Problem sheet 2

Thursday 16th May 2024

Submanifolds of $\mathbb{H}^{p,q}$

Exercise 1 : Examples of minimal and maximal surfaces

The goal of this exercise is to study examples of minimal and maximal surfaces : the Clifford torus in $\mathbb{S}^3$ and its analogue in $\mathbb{H}^{2,1}$, called the Barbot surface.

1. Let P a 2-dimensional subspace of $\mathbb{R}^4$, let $\gamma = P \cap \mathbb{S}^3$, and let $\gamma^* = P^\perp \cap \mathbb{S}^3$. Show that the set of midpoints of geodesic segments in $\mathbb{S}^3$ connecting points of γ and γ^* is a minimal surface (called Clifford torus).
2. Do the same for P a plane of signature $(1,1)$ in $\mathbb{H}^{2,1}$, finding a maximal surface (sometimes called Barbot surface).
3. Find the subgroup of isometries of $\mathbb{S}^3$ and $\mathbb{H}^{2,1}$ preserving the Clifford torus and the Barbot surface.
4. Show that the Clifford torus is a torus, the Barbot surface is a plane, and both are intrinsically flat.
5. Compute the second fundamental forms in coordinates that make the first fundamental form the standard Euclidean metric.
6. Compute the mean curvature of the surface at distance d from γ, for $d \in (0, \pi/2)$.

Exercise 2 : Anti-de Sitter geometry and $\mathrm{PSL}(2,\mathbb{R})$ — continued

Recall from Sheet 1 that there is an isometry $\mathbb{H}^{2,1} \cong \mathrm{PSL}(2,\mathbb{R})$, that induces a natural identification $\partial_\infty \mathbb{H}^{2,1} \cong \mathrm{P}\{\text{rank one matrices}\} \cong \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$.

1. Show that, under the above identification, a triple $((x_1,x_2),(y_1,y_2),(z_1,z_2)) \in (\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1)^3$ is positive (resp. negative) if and only if both triples (x_1,y_1,z_1) and (x_2,y_2,z_2) consist of pairwise distinct elements and they have the same cyclic order (resp. the opposite cyclic order) in $\mathbb{R}\mathrm{P}^1$. (Hint : to simplify computations, use the transitivity of the action of $\mathrm{PSL}(2,\mathbb{R})$ on ordered triples in $\mathbb{R}\mathrm{P}^1$.)
2. Prove that a subset $\Lambda \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ homeomorphic to S^1 is positive (resp. negative) if and only if it is the graph of an orientation-preserving (resp. orientation-reversing) self-homeomorphism of $\mathbb{R}\mathrm{P}^1$.

In the second part, we wish to show that if $\rho = (\rho_1, \rho_2) : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ acts freely, properly discontinuously and cocompactly on a spacelike surface $\Sigma \subset \mathbb{H}^{2,1} \cong \mathrm{PSL}(2,\mathbb{R})$, then both ρ_1 and ρ_2 are Fuchsian representations.

3. Given Σ as above, show that the map $\Phi : T^1\Sigma \to \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ sending (x,v) to $\lim_{t\to+\infty} \exp_x(tv)$ is well-defined and ρ-equivariant.
4. Letting π_i be the projections on each factor, show that the maps $(x,v) \to (x, \pi_i \circ \Phi)$ induces bundle isomorphisms between T^1S and $S \times_{\rho_i} \mathbb{R}\mathrm{P}^1$.
5. Conclude that ρ_1 and ρ_2 have maximal Euler class and are thus Fuchsian representations.
6. Finally, explain that $\partial_\infty\Sigma$ is the graph of the unique ρ_1-ρ_2-equivariant homeomorphism of $\mathbb{R}\mathrm{P}^1$.

Exercise 3 : The Poincaré model of $\mathbb{H}^{p,q}$

The goal is to compute the pseudo-hyperbolic metric as the following metric on $\mathbb{D}^p \times \mathbb{S}^q$:

$$f^2 \left(g_{\mathbb{S}^p_+} - g_{\mathbb{S}^q} \right)$$

where $\mathbb{S}^p_+$ is the hemisphere in $\mathbb{S}^p$ and f is a function of the distance from the center of $\mathbb{S}^p_+$.

1. Check that the stereographic projection $\varphi : \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}$ can be expressed as :
$$\varphi(u) = \frac{\|u\|^2 - 1}{\|u\|^2 + 1} \left(\frac{2u}{\|u\|^2 - 1}, 1 \right)$$
2. Show that the pull-back of the spherical metric via φ is $4du^2/(1 + \|u\|^2)^2$.
3. Do the analogue of items 1 and 2 for the stereographic projection $\varphi : \mathbb{D}^n \to \mathbb{H}^n \subset \mathbb{R}^{n,1}$.
4. Now consider the map $\varphi : \mathbb{D}^p \times \mathbb{S}^q \to \widehat{\mathbb{H}}^{p,q}$ defined by
$$\varphi(u, w) = \frac{1 + \|u\|^2}{1 - \|u\|^2} \left(\frac{2u}{1 + \|u\|^2}, w \right) .$$
Check that, if $q = 0$, this coincides with the stereographic projection as in item 3.
5. Show that φ is a diffeomorphism.
6. Show that
$$\varphi^* g_{\widehat{\mathbb{H}}^{p,q}} = \left(\frac{1 + \|u\|^2}{1 - \|u\|^2} \right)^2 \left(\frac{4\|du\|^2}{(1 + \|u\|^2)^2} - g_{\mathbb{S}^q} \right)$$
and thus conclude the claim.
7. What is the analogue of the above construction for $\mathbb{S}^n$, i.e. a map $\varphi : \mathbb{R}^p \times \mathbb{S}^{n-p} \to \mathbb{S}^n$?

Exercise 4 : Differential geometry of submanifolds

Let Σ be a p-dimensional spacelike submanifold of $\mathbb{H}^{p,q}$. Recall that the second fundamental form $\mathrm{II} \in \Gamma^\infty(\mathrm{Sym}^2(T^*\Sigma, N\Sigma))$ is defined by :

$$\nabla_X Y = \nabla^\Sigma_X Y + \mathrm{II}(X, Y)$$

and its norm $\|\mathrm{II}\| \in C^\infty(\Sigma)$ is defined by

$$\|\mathrm{II}\|^2 = -\sum_{i,j=1}^{p} \langle \mathrm{II}(e_i, e_j), \mathrm{II}(e_i, e_j) \rangle$$

where $\{e_1, \ldots, e_p\}$ is an orthonormal basis for Σ.

1. Show that the definition of $\|\mathrm{II}\|$ does not depend on the choice of the orthonormal frame.
2. Prove the following identity between the Riemann tensors of $\mathbb{H}^{p,q}$ and Σ :
$$\langle R^\Sigma(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle + \langle \mathrm{II}(X, W), \mathrm{II}(Y, Z) \rangle - \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W) \rangle$$
3. Deduce the Gauss' equation in $\mathbb{H}^{p,q}$:
$$K(e_i, e_j) = -1 + \langle \mathrm{II}(e_i, e_i), \mathrm{II}(e_j, e_j) \rangle - \langle \mathrm{II}(e_i, e_j), \mathrm{II}(e_i, e_j) \rangle$$
Deduce that, if $p = 2$, then a *maximal* surface has curvature ≥ -1.
4. Using the previous item, show that, if Σ is a *maximal* submanifold of dimension p, then
$$\mathrm{Ric}_\Sigma(e_i, e_i) = -(p - 1) - \sum_{j=1}^{p} \langle \mathrm{II}(e_i, e_j), \mathrm{II}(e_i, e_j) \rangle$$
and
$$\mathrm{Scal}_\Sigma = -p(p - 1) + \|\mathrm{II}\|^2 .$$
Conclude that $\mathrm{Scal}_\Sigma \geq -p(p - 1)$, and explain how this inequality generalises the conclusion of the previous item.