Problem sheet 3

Friday 17th May 2024

Further topics

Exercise 1 : Isometric actions of non-hyperbolic groups

Recall the definitions of Clifford torus and Barbot surface in Sheet 2.

1. Find some discrete subgroup of $\mathrm{Isom}(\mathbb{S}^3)$ acting freely, properly discontinuously and cocompactly on the Clifford torus. Do the same for a discrete subgroup $\mathrm{Isom}(\mathbb{H}^{2,1})$ acting on the Barbot surface.
2. Classify all free, properly discontinuously and cocompact isometric actions of $\mathbb{Z}^2$ on the Barbot surface.
3. Are these actions $\mathbb{H}^{2,1}$-convex-cocompact ?

Exercise 2 : Guichard-Wienhard domain of discontinuity

The goal of this exercise is to provide another proof of the fact that Guichard-Wienhard domain of discontinuity is empty for $G = \mathrm{PO}(2,2)_0 \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, relying on the previous exercises on Anti-de Sitter geometry.

1. Check that the following is an orthonormal basis for the space of 2-by-2 real matrices, the quadratic form being $q = -\det$ as usual.

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

2. Show that, up to applying the identity component of the isometry group, every maximally isotropic subspace V is the graph of $f : \mathrm{Span}(e_1, e_2) \to \mathrm{Span}(e_3, e_4)$ of one of the following forms :

$$f\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad \text{or} \qquad f\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

3. Now show that $\mathrm{P}(V) \subset \partial_\infty \mathbb{H}^{2,1} \cong \mathbb{RP}^1 \times \mathbb{RP}^1$ is either of the form $\{\star\} \times \mathbb{RP}^1$ or $\mathbb{RP}^1 \times \{\star\}$.
4. Using Sheet 2, conclude that $\mathrm{P}(V)$ intersects every positive curve Λ.