LECTURE 3

Reference : A. Seppi, G. Smith, J. Toulisse, On complete maximal submanifolds in pseudo-hyperbolic space, Preprint (ArXiv) 2023

Goal of the lecture

Corollary (S.-Smith-Toulisse '23)

Let Γ be a hyperbolic group with $\partial\Gamma \simeq S^{p-1}$, and let $\rho: \Gamma \to PO(p, q+1)$ be a $\mathbb{H}^{p,q}$-convex-cocompact representation.

Then there exists a unique $\rho(\Gamma)$-invariant smooth, complete, spacelike, maximal submanifold $\Sigma \subset \mathbb{H}^{p,q}$.

Moreover, $\rho(\Gamma)$ acts properly discontinuously and cocompactly on Σ.

I. Maximal submanifolds

Let $\Sigma \subset \mathbb{H}^{p,q}$ (or $M^{p,q}$) a p-dimensional submanifold

- Σ is spacelike if $I = \iota^* g_{\mathbb{H}^{p,q}}$ is a Riemannian metric.
- Σ is complete if I is a complete Riemannian metric.

Recall the def of second fundamental form;

given X, Y vector fields in $U \subset \Sigma$, $p \in U$

$$\nabla_X Y(p) = \underbrace{\nabla^\Sigma_X Y(p)}_{\in T_p\Sigma} + \underbrace{\mathrm{II}(X(p), Y(p))}_{\substack{\in N_p\Sigma \\ \| \\ (T_p\Sigma)^\perp}}$$

Σ is totally geodesic if $\mathrm{II} \equiv 0$.

The vector $H = \operatorname{tr}_{I} \mathrm{II}$ is the mean curvature vector

$$\mathrm{II}_p : T_p\Sigma \times T_p\Sigma \longrightarrow N_p\Sigma \qquad N_p\Sigma \simeq \mathbb{R}^q$$

Take $\{v_1, \dots, v_q\}$ orthonormal basis for $N_p\Sigma$

$\langle \mathrm{II}_p, v_j \rangle$ is a $\mathbb{R}$-valued 2-form $\rightsquigarrow$ trace using I
(raise an index and trace)

Concretely, $H(p) = \sum_{i=1}^{n} \mathrm{II}(e_i, e_i)$ for $\{e_i\}$ any orthonormal basis of $T_p\Sigma$

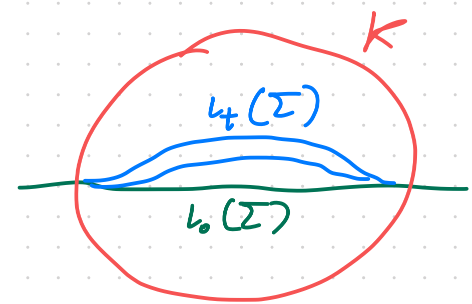
Finally, Σ is maximal if $H \equiv 0$.

Fact Σ is maximal

$\Leftrightarrow$ for every smooth variation $\iota_t : \Sigma \to \mathbb{H}^{p,q}$, $\iota_0 = \iota$,

such that $\iota_t = \iota$ outside a compact subset K,

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}\,(\Sigma_t \cap K) = 0$$



Indeed, setting $\xi(p) := \frac{d}{dt} \iota_t(p)$,

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}\,(\Sigma_t \cap K) = -2 \int_{\Sigma \cap K} \langle H, \xi \rangle \, d\mathrm{Vol}_\Sigma$$

Hence, if $H(p) \neq 0$, let $f : \Sigma \to \mathbb{R}$ a compactly supported function with $f(p) > 0$, $f \geq 0$

let ι_t be a variation such that

$$\xi = \frac{d}{dt}\Big|_{t=0} \iota_t = fH$$

Then $$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\Sigma_t \cap K) = -2 \int_{\Sigma \cap K} f \underbrace{\langle H, H \rangle}_{< 0} d\mathrm{Vol}_\Sigma \neq 0$$

Moreover, $\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{vol}(\Sigma_t \cap K) > 0$

uses signature
and sectional curvature
of $\mathbb{H}^{p,q}$

Hence maximal submanifold are local maxima of volume

II. Asymptotic boundaries

Recall that a subset $\Lambda \subset \partial_\infty \mathbb{H}^{p,q} \subset \mathbb{RP}^{p+q}$ is

- **positive** if $\forall x, y, z \in \Lambda$, $x \oplus y \oplus z \subset \mathbb{R}^{p,q+1}$ has dimension 3 and signature $(2,1)$

- **non-negative** if $\forall x, y, z \in \Lambda$, $x \oplus y \oplus z$ does not contain a negative definite 2-plane

If Λ is homeomorphic to S^{p-1} and positive (non-negative), then it will be called a positive (non-negative) $(p-1)$-sphere

Theorem (S. - Smith - Toulisse '23)

Let $\Lambda \subset \partial_\infty \mathbb{H}^{p,q}$ be a non-negative $(p-1)$-sphere. Then there exists a unique complete maximal submanifold $\Sigma^p \subset \mathbb{H}^{p,q}$ such that $\partial_\infty \Sigma = \Lambda$.

Moreover, Σ is contained in the convex hull $\mathcal{C}(\Lambda)$.

Fact $\Lambda \subset \partial_\infty \mathbb{H}^{p,q}$ is a non-negative $(p-1)$-sphere

$\Longleftrightarrow$ $\exists\, \Sigma^p$ complete spacelike submanifold such that $\partial_\infty \Sigma = \Lambda$.

("complete" can be replaced by "properly embedded")

<u>Rmk</u> The theorem does not hold for $\Sigma^k \subset \mathbb{H}^{p,q}$, $k<p$, $p\geq 3$.

In fact, there exist Jordan curves $\Lambda \subset \partial_\infty \mathbb{H}^3$ that admit several minimal disks $\Sigma_i \subset \mathbb{H}^3$, $i\in I$, with $\partial_\infty \Sigma = \Lambda$

(Anderson '86, Huang-Wang '15, Lowe-Huang-S. '23)

Including $\mathbb{H}^3 \overset{\iota}{\hookrightarrow} \mathbb{H}^{p,q}$ as a totally geodesic subspace, $\iota(\Sigma_i)$ are maximal 2-dimensional submanifolds with $\partial_\infty \iota(\Sigma_i) = \partial\iota(\Lambda)$.

III. Invariant submanifolds

Corollary (S.-Smith-Toulisse '23)

Let Γ be a hyperbolic group with $\partial\Gamma \simeq S^{p-1}$, and let $\rho: \Gamma \to PO(p, q+1)$ be a $\mathbb{H}^{p,q}$-convex-cocompact representation.

Then there exists a unique $\rho(\Gamma)$-invariant smooth, complete, spacelike, maximal submanifold $\Sigma \subset \mathbb{H}^{p,q}$.

Moreover, $\rho(\Gamma)$ acts properly discontinuously and cocompactly on Σ.

Proof of "Thm SST $\Rightarrow$ Cor SST"

non-negative

Let $\Lambda :=$ proximal limit set of ρ = a positive $(p-1)$-sphere.

Let Σ be the complete maximal submanifold with $\partial_\infty \Sigma = \Lambda$.

- Since Λ is $\rho(\Gamma)$-invariant, by uniqueness, Σ is $\rho(\Gamma)$-invariant.

- For uniqueness, by DGK, if Σ is a complete submanifold which is $\rho(\Gamma)$-invariant, then $\partial_\infty \Sigma = \Lambda$, so one can apply uniqueness in SST.

- Finally, $\Sigma \subset \mathcal{C}(\Lambda)$ and is a closed subset

  $\rightarrow$ $\rho(\Gamma) \curvearrowright \Sigma$ is prop. disc. and cocompact.