LECTURE 6

References:

- O. Guichard and A. Wienhard, Anosov representations: domains of discontinuity and applications, Invent. Math., 190(2), 2012
- B. Collier, N. Tholozan, J. Toulisse, The geometry of maximal reps of surface groups into SO(2,n), Duke Math. J., 168(15), 2019
- A. Seppi, G. Smith, J. Toulisse, On complete maximal submanifolds in pseudo-hyperbolic space, Section 9, Preprint (ArXiv) 2023

Let $p, q \geq 1$ and

$$\mathrm{Isot}(\mathbb{R}^{p,q+1}) = \left\{ \begin{array}{c} \text{maximally isotropic subspaces} \\ \text{of } \mathbb{R}^{p,q+1} \end{array} \right\}$$

If $V \in \mathrm{Isot}(\mathbb{R}^{p,q+1})$, then $\dim V = \min\{p, q+1\}$.

Indeed, write $\mathbb{R}^{p+q+1} = \underset{>0}{\mathbb{R}^{p}} \oplus \underset{<0}{\mathbb{R}^{q+1}}$ an orthogonal decomposition.

- if $q+1 \geq p$, then $V = \mathrm{graph}(L : \mathbb{R}^{p} \to \mathbb{R}^{q+1})$
  L linear isometry onto its image
- if $p \geq q+1$, then $V = \mathrm{graph}(M : \mathbb{R}^{q+1} \to \mathbb{R}^{p})$
  M linear isometry onto its image

Theorem (Guichard-Wienhard '12)

Let Γ be a hyperbolic group, and let $\rho : \Gamma \to PO(p, q+1)$ a P_1-Anosov representation ($\Leftrightarrow$ ($\mathbb{H}^{p,q}$ or $\mathbb{H}^{p-1,q+1}$)- convex - cocompact).

let $\Lambda_\rho :=$ proximal limit set of ρ

and $\Omega_\rho := \{ V \in \mathrm{Isot}(\mathbb{R}^{p,q+1}) \mid \mathbb{P}(V) \cap \Lambda = \phi \}$,

Then $\rho(\Gamma) \curvearrowright \Omega_\rho$ properly discontinuously and cocompactly.

Question: What is the topology of $\Omega_\rho / \rho(\Gamma)$?

Suppose from now on that $\partial\Gamma \simeq S^{p-1}$ and ρ is positive P_1-Anosov

Prop If $p \geq q+1$, then $\Omega_p = \emptyset$. $\Leftrightarrow \forall V \quad \mathbb{P}(V) \cap \Lambda \neq \emptyset$

Pf: let $\tilde{\Lambda}$ be a lift to $\partial_\infty \hat{\mathbb{H}}^{p,q}$ of $\Lambda \subset \partial_\infty \mathbb{H}^{p,q}$.

Then $\tilde{\Lambda} = \text{graph}(f : \mathbb{S}^{p-1} \to \mathbb{S}^q)$, f 1-Lipschitz map, $\text{im}(f)$ does not contain antipodal points.

Moreover, $\widetilde{\mathbb{P}(V)} \cap \partial_\infty \hat{\mathbb{H}}^{p,q} = \text{graph}(M|_{\mathbb{S}^q} : \mathbb{S}^q \to \mathbb{S}^{p-1})$

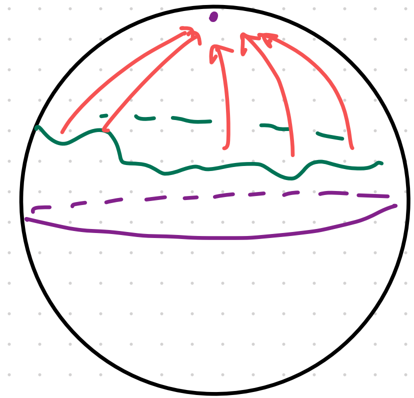
for $M|_{\mathbb{S}^q} : \mathbb{S}^q \to \mathcal{S} := M(\mathbb{S}^q)$ isometry

Consider $\phi := M \circ f|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$, which is 1-Lipschitz and $\text{im}(\phi)$ does not contain antipodal points.

$\Rightarrow \phi$ has a fixed point x_0.

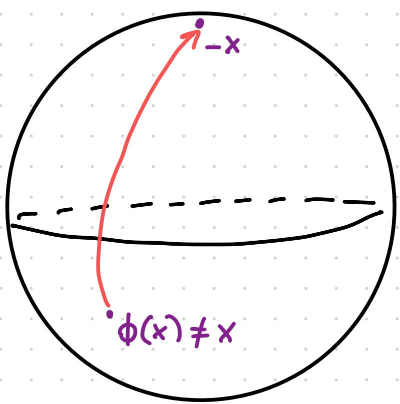
This will imply:

$$\Rightarrow \underbrace{[x_0, f(x_0)]}_{\in \tilde{\Lambda}} = \underbrace{[M(f(x_0)), f(x_0)]}_{\in \widetilde{\mathbb{P}(V)}} \in \tilde{\Lambda} \cap \widetilde{\mathbb{P}(V)}$$



To show ϕ has a fixed point;

- $\deg(\phi) = 0$: ϕ can be continuously deformed to a constant map. because $\mathrm{im}(\phi) \subset$ an open hemisphere.



- $\deg(\phi) = \pm 1$ if ϕ has no fixed point: if $\phi(x) \neq x$ $\forall x \in \mathcal{S}$, then ϕ can be continuously deformed to the antipodal map α

$$\Rightarrow \deg(\phi) = \deg(\alpha) = \pm 1.$$

#

So we only consider $q \geq p$.

We construct a ρ-equivariant map

$\pi : \Omega_\rho \longrightarrow \Sigma = \rho(\Gamma)$-invariant spacelike submanifold

$\pi(V) :=$ the unique $x \in \Sigma^p \subset \mathbb{H}^{p,q}$ such that $V \subset x^\perp$

Claim Such x exists and is unique

$\Updownarrow$

$x \subset V^\perp$

Then $\pi^{-1}(x) = \mathrm{Isot}(x^\perp) := \left\{ \begin{array}{l} \text{maximally isotropic} \\ \text{subspaces of } x^\perp \end{array} \right\}$

$$\cong \mathrm{Isot}(\mathbb{R}^{p,q}) \cong V_{p,q}$$

Stiefel manifold

$$V_{p,q} = \{ (e_1, \ldots, e_p) \in (\mathbb{R}^q)^p \mid \langle e_i, e_j \rangle = \delta_{ij} \}$$

Then $\Omega_\rho / \rho(\Gamma)$ is a fiber bundle over $\Sigma / \rho(\Gamma)$

with fiber $\mathcal{V}_{p,q}$.

Rmk $\mathcal{V}_{p,q}$ is connected unless $p=q$ $(\mathcal{V}_{p,p} = O(p))$

$\Rightarrow$ $\Omega_\rho / \rho(\Gamma)$ is connected unless $p=q$ and

$N\Sigma / \rho(\Gamma)$ admits a reduction from an $O(p)$-bundle

to an $SO(p)$-bundle

i.e. the Stiefel-Whitney class vanishes.

It remains to show the claim.

Let $W = V^\perp$. Then W is a degenerate subspace of $\mathbb{R}^{p,q+1}$,

$\dim V = p \implies \dim W = q+1 > p$

$W = V^\perp \supset V$, $V = \ker(\langle \cdot, \cdot \rangle|_W)$

W does not contain any positive definite line

We have $W = \lim_{n\to+\infty} W_n$, where $W_n \subset \mathbb{R}^{p,q+1}$ is negative definite

($\dim W = \dim W_n = q+1$)

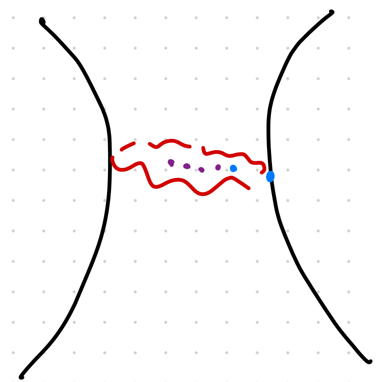
$\Rightarrow W_n \cap \hat{\mathbb{H}}^{p,q} \simeq -\mathbb{S}^q$ (totally geodesic)

Hence $\exists\, x_n \in \mathbb{P}(W_n) \cap \Sigma$. Up to a subsequence, $x_n \to x \in \Sigma \cap \partial_\infty \Sigma$.

- if $x \in \partial_\infty \Sigma = \Lambda$, then $\Lambda \cap \mathbb{P}(V^\perp) \neq \phi$

  $\Rightarrow \Lambda \cap \mathbb{P}(V) \neq \phi$. #

- so if $\Lambda \cap \mathbb{P}(V) = \phi$, then $x \in \Sigma \cap \mathbb{P}(V^\perp)$.



Uniqueness follows since Σ is spacelike.