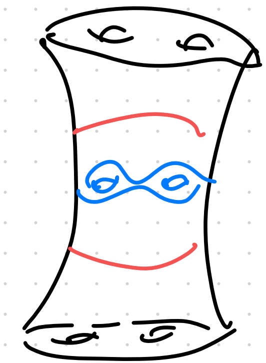
I = first fund. form

II = second fund. form

$$\mathrm{II}(X, Y) = \langle \nabla^M_X Y, N \rangle$$

- Sacks-Uhlenbeck, Schoen-Yau ~ 80'

If $(M \simeq \Sigma \times \mathbb{R}, h)$ is quasi-Fuchsian, then M contains a closed minimal surface homotopic to $\Sigma \times \{*\}$.



- Anderson '86, Huang-Wang '15

The minimal surface is not unique in general.

Def $(M \simeq \Sigma \times \mathbb{R}, h)$ complete, hyperbolic is

[weakly] almost-Fuchsian if M contains a closed

minimal surface homotopic to $\Sigma \times \{*\}$

with principal curvatures in $(-1, 1)$

$[-1, 1]$

eigenvalues of $\mathrm{I}^{-1}\mathrm{II}$

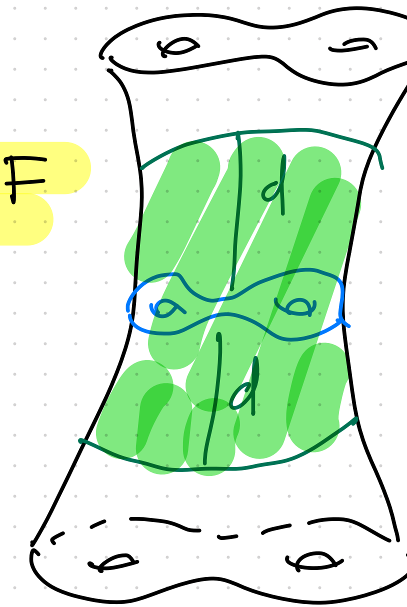
concretely, one can find an I-orthonormal frame

such that $\lambda := \mathrm{II}(e_1, e_1)$, $\mu := \mathrm{II}(e_2, e_2)$

minimal $\lambda = -\mu$ $\mathrm{II}(e_1, e_2) = \mathrm{II}(e_2, e_1) = 0$

Uhlenbeck's observations (1983)

1) for $M \simeq \Sigma \times \mathbb{R}$, AF $\Longrightarrow$ QF



Question: does WAF imply QF?

Answer will be YES

2) for $M \simeq \Sigma \times \mathbb{R}$

WAF $\Longrightarrow$ uniqueness of the minimal surface in the homotopy class of $\Sigma \times \{*\}$

The results (with M.T. Nguyen & J.M. Schlenker)

"Conjecture" (~ 2000's)

In the definition of AF, one can remove the adjective "minimal"

Def $(M \simeq \Sigma \times \mathbb{R}, h)$ complete, hyperbolic is nearly-Fuchsian if it contains a closed surface with principal curvatures in $(-1, 1)$

"Conjecture" $\quad NF \iff AF$

Theorem (Nguyen - Schlenker - S. '25)

If $(M \simeq \Sigma \times \mathbb{R}, h)$ is weakly almost-Fuchsian, then M is nearly-Fuchsian.

Rmk The theorem still holds true in a neighborhood of WAF manifolds.

later today   openness   Davalo '24

$$AF \overset{\not\Leftarrow}{\Longrightarrow} WAF \overset{\not\Leftarrow}{\Longrightarrow} NF \overset{\not\Leftarrow}{\Longrightarrow} QF$$

Corollary 1 Every WAF manifold is QF.

Corollary 2 There are nearly-Fuchsian manifolds that are <u>not</u> almost-Fuchsian

Next: There exist weakly almost-Fuchsian manifolds that are not almost-Fuchsian

3) Back to Uhlenbeck: Parametrization by holomorphic quadratic differentials

Fact If $S \subset \mathbb{H}^3$ is a minimal surface, $(\mathrm{I}, \mathrm{II})$, then $\mathrm{II} = \mathrm{Re}(q)$, for q a X-holomorphic quadratic differential

$$X = [\mathrm{I}]$$

conformal class

What is a HQD?

$q \in H^0(X, K^2)$

$q \overset{\text{locally}}{=} q(z)\,dz^2$

$z = x + iy$

$I = e^{2u}(dx^2 + dy^2)$

$$q = (a + ib)\,dz^2$$
$$= (a + ib)(dx^2 - dy^2 + 2i\,dx\,dy)$$

$$\mathrm{Re}(q) = \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}$$

is symmetric and traceless

Moreover, Codazzi equation $\Longleftrightarrow$ CR equations for q

The map $(\mathrm{I}, \mathrm{II}) \longmapsto (X, q)$

$$AF(\Sigma) \longrightarrow T^*\mathcal{T}(\Sigma)$$

is injective (diffeo onto its image)

Bronstein - Smith '24

The image of this map is convex
in every fiber!

How to deal with this problem?

For (X, q) fixed, we wish to construct $(\mathrm{I}, \mathrm{II})$ satisfying the Gauss-Codazzi equations

$$\mathrm{I} = e^{2u} h_0$$

$$\mathrm{II} = \mathrm{Re}(q)$$

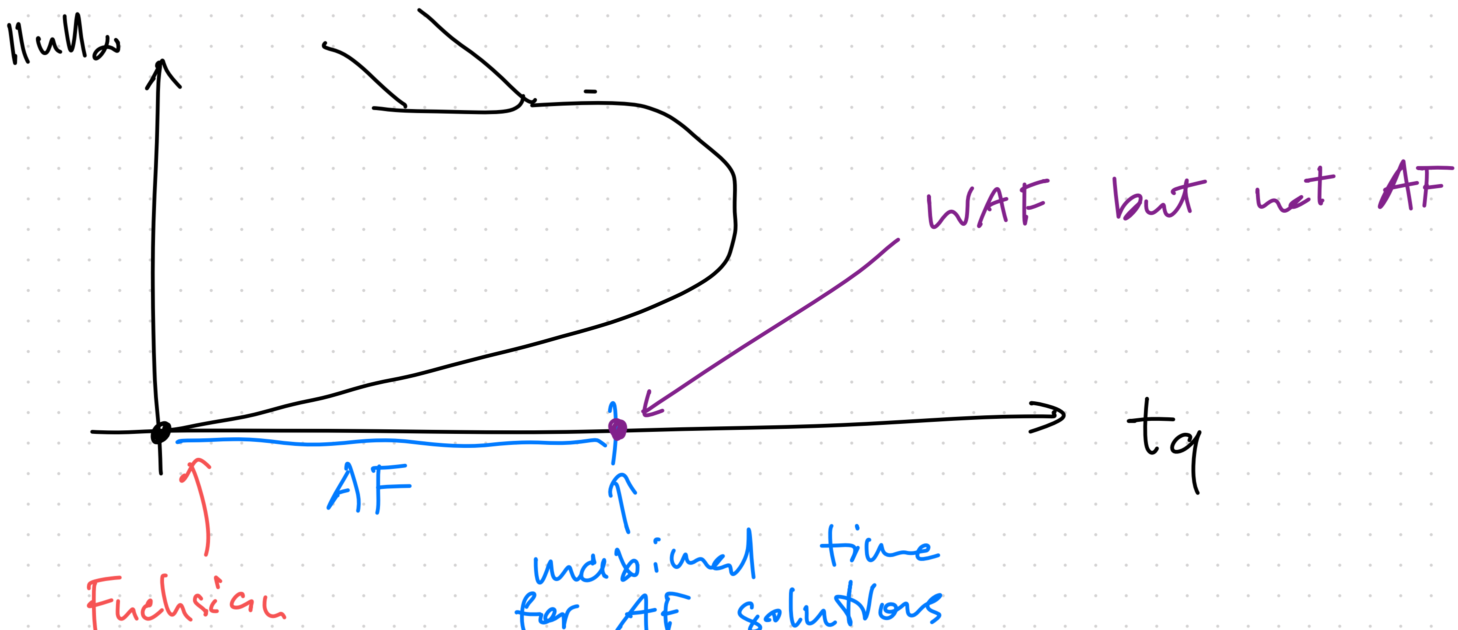
h_0 hyperbolic metric (Uniformization Theorem)

Need to solve Gauss' equation for u:

$$K_{\mathrm{I}} = -1 + \det_{\mathrm{I}} \mathrm{II} \iff e^{-2u}(-1-\Delta u) = -1 - e^{-4u}\|q\|^2_{h_0}$$

$$\iff \Delta u = -1 + e^{2u} + e^{-2u}\|q\|^2_{h_0}$$

Fix $q \in T^*_X \mathcal{T}(\Sigma) = H^0(X, K_X^{\,2})$. Look at the ray $\mathbb{R}_{>0} \cdot q$

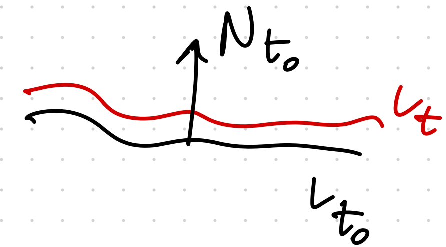


"Conjecture" : was motivated from <u>Geometric Flows</u>

Ben Andrews' ICM paper from 2002:

Geometric flows in $\mathbb{S}^3$ that evolve by functions of principal curvatures

$$\frac{d}{dt}\Big|_{t=t_0} \iota_t = (\arctan \lambda_{t_0} + \arctan \mu_{t_0}) N_{t_0}$$



with good properties

$\Updownarrow$

evolution of the Gauss map in $\mathbb{S}^2 \times \mathbb{S}^2$ by mean curvature flow

Analogous flow in $\mathbb{H}^3$:

$$\left.\frac{d}{dt}\right|_{t=t_0} \iota_t = (\operatorname{arctanh}\lambda_{t_0} + \operatorname{arctanh}\mu_{t_0}) N_{t_0}$$

$\Updownarrow$

evolution of the Gauss map by mean curvature flow in a para-Kähler manifold

<u>Problem</u>: only defined if $\lambda, \mu \in (-1,1)$

As long as the flow exists, $\operatorname{arctanh}\lambda + \operatorname{arctanh}\mu$ decays exponentially

So, long-time existence $\Longrightarrow$ for $t=\infty$

$\operatorname{arctanh}\lambda + \operatorname{arctanh}\mu = 0 \rightarrow \lambda = -\mu$