LECTURE 1

Reference:

- A. Wienhard, An invitation to higher Teichmüller theory, ICM 2018 invited lecture.

Motivation

For S a closed oriented surface, the Teichmüller space

$$\mathcal{T}(S) = \left\{ \begin{array}{c} \text{hyperbolic metrics} \\ \text{on } S \end{array} \right\} \Big/ \mathrm{Diff}_0(S)$$

$\varphi \cdot h := \varphi^* h$

is identified to a connected component of

$$\mathcal{X}(\pi_1 S, PSL(2,\mathbb{R})) = \mathrm{Hom}(\pi_1 S, PSL(2,\mathbb{R})) \Big/ \text{conjugation}$$

via the holonomy map

$$\mathrm{hol} : \mathcal{T}(S) \longrightarrow \mathcal{X}(\pi_1 S, PSL(2,\mathbb{R}))$$

The construction of hol is as follows:

given h hyperbolic metric on S, let $\tilde{h} := \pi^* h$ on $\tilde{S}$

$(\tilde{S}, \tilde{h})$ is a complete simply connected hyperbolic surface

$\Rightarrow \exists\ dev : (S, \tilde{h}) \longrightarrow \mathbb{H}^2$ orientation-preserving isometry

Then $\exists\ \rho : \pi_1 S \longrightarrow \mathrm{Isom}\,\mathbb{H}^2$ "holonomy representation"

such that $\forall \gamma \in \pi_1 S \quad dev \circ \gamma = \rho(\gamma) \circ dev$

$\Rightarrow \rho(\pi_1 S) \curvearrowright \mathbb{H}^2$ is free, properly discontinuous, $\mathbb{H}^2 / \rho(\pi_1 S) \simeq S$.

<u>Goldman</u> $\mathrm{hol}(\mathcal{T}(S))$ is a connected component

(maximal Euler number)

So,

$\left\{ \begin{array}{c} \text{holonomies of hyperbolic} \\ \text{metrics on } S \end{array} \right\} \subset \mathrm{Hom}(\pi_1 S, PSL(2,\mathbb{R}))$

has 2 connected components

consists entirely of discrete and faithful representations.

Now, let G a real semi-simple Lie group of rank ≥ 2.

<u>Def</u> (Wienhard)

A higher Teichmüller space is a connected component of $\mathrm{Hom}(\pi_1 S, G)$ that consists entirely of discrete and faithful representations.

Examples

- Hitchin components for G real split simple Lie group
  [e.g. $G = SL(n,\mathbb{R}), Sp(2n,\mathbb{R}), SO(n,n+1)$]
  namely the connected components containing the representations $i \circ \rho$, for $\rho : \pi_1 S \to SL(2,\mathbb{R})$ in the Teichmüller component and $i : SL(2,\mathbb{R}) \hookrightarrow G$ irreducible embedding.
  (Choi-Goldman for $SL(3,\mathbb{R})$, Fock-Goncharov, Labourie)
- Maximal representations for G real simple of Hermitian type
  [e.g. $G = O(2,q)$] (Bradlow-García Prada-Gothen, Burger-Iozzi-Wienhard)
- θ-positive representations for, in addition, $G = SO(p,q)$, $p \neq q$, and an exceptional family of rank 4
  (Guichard-Wienhard, Guichard-Labourie-Wienhard)

conjecturally, these are all higher Teich.

Let now M be a closed (topological) manifold of dimension $n > 2$.

<u>Goal</u>: look for connected components of $\mathrm{Hom}(\pi_1 M, G)$ consisting entirely of discrete and faithful representations

"higher higher Teichmüller spaces"

higher dimensional

higher rank

<u>Rmk</u> By Mostow rigidity, If M^n is hyperbolic the holonomies $\pi_1 M \to SO(n,1)$ of hyperbolic structures on M form a $SO(n,1)$-conjugacy orbit of a point.

<u>Rmk</u> Moreover, we would like to have a representation with Zariski dense image in a higher higher Teichmüller space.

There are examples where

$$\rho : \pi_1 M \longrightarrow \underset{\text{rank one}}{H} \longrightarrow G \qquad \text{is rigid}$$

"Old" examples ; Convex projective structures

$\exists\ \Gamma = \Gamma_0 \triangleright \Gamma_1 \triangleright \cdots \triangleright \Gamma_n = \{1\}$

$\Gamma_{i+1} = [\Gamma, \Gamma_i]$

Theorem (Benoist '05) $\dim M = n$

If $\pi_1 M$ does not contain an infinite nilpotent normal subgroup, then

$$\left\{ \begin{array}{c} \text{holonomies of convex projective} \\ \text{structures on } M \end{array} \right\} \subset \mathrm{Hom}(\pi_1 M, PGL(n+1, \mathbb{R}))$$

is a union of connected components.

i.e. $\rho(\pi_1 M) \curvearrowright \Omega$ properly discontinuously, freely and cocompactly

$\Omega / \rho(\pi_1 M) \cong M$, $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$ properly convex

- openness: Koszul '68
- closedness: Choi-Goldman for $n=2$; Kim for $n=3$ } 2000s

"New" examples : $\mathbb{H}^{p,q}$-convex-cocompactness

Theorem (Beyrer - Kassel '23)

Let $p,q \in \mathbb{N}$, $p \geq 2$, $q \geq 1$. If M is a closed negatively curved manifold, $\dim M = p$, then

$$\left\{ \begin{array}{c} \mathbb{H}^{p,q}\text{-convex cocompact representations} \\ \rho : \pi_1 M \longrightarrow PO(p,q+1) \end{array} \right\} \subset \mathrm{Hom}(\Gamma, PO(p,q+1))$$

is a union of connected components.

- openness: Danciger - Guéritaud - Kassel '18 by the P_1-Anosov condition
- closedness: Barbot '15 for M hyperbolic and $q = 1$

  (maximal globally hyperbolic Anti-de Sitter structures on $M \times \mathbb{R}$)
- for $p = 2$, there are maximal components in $PO(2,q)$

Rmk For $q=0$, $\mathbb{H}^{p,0}$-convex-cocompact $\Longleftarrow$ cocompact
- $p=2 \rightsquigarrow$ true because of Teichmüller spaces
- $p>2 \rightsquigarrow$ true by Mostow rigidity

Rmk By the following theorem of Kleiner-Leeb and Quint, convex-cocompactness in the Riemannian symmetric space is not more interesting than cocompactness, for rank $G \geq 2$.

Theorem Let G be a real semi-simple Lie group, rank $G \geq 2$. If $\Gamma < G$ discrete and Zariski dense acts cocompactly on $C \subseteq G/K$ non-empty, closed, convex, then Γ is a uniform lattice.

$\uparrow$ Riemannian symmetric space

$\rightarrow$ Look for convex-cocompactness in a non-Riemannian symmetric space.