LECTURE 2

Reference : J. Danciger, F. Guéritaud, F. Kassel, Convex-cocompactness in pseudo-Riemannian hyperbolic spaces, Geom. Dedicata 192, 87-126, 2018

I. Pseudo-hyperbolic space

Given $x, y \in \mathbb{R}^{p+q+1}$, let

$$\langle x, y \rangle := \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{p+q+1} x_j y_j$$

$$\left(\begin{array}{c|c} id_p & 0 \\ \hline 0 & -id_q \end{array} \right)$$

We define:

$$\hat{\mathbb{H}}^{p,q} := \{ x \in \mathbb{R}^{p+q+1} \mid \langle x, x \rangle = -1 \} \cong \mathbb{D}^p \times \mathbb{S}^q$$

The restriction of $\langle \cdot, \cdot \rangle$ endows $\hat{\mathbb{H}}^{p,q}$ with a pseudo-Riemannian metric of signature (p,q) and constant sectional curvature -1.

$$\mathrm{Isom}\, \hat{\mathbb{H}}^{p,q} = O(p, q+1)$$

Then define

$$\mathbb{H}^{p,q} := \hat{\mathbb{H}}^{p,q} / \{\pm id\}$$

$\cong \mathbb{D}^p \times \mathbb{S}^q / \langle(-id, -id)\rangle$

$\mathbb{RP}^{p+q}$

endowed with the metric induced from $\hat{\mathbb{H}}^{p,q}$

$$\operatorname{Isom} \mathbb{H}^{p,q} = PO(p, q+1)$$

We have moreover

$$\partial_\infty \hat{\mathbb{H}}^{p,q} := \{ x \in \mathbb{R}^{p+q+1} \mid \langle x, x \rangle = 0 \} / \mathbb{R}_{>0}$$

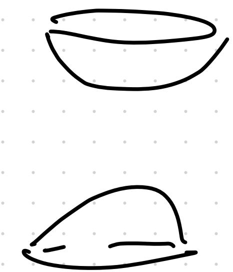
$\cong \mathbb{S}^p \times \mathbb{S}^q$

$$\partial_\infty \mathbb{H}^{p,q} := \{ x \in \mathbb{R}^{p+q+1} \mid \langle x, x \rangle = 0 \} / \mathbb{R}^*$$

"$\partial(\mathbb{H}^{p,q}) \subset \mathbb{R}^{p+q}$"

$\cong \mathbb{S}^p \times \mathbb{S}^q / \langle(-id, -id)\rangle$

<u>Examples</u>



- $\hat{\mathbb{H}}^{p,0}$ = two-sheeted hyperboloid $\sum_{i=1}^{p} x_i^2 - x_{p+1}^2 = -1$

$\mathbb{H}^{p,0} \cong \mathbb{H}^p$

- $\hat{\mathbb{H}}^{0,q} = \left\{ \sum_{j=1}^{q+1} x_j^2 = 1 \right\} = (\mathbb{S}^q, -(\text{spherical metric}))$

$\mathbb{H}^{0,q} = (\mathbb{RP}^q, -(\text{spherical metric}))$

<u>Rmk</u> Both $\mathbb{H}^p$ and $-\mathbb{RP}^q$ embed in $\mathbb{H}^{p,q}$ as totally geodesic submanifolds $\left(\begin{array}{c} x_{p+2} = \dots = x_{p+q+1} = 0 \\ \& \\ x_1 = \dots = x_p = 0 \end{array} \right)$

Affine charts

In the affine chart $A_{p+q+1} = \{x^{p+q+1} \neq 0\}$, we have

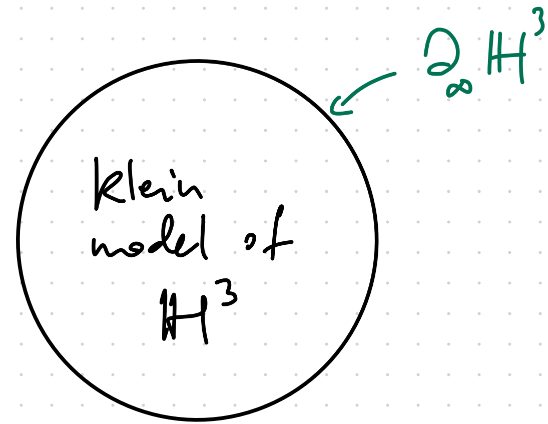
$$\mathbb{H}^{p,q} \cap A_{p+q+1} = \mathbb{P}\left\{ \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 - x_{p+q+1}^2 < 0,\ x_{p+q+1} \neq 0 \right\}$$

$$= \mathbb{P}\left\{ x_{p+q+1} = 1,\ \sum_{i=1}^{p} t_i^2 - \sum_{j=p+1}^{p+q} t_j^2 < 1 \right\}$$

$$t_i := \frac{x_i}{x_{p+q+1}}$$

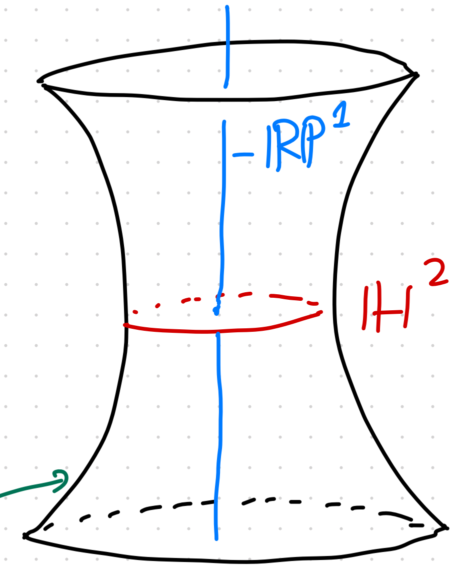
Some pictures in A_4 $(p+q=3)$

- $\mathbb{H}^{3,0} \cap A_4 = \{ \sum_{i=1}^{3} t_i^2 < 1 \}$



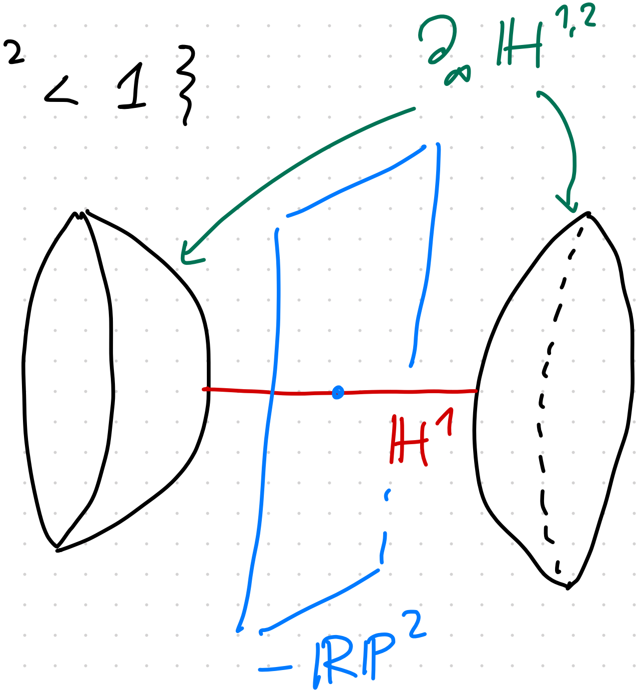
- $\mathbb{H}^{2,1} \cap A_4 = \{ t_1^2 + t_2^2 - t_3^2 < 1 \}$

($\mathbb{H}^{p,1}$ is also called Anti-de Sitter space of dimension $p+1$)



- $\mathbb{H}^{1,2} \cap A_4 = \{ t_1^2 - t_2^2 - t_3^2 < 1 \}$

($\mathbb{H}^{1,p}$ is also called (minus) de Sitter space of dim $p+1$)



- $\mathbb{H}^{0,3} \cap A_4 = \mathbb{R}^3$

  $=$ affine chart for $\mathbb{RP}^3$

II. Convex cocompactness

Definition (Danciger – Guéritaud – Kassel '18)

A discrete subgroup $\Gamma < PO(p,q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact if $\exists\, C \subset \mathbb{H}^{p,q}$ such that:

1) C is closed and properly convex ← i.e., C is convex and bounded in an affine chart

2) C has non-empty interior ← Automatic if Γ irreducible (i.e. does not preserve any projective subspace of $\mathbb{R}^{p+q}$)

3) $\partial_\infty C = \overline{C}^{\mathbb{RP}^{p+q}} \setminus C$ does not contain any projective line segment

4) Γ preserves C and $\Gamma \curvearrowright C$ is properly discontinuous and cocompact.

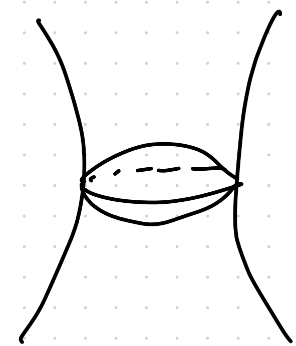
Automatic if $q=0$

Rmk We will work with representations $\rho : \Gamma \to PO(p,q+1)$ with $\rho(\Gamma)$ $\mathbb{H}^{p,q}$-convex-cocompact. We will assume:

- ρ has always finite kernel,
- ρ is sometimes also torsion-free

$\Big\} \Rightarrow \rho$ injective

Example $\rho : \pi_1 M \xrightarrow{hol} O(p,1) \xrightarrow{\left[\begin{pmatrix} * & \\ & 1 \end{pmatrix}\right]} PO(p,q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact (take C = thickening of $\mathbb{H}^p \subset \mathbb{H}^{p,q}$)



Examples come from AdS geometry too.

Theorem (Danciger - Guéritaud - Kassel '18)

The space

$$\left\{ \begin{array}{c} \rho : \Gamma \to PO(p, q+1) \\ \text{of finite kernel} \\ \rho(\Gamma)\ \mathbb{H}^{p,q}\text{-convex cocompact} \end{array} \right\} \subset \mathrm{Hom}(\Gamma, PO(p, q+1))$$

is open.

This will follow from stability of the Anosov property.

III. Anosov representations

$P_1 = \mathrm{Stab}(\text{Isotropic line in } \mathbb{R}^{p,q+1})$

$\downarrow$

Let Γ be a word hyperbolic group.

A representation $\rho : \Gamma \longrightarrow PO(p,q+1)$ is P_1-Anosov if there exists a continuous, ρ-equivariant map

$$\xi : \partial\Gamma \longrightarrow \partial_\infty \mathbb{H}^{p,q} \qquad \left(= PO(p,q+1) / P_1 \right)$$

such that:

i) ξ is transverse: $\forall \eta_1, \eta_2 \in \partial\Gamma,\ \eta_1 \neq \eta_2 \Rightarrow \xi(\eta_1) \notin \xi(\eta_2)^\perp$

$(\Rightarrow \xi$ injective$)$

ii) ξ has an associated flow with a uniform contraction property

Automatic if ρ irreducible (Guichard – Wienhard)

Automatic if $q=0$

Rmk The most important consequence of ii) is:

ξ is dynamics-preserving

i.e. if γ has infinite order,

$$\xi\left(\begin{matrix}\text{attracting fixed point}\\ \text{of } \gamma \text{ in } \partial\Gamma\end{matrix}\right) = \begin{matrix}\text{attracting fixed point}\\ \text{of } \rho(\gamma) \text{ in } \partial_\infty \mathbb{H}^{p,q}\end{matrix}$$

$\rho(\gamma)$ is proximal, i.e. has a unique attracting fixed point

Moreover, $\xi(\partial\Gamma) =$ proximal limit set of $\rho(\Gamma)$

$= \{$attracting fixed points of proximal $\rho(\gamma)$'s$\}$

Fact If $q=0$, $\rho:\Gamma\to PO(p,1)$ is P_1-Anosov

$\iff$ ρ is $\mathbb{H}^{p,0}$-convex-cocompact in the classical sense.

Definition A subset $\Lambda\subset\partial_\infty\mathbb{H}^{p,q}\subset\mathbb{RP}^{p+q}$ is

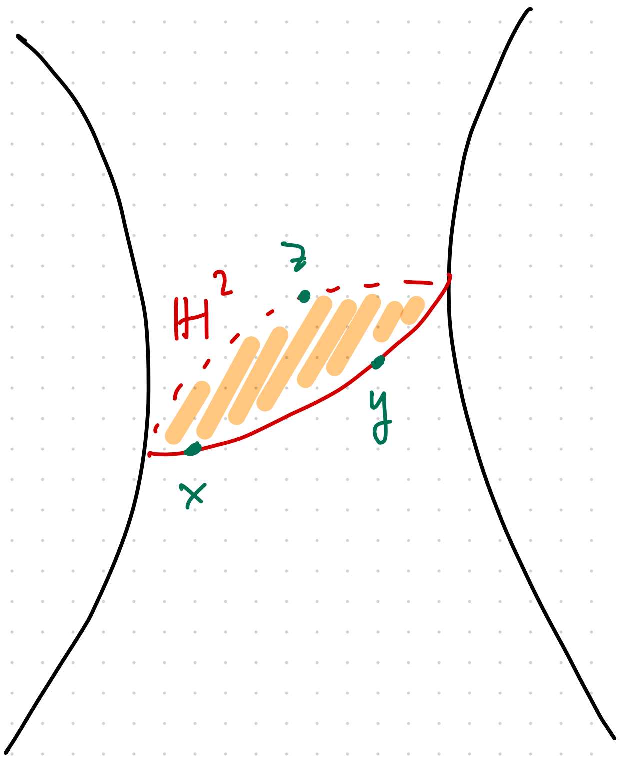
- positive if $\forall x,y,z\in\Lambda$, $x\oplus y\oplus z\subset\mathbb{R}^{p,q+1}$ has dimension 3 and signature $(2,1)$

  ($\iff (x\oplus y\oplus z)\cap\mathbb{H}^{p,q}$ is a copy of $\mathbb{H}^2$)

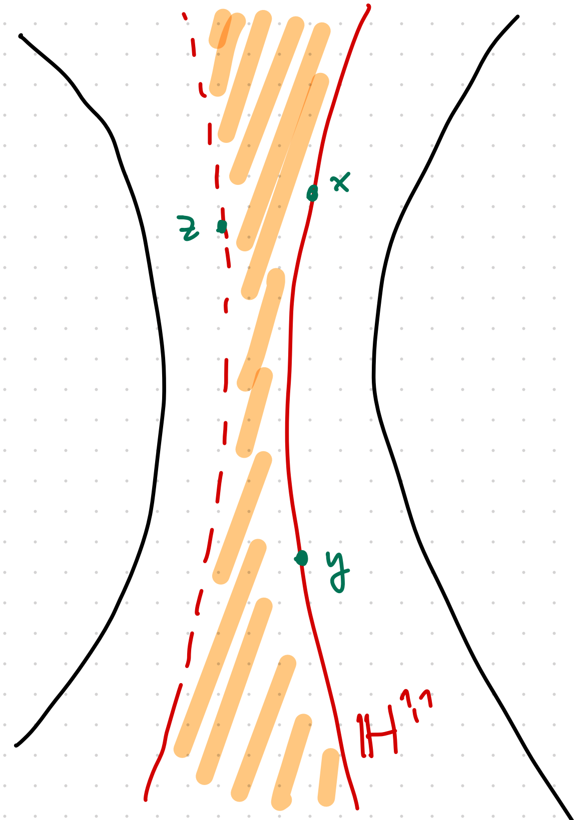
- negative if $\forall x,y,z\in\Lambda$, $x\oplus y\oplus z\subset\mathbb{R}^{p,q+1}$ has dimension 3 and signature $(1,2)$

  ($\iff (x\oplus y\oplus z)\cap\mathbb{H}^{p-1,q+1}$ is a copy of $\mathbb{H}^2$)

Geometrically,



positive



not positive

Fact If $\Lambda \subset \partial_\infty \mathbb{H}^{p,q}$ is closed, connected and transverse, then Λ is either positive or negative.

Theorem (Danciger - Guéritaud - Kassel '18)

Let $\rho : \Gamma \to PO(p,q+1)$ a discrete representation. Then

ρ has finite kernel & $\rho(\Gamma)$ is $\mathbb{H}^{p,q}$-convex-cocompact $\iff$ Γ is word hyperbolic & ρ is positive P_1-Anosov

the proximal limit set is positive

Similarly,

$\rho(\Gamma)$ is $\mathbb{H}^{p-1,q+1}$-convex-cocompact $\iff$ ρ is negative P_1-Anosov

Corollary

$$\left\{ \begin{array}{c} \rho : \Gamma \longrightarrow PO(p, q+1) \\ \text{of finite kernel} \\ \rho(\Gamma)\ \mathbb{H}^{p,q}\text{-convex-cocompact} \end{array} \right\} \text{ is open in } \mathrm{Hom}(\Gamma, PO(p, q+1))$$

(P_1-Anosov property is open)

Proof Idea:

"$\Leftarrow$" $\rho(\Gamma) \curvearrowright \mathcal{C}(\Lambda)$ = convex hull of the proximal limit set of $\rho(\Gamma)$

is properly discontinuous and cocompact.

" $\Rightarrow$ " Use the Hilbert metric d on

Ω = maximal invariant proper convex domain (= dual of the convex hull of Λ)

$\rightsquigarrow$ (C, d) is Gromov hyperbolic

$\rightsquigarrow$ Γ is hyperbolic (Milnor-Švarc)

Moreover, the orbit map $\Gamma \longrightarrow (C, d)$ extends to a transverse continuous map

$$\xi : \partial\Gamma \longrightarrow \partial C \cong \partial_\infty C = \Lambda \subset \partial_\infty \mathbb{H}^{p,q}.$$