LECTURE 4

Reference : A. Seppi, G. Smith, J. Toulisse, On complete maximal submanifolds in pseudo-hyperbolic space, Preprint (ArXiv) 2023

<u>Goal</u>: sketch of the proof of the following theorem:

<u>Theorem</u> (SST)

Let $\Lambda \subset \partial_\infty \mathbb{H}^{p,q}$ be a non-negative $(p-1)$-sphere.
Then there exists a unique complete maximal submanifold $\Sigma^p \subset \mathbb{H}^{p,q}$ such that $\partial_\infty \Sigma = \Lambda$.

Moreover, Σ is contained in the convex hull $\mathcal{C}(\Lambda)$.

Global strategy for existence: continuity method

Construct $\{\Lambda_t\}_{t\in[0,1]}$ a continuous family of non-negative $(p-1)$-spheres, such that $\Lambda_0 = \partial_\infty$ (totally geodesic $\mathbb{H}^p$)

Let $J := \{ t \in [0,1] \mid \exists\, \Sigma$ complete maximal submanifold, $\partial_\infty \Sigma = \Lambda_t \}$

Then $J \neq \emptyset$ because $0 \in J$.

Want to prove that J is closed and open.

easier

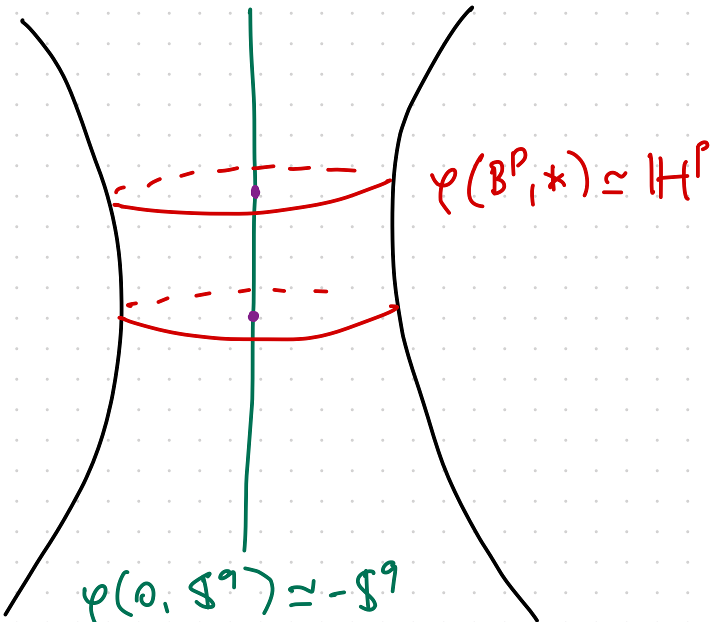
harder

- **Proof of "existence" part**

We need to work with another model of $\hat{\mathbb{H}}^{p,q}$:

Define $\varphi : B^p \times \mathbb{S}^q \longrightarrow \hat{\mathbb{H}}^{p,q}$, where $B^p \subset \mathbb{R}^p$, $\mathbb{S}^q \subset \mathbb{R}^{q+1}$, $\hat{\mathbb{H}}^{p,q} \subset \mathbb{R}^p \oplus \mathbb{R}^{q+1}$

$$\varphi(u,w) = \frac{1+\|u\|^2}{1-\|u\|^2}\left(\frac{2u}{1+\|u\|^2}, w\right)$$



$\|du\|^2$

Then:

$$\varphi^* g_{\mathbb{H}^{p,q}} = \frac{4\, g_{B^p}}{(1-\|u\|^2)^2} - \left(\frac{1+\|u\|^2}{1-\|u\|^2}\right)^2 g_{\mathbb{S}^q}$$

$$= \left(\frac{1+\|u\|^2}{1-\|u\|^2}\right)^2 \left(g_{\mathbb{S}^p_+} - g_{\mathbb{S}^q}\right)$$

$$= f^2 \left(g_{\mathbb{S}^p_+} - g_{\mathbb{S}^q}\right)$$

Consequences.

- A spacelike submanifold $\Sigma^p \subset \hat{\mathbb{H}}^{p,q}$ is locally the graph of $f: \mathbb{S}^p_+ \to \mathbb{S}^q$, $\|df\| < 1$

Pf: being spacelike only depends on the conformal class of the ambient metric

- A complete submanifold $\Sigma^p \subset \hat{\mathbb{H}}^{p,q}$ is globally the graph of $f: \mathbb{S}^p_+ \to \mathbb{S}^q$, $\|df\| < 1$

Pf: $\pi|_\Sigma : \Sigma \to \mathbb{H}^p$ is a local diffeo and increases distances

$\Rightarrow$ π has the path lifting property

$\Rightarrow$ π is a covering map $\Rightarrow$ π is a diffeomorphism

Moreover, the map φ extends to the boundary:

$$\partial\varphi : \mathbb{S}^{p-1} \times \mathbb{S}^{q} \longrightarrow \partial_\infty \hat{\mathbb{H}}^{p,q}$$

$$\partial\varphi(\underset{\mathbb{R}^p}{\underset{\cap}{u}}, \underset{\mathbb{R}^{q+1}}{\underset{\cap}{w}}) = (u, w) \in \mathbb{R}^{p,q+1}$$

Like for $\mathbb{H}^n$, $\partial_\infty \overset{(\wedge)}{\mathbb{H}}{}^{p,q}$ has a conformal pseudo-Riemannian structure of signature $(p-1, q)$ preserved by the group of isometries.

Under $\partial\varphi$, the conformal metric of $\partial_\infty \hat{\mathbb{H}}^{p,q}$ is given by

$$\left[g_{\mathbb{S}^{p-1}} - g_{\mathbb{S}^q} \right]$$

Consequences:

Let $\Lambda \subset \partial_\infty \mathbb{H}^{p,q}$ be a subset homeomorphic to S^{p-1}. TFAE:

i) $\Lambda \subset \partial_\infty \mathbb{H}^{p,q}$ is a non-negative $(p-1)$-sphere

ii) Λ admits a lift $\tilde{\Lambda} \subset \partial_\infty \hat{\mathbb{H}}^{p,q}$ such that $\forall x, y \in \tilde{\Lambda}$, $\langle x, y \rangle \leq 0$

iii) Λ admits a lift $\tilde{\Lambda} = \text{graph}(f : S^{p-1} \to S^q)$

f 1-Lipschitz, $\text{im}(f)$ does not contain antipodal points

iv) Λ admits a lift $\tilde{\Lambda} = \text{graph}(f : S^{p-1} \to S^q)$

f 1-Lipschitz, such that $f(-x) \neq f(x)$ $\forall x \in S^{p-1}$

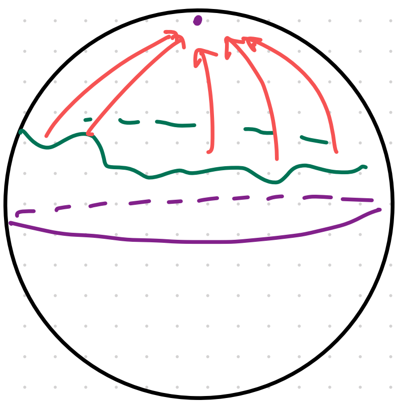
ii) $\Leftrightarrow$ iii) : $\langle (p, f(p)), (q, f(q)) \rangle = \langle p, q \rangle - \langle f(p), f(q) \rangle =$

$= \cos d(p,q) - \cos d(f(p), f(q)) \leq 0 \Leftrightarrow d(f(p), f(q)) \leq d(p,q)$

- **Deforming Λ**

Now, given $\tilde{\Lambda} = \text{graph}(f : S^{p-1} \to S^q)$, we want to deform it continuously to $\tilde{\Lambda}_0 = \text{graph}(\text{constant map})$

<u>Fact</u>: If $f : S^{p-1} \longrightarrow S^q$ is 1-Lipschitz and $\text{im}(f)$ does not contain antipodal points, then $\text{im}(f)$ is contained in an open hemisphere.



Then we can deform f to the map identically equal to the center C of the hemisphere.

$f_t(x) =$ geodesic between $f(x)$ and C parametrized at constant speed

$\forall t \in [0,1]$, f_t is 1-Lipschitz.

- **Closedness of $\mathcal{J}$**

Let Σ_n be a sequence of complete maximal submanifolds of $\widehat{\mathbb{H}}^{p,q}$, let $\tilde{\Lambda}_n := \partial_\infty \Sigma_n$. Up to a subsequence we can assume that

$$\Sigma_n \longrightarrow \Sigma_\infty \quad \text{and} \quad \tilde{\Lambda}_n \longrightarrow \tilde{\Lambda}_\infty$$

Ascoli-Arzelà for 1-Lipschitz maps

in the Hausdorff topology.

Moreover,
$$\left.\begin{aligned} \Sigma_\infty &= \mathrm{gr}\,(f : \mathbb{S}^p_+ \to \mathbb{S}^q) \\ \tilde{\Lambda}_\infty &= \mathrm{gr}\,(\partial f : \mathbb{S}^{p-1} \to \mathbb{S}^q) \end{aligned}\right\} \text{1-Lipschitz}$$

$$\partial_\infty \Sigma_\infty = \tilde{\Lambda}_\infty .$$

Then we have the following dichotomy ;

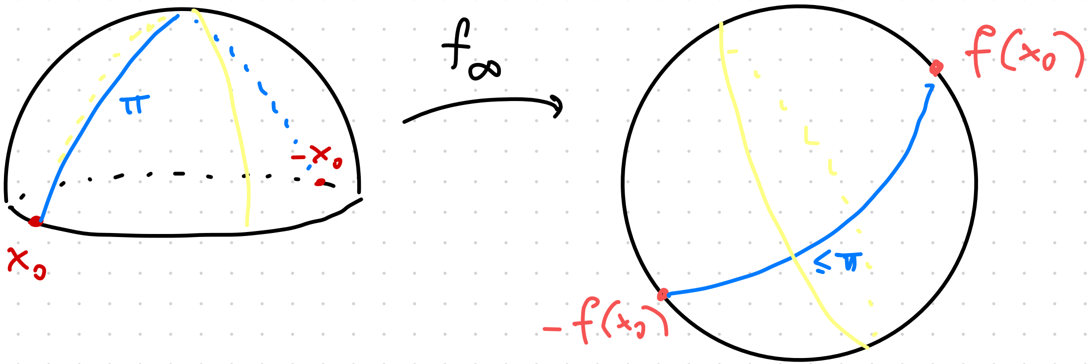
1) if $\tilde{\Lambda}_\infty$ contains antipodal points, then Σ_∞ is a Lipschitz submanifold foliated by lightlike geodesics

2) if $\tilde{\Lambda}_\infty$ does not contain antipodal points, then Σ_∞ is a complete maximal submanifold, and the convergence is actually $\mathcal{C}^\infty$.

So, if $t_n \in J$ (i.e. there exists Σ_{t_n} with $\partial_\infty \Sigma_{t_n} = \tilde{\Lambda}_{t_n}$), $t_n \to \bar{t}$, since $\Lambda_{\bar{t}}$ is a non-negative $(p-1)$-sphere, $\tilde{\Lambda}_{\bar{t}}$ does not contain antipodal points, and we are in case 2.

Comments on case 1:

If $f_\infty : \mathbb{S}^{p-1} \longrightarrow \mathbb{S}^q$ is 1-Lipschitz and $f_\infty(-x_0) = -f_\infty(x_0)$, then f_∞(geodesic connecting x_0 and $-x_0$) = geodesic connecting $f(x_0)$ and $-f(x_0)$



So f_∞ is isometric when restricted to these geodesics

$\Longrightarrow$ graph(f_∞) is foliated by complete lightlike geodesics.

Comments on case 2: we use

- A theorem of Ishihara giving a uniform bound on $\|\mathrm{II}\|$ for any complete maximal p-submanifold

  $\rightsquigarrow$ Ascoli-Arzelà type argument assuming $T_{p_n}\Sigma_n \longrightarrow$ spacelike p-subspace of $T_{p_\infty}\mathbb{H}^{p,q}$

- Elliptic regularity to infer $\mathcal{C}^\infty$-convergence
- A Lie-theoretic argument to show that, if $\tilde{\Lambda}_\infty$ does not contain antipodal points, then $T_{p_n}\Sigma_n$ cannot become degenerate in the limit.

- **Openness of J**

<u>Thm</u> Let $(\tilde{\Lambda}_t)_{t\in(-\varepsilon,\varepsilon)}$ be a smoothly varying family of smooth spacelike $(p-1)$-spheres in $\partial_\infty \hat{\mathbb{H}}^{p,q}$.
If there exists a complete maximal submanifold $\Sigma_0^p \subset \mathbb{H}^{p,q}$ with $\partial_\infty \Sigma_0 = \tilde{\Lambda}_0$, then, up to restricting ε, there exists a smooth family $(\Sigma_t)_{t\in(-\varepsilon,\varepsilon)}$ of complete maximal submanifolds with $\partial_\infty \Sigma_t = \tilde{\Lambda}_t$.

So, we apply continuity method only to Λ smooth.

The general case follows by approximating Λ by smooth spheres + dichotomy.

- Proof of "uniqueness" part

Suppose Σ_1, Σ_2 are two complete maximal submanifolds in $\hat{\mathbb{H}}^{p,q}$ with $\partial_\infty \Sigma_1 = \partial_\infty \Sigma_2$.

Consider the function

$$\Sigma_1 \times \Sigma_2 \xrightarrow{f} \mathbb{R}$$
$$(x, y) \longmapsto \langle x, y \rangle$$

If $\Sigma_1 \neq \Sigma_2$, then $\sup f \in (-1, 0]$

- using dichotomy, reduce to the case where the sup is attained
- at the maximum, $\mathrm{Hess} f \leq 0$

$\rightsquigarrow$ contradiction using maximal condition

- Proof of "moreover" part

Given two vector fields V, W, we have (in $\mathbb{R}^{p+q+1}$)

$$(D_V W)_x = (\nabla_V W)_x + \langle V, W \rangle x$$
$$= (\nabla^{\Sigma}_V W)_x + \mathrm{II}(V, W) + \langle V, W \rangle x$$

Now, let $\varphi : \mathbb{R}^{p+q+1}$ be a linear form such that $\varphi|_\Lambda > 0$.

We have

$$\mathrm{Hess}^{\Sigma}(\varphi|_{\Sigma})(V, V) = \frac{d^2}{dt^2}\Big|_{t=0} \varphi(\gamma(t)) = \varphi\left(\frac{d^2}{dt^2}\Big|_{t=0} \gamma(t)\right)$$

γ geodesic, $\gamma(0) = x$, $\gamma'(0) = V$

$$= \varphi(\mathrm{II}(V, V)) + \langle V, V \rangle \varphi(x)$$

$\nabla^{\Sigma}_{\gamma'} \gamma' = 0$

Taking the trace wrt I,

$\{e_i\}$ orthonormal basis for I

$$(\Delta^{\Sigma}\varphi)(x) = \sum_{i=1}^{p} \mathrm{Hess}^{\Sigma}(\varphi|_{\Sigma})(e_i, e_i)$$

$$= \varphi\Big(\underbrace{\sum_{i=1}^{p} \mathrm{II}(e_i, e_i)}_{=0}\Big) + \sum_{i=1}^{p} \langle e_i, e_i \rangle \varphi(x) = p\varphi(x)$$

So if $\varphi|_{\Lambda} \geq 0$, then $\varphi|_{\Sigma} \geq 0$;

if $\varphi(x) < 0$ for some x, then φ has a negative minimum x_{min}

$$\Rightarrow \underbrace{(\Delta^{\Sigma}\varphi)(x_{min})}_{\geq 0} = p\underbrace{\varphi(x_{min})}_{<0} \quad \#$$