LECTURE 5

Reference: J. Beyrer and F. Kassel, $\mathbb{H}^{p,q}$-convex-cocompactness and higher higher Teichmüller spaces, Preprint (ArXiv) 2023

We will discuss several applications / consequences of:

Corollary (S. - Smith - Toulisse '23)

Let Γ be a hyperbolic group with $\partial\Gamma \simeq S^{p-1}$, and let $\rho : \Gamma \to PO(p, q+1)$ be a $\mathbb{H}^{p,q}$-convex-cocompact representation.

Then there exists a $\rho(\Gamma)$-invariant spacelike submanifold $\Sigma^p \subset \mathbb{H}^{p,q}$, on which the action of $\rho(\Gamma)$ is properly discontinuous and cocompact.

[Σ is constructed as the unique complete maximal submanifold with $\partial_\infty \Sigma$ = proximal limit set.]

I. About the hypotheses on Γ

Let Γ be hyperbolic, torsion-free, $\partial\Gamma \simeq S^{p-1}$.

Theorem (Bartels - Lück - Weinberger '10)

If $p \geq 6$, then $\Gamma \cong \pi_1 M$ for

M = topological closed p-manifold with $\tilde{M} \underset{\text{homeo}}{\cong} \mathbb{R}^p$

Corollary (SST '23)

If $\rho : \Gamma \to PO(p, q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact,

then $\Gamma \cong \pi_1 M$ for

M = smooth closed p-manifold with $\tilde{M} \underset{\text{diffeo}}{\cong} \mathbb{R}^p$

Proof: $M := \Sigma / \rho(\Gamma)$

($\rho(\Gamma) \cap \Sigma$ is free)

Fact (Bartels - Lück - Weinberger '10)

For every $k \geq 2$, there exists Γ hyperbolic, torsion-free, $\partial\Gamma \simeq S^{4k-1}$, such that Γ is not isomorphic to the π_1 of a smooth closed aspherical manifold.

Corollary (SST)

Γ as above does not admit any $\mathbb{H}^{4k,q}$ - convex-cocompact representation for any $q \geq 1$.

II. Curvature properties

Question: What can we say on Σ, hence on M and Γ?

Labourie-Toulisse: if $p=2$, then $-1 \leq K_\Sigma \leq 0$

and if $\exists x \mid K_\Sigma(x)=0$ then $K_\Sigma \equiv 0$

$\Rightarrow \Sigma$ is a Barbot surface in $\mathbb{H}^{2,1} \subset \mathbb{H}^{2,q}$

Conjecture (?) Σ has non-positive sectional curvature for any p

This would imply

$\Gamma \cong \pi_1$(non-positively curved closed smooth manifold)

Moriani - Trebeschi For any p, q, $-p(p-1) \leq \mathrm{Scal}_\Sigma \leq 0$

and $\mathrm{Scal}_\Sigma(x) = 0 \implies \mathrm{Scal}_\Sigma \equiv 0$

classified

Rmk $\mathrm{Scal}_\Sigma = -p(p-1) + \|\mathrm{II}\|^2.$

III. Closedness of $\mathbb{H}^{p,q}$-convex-cocompact representations

<u>Theorem</u> (Beyrer-Kassel '23)

Let $p,q \in \mathbb{N}$, $p \geq 2$, $q \geq 1$.
Let Γ be a hyperbolic group with $\partial\Gamma \cong S^{p-1}$. Then

$$\left\{ \begin{array}{c} \mathbb{H}^{p,q}\text{-convex cocompact representations} \\ \rho : \Gamma \to PO(p,q+1) \\ \text{of finite kernel} \end{array} \right\} \subset \mathrm{Hom}(\Gamma, PO(p,q+1))$$

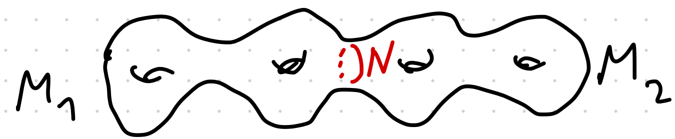
is closed, hence a union of connected components.

Rmk The theorem does not hold if $\partial\Gamma \simeq S^{k-1}$, $k<p$.

Indeed, one can take M a closed hyperbolic manifold admitting a totally geodesic closed separating hypersurface N.

$$M \setminus N = M_1 \sqcup M_2$$

e.g. if $k=2$, $M=$ closed surface, $N=$ simple closed geodesic



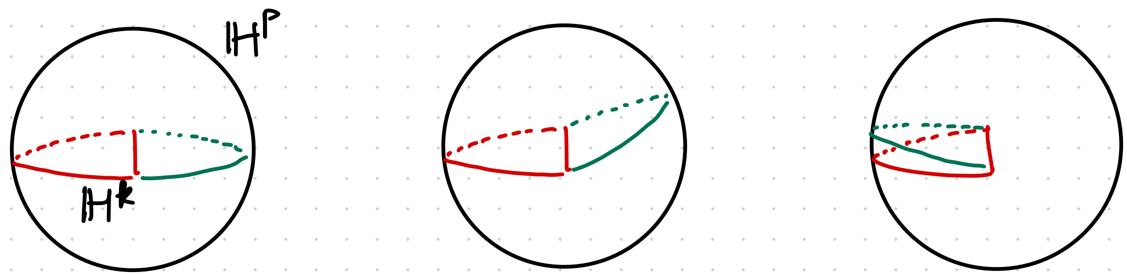
Consider:

$$\rho : \pi_1 M \xrightarrow{\mathrm{hol}} O_0(k,1) \longrightarrow PO(p,q+1)$$

$$\left[\begin{pmatrix} \mathrm{id}_{p-k} & & \\ & * & \\ & & \mathrm{id}_q \end{pmatrix}\right]$$

Idea: construct ρ_θ as ρ "bent" along N of an angle θ.

(Johnson–Millson)



$\theta = 0$ $\qquad$ $\theta = \varepsilon$ $\qquad$ $\theta \to \pi$

Concretely, $\pi_1 M = \pi_1 M_1 *_{\pi_1 N} \pi_1 M_2$

R_θ = rotation around the $(k-1)$-subspace fixed by $\rho(\pi_1 N)$

$$\rho_\theta(\gamma) := \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1 M_1 \\ R_\theta \rho(\gamma) R_\theta^{-1} & \text{if } \gamma \in \pi_1 M_2 \end{cases}$$

Well-defined since R_θ commutes with $\rho(\pi_1 N)$

For $\theta = \pi$, the representation fails to be discrete and faithful.

Examples of $\mathbb{H}^{p,q}$-convex-cocompact components

- If M^p is closed hyperbolic and contains a totally geodesic closed separating hypersurface N, then the component of

$$\rho: \pi_1 M \xrightarrow{\text{hol}} O(p,1) \longrightarrow PO(p,q+1) \qquad \left(\begin{array}{c|c} * & \\ \hline & \mathrm{id}_q \end{array}\right)$$

admits Zariski dense representations.

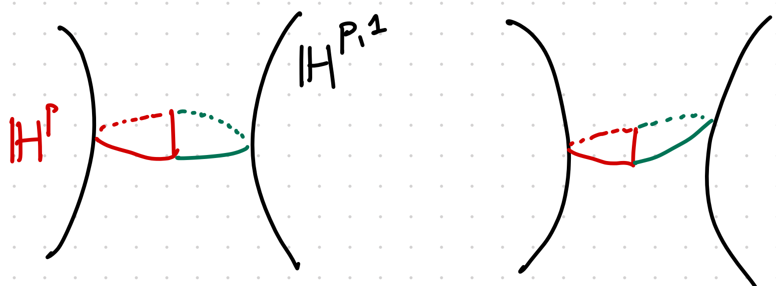
"non-trivial higher higher Teichmüller space"

Idea: bending as before.

If $q = 1$,

$$\rho_t(\gamma) := \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1 M_1 \\ R_t \rho(\gamma) R_t^{-1} & \text{if } \gamma \in \pi_1 M_2 \end{cases}$$

for $R_t \sim \left(\begin{array}{c|c|c} \cosh t & & \sinh t \\ \hline & \mathrm{id}_p & \\ \hline \sinh t & & \cosh t \end{array}\right)$



For higher $q > 1$, need q totally geodesic hypersurfaces.

- M need not be hyperbolic:

→ Monclair - Tholozan - Schlenker '23

M^p = Gromov - Thurston manifold (non-hyperbolic if $p > 3$)

$\rho : \pi_1 M \longrightarrow PO(p,2)$ $\mathbb{H}^{p,1}$-convex-cocompact.

→ Marquis - Lee

Γ = (finite-index subgroup of) a Coxeter group ← not a hyperbolic lattice

$\rho : \Gamma \longrightarrow PO(p,2)$ $\mathbb{H}^{p,1}$-convex-cocompact.

- Marquis - Lee: examples of Γ, $\partial\Gamma \simeq S^{p-1}$, admitting a $\mathbb{H}^{p,q}$-convex-cocompact representation for $q = 2$
but not for $q = 0, 1$.

Outline of the strategy of Beyrer-Kassel

Suppose $\rho_n \longrightarrow \rho_\infty$ where each ρ_n is $\mathbb{H}^{p,q}$-convex-cocompact. We want to show that so too is ρ_∞.

A) Each ρ_n preserves a complete maximal submanifold Σ_n. By the dichotomy, $\Sigma_n \longrightarrow \Sigma_\infty$, where Σ_∞ is complete maximal, or degenerate "weakly spacelike".

B) In any case, $\rho_\infty(\Gamma)$ acts properly discontinuously and cocompactly on Σ_∞ (uses the fact that $\Sigma_\infty \subset \Omega(\partial_\infty \Sigma)$)

c) If $\rho : \Gamma \to PO(p,q+1)$ is discrete and preserves a properly embedded (weakly) spacelike Σ with $\Sigma \subset \Omega(\partial_\infty \Sigma)$, then TFAE:

i) Γ is hyperbolic

ii) $\forall x,y \in \partial_\infty \Sigma \quad \langle x,y \rangle \neq 0$

iii) $\rho(\Gamma)$ is $\mathbb{H}^{p,q}$-convex-cocompact.

satisfied by Σ_∞ as in A), or by any Σ spacelike

<u>Rmk</u> On the Barbot surface ($\cong \mathbb{R}^2$), there are actions of $\Gamma \cong \mathbb{Z}^2$.

Corollary (BK + SST)

Let Γ be a hyperbolic group with $\partial\Gamma \simeq S^{p-1}$.
Then $\rho : \Gamma \to PO(p, q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact $\Longleftrightarrow$
$\rho(\Gamma)$ acts properly discontinuously and cocompactly on a complete spacelike submanifold $\Sigma^p \subset \mathbb{H}^{p,q}$.

can add "maximal"