Problem sheet 1

Wednesday 15th May 2024

About the geometry of $\mathbb{H}^{p,q}$

Exercise 1 : Differential geometry in the hyperboloid model

Consider the "hyperboloid model" of the pseudo-hyperbolic space, namely

$$\widehat{\mathbb{H}}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x\rangle = -1\}$$

1. Show that, if D is the flat connection of $\mathbb{R}^{p+q+1}$ and ∇ is the Levi-Civita connection of $\widehat{\mathbb{H}}^{p,q}$, then
$$(\nabla_X Y)(p) = (D_X Y)(p) - \langle X, Y\rangle p$$
for any point p and any vector fields X, Y defined in a neighbourhood of p.
2. Show the following identify for the Riemann tensor R of $\widehat{\mathbb{H}}^{p,q}$:
$$R(X,Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X$$
3. Conclude that $\widehat{\mathbb{H}}^{p,q}$ has constant sectional curvature -1.
4. Show that the intersection of $\widehat{\mathbb{H}}^{p,q}$ with any linear subspace of $\mathbb{R}^{p+q+1}$ is totally geodesic.
5. Show that the isometry group of $\widehat{\mathbb{H}}^{p,q}$ is $\mathrm{O}(p,q)$, and that every linear isometry $T_x\widehat{\mathbb{H}}^{p,q} \to T_y\widehat{\mathbb{H}}^{p,q}$ is realised as the differential $d\varphi$ of a global isometry $\varphi : \widehat{\mathbb{H}}^{p,q} \to \widehat{\mathbb{H}}^{p,q}$.
6. Deduce that $\widehat{\mathbb{H}}^{p,q}$ is geodesically complete.
7. Find the geodesics $t \mapsto \exp_p(tX)$, for given $p \in \widehat{\mathbb{H}}^{p,q}$ and $X \in T_p\widehat{\mathbb{H}}^{p,q}$. (Hint : distinguish the cases $\langle X, X\rangle = 1, 0$ and -1.) Use the result to deduce again that $\widehat{\mathbb{H}}^{p,q}$ is geodesically complete.

Exercise 2 : Anti-de Sitter geometry and $\mathrm{PSL}(2,\mathbb{R})$

Let $\widehat{G} = \mathrm{SL}(2,\mathbb{R})$ and $G = \mathrm{PSL}(2,\mathbb{R})$. Throughout the exercise, do not forget that $G \cong \mathrm{Isom}(\mathbb{H}^2)$ via the upper half-plane model of $\mathbb{H}^2$.

1. Show that $\mathbb{R}^{2,2}$ is isomorphic to the vector space of 2-by-2 real matrices endowed with the quadratic form $q = -\det$. Deduce that $\widehat{\mathbb{H}}^{2,1}$ is isometric to $\widehat{G}$, and $\mathbb{H}^{2,1}$ is isometric to G, when the latter are endowed with the Lorentzian metric induced by q.
2. Show that the identity component of the group of isometries of G equals $G \times G$ acting by left and right multiplication. What is the identity component of the group of isometries of $\widehat{G}$?
3. Use the previous exercise to compute the geodesics of G. Try to use your knowledge of Lie groups, and in particular the exponential map, to find the same result.
4. Show that the subset of elements of G of order two is a totally geodesic copy of $\mathbb{H}^2$, and find an explicit isometry.
5. Check that, in the matrix model above, $\partial_\infty\mathbb{H}^{2,1}$ is identified with the projectivisation of the space of rank one 2-by-2 real matrices.
6. Now show that the map $[M] \mapsto (\mathrm{Im}(M), \mathrm{Ker}(M))$ identifies $\partial_\infty\mathbb{H}^{2,1}$ with $\mathbb{RP}^1 \times \mathbb{RP}^1$. How does the isometry group $G \times G$ act on $\mathbb{RP}^1 \times \mathbb{RP}^1$ under this identification ? Can you find a criterion for the convergence of a sequence $\{\gamma_n\}$ in G to an element $(x, y) \in \mathbb{RP}^1 \times \mathbb{RP}^1$?

Further reading : Bonsante-Seppi, "Anti-de Sitter geometry and Teichmüller theory", section 3, In the tradition of Thurston : Geometry and Topology 2020, https ://arxiv.org/abs/2004.14414.

Exercise 3 : An upper half-space model

Consider the following "upper half-space model" of $\mathbb{H}^{p,q}$:

$$\mathcal{H}^{p,q} := \left(\{(x_1, \dots, x_{p+q}), x_0 > 0\}, \frac{dx_1^2 + \dots + dx_p^2 - dx_{p+1}^2 - \dots - dx_{p+q}^2}{x_1^2} \right)$$

1. Can you find some geodesics and some totally geodesic subspaces of $\mathcal{H}^{p,q}$?
2. We now want to find all lightlike geodesics (i.e. such that $\langle \gamma', \gamma' \rangle = 0$). As a first step, show the following formula relating the Levi-Civita connections $\widehat{\nabla}$ and ∇ of two pseudo-Riemannian metrics $\widehat{g}$ and g with $\widehat{g} = e^{2u} g$:

$$\widehat{\nabla}_X Y = \nabla_X Y + du(X)Y + du(Y)X - g(X,Y)\mathrm{grad}u \ .$$

3. Deduce that unparameterised lightlike geodesics are invariant under a conformal change of the metric.
4. Using the above point, find all lightlike geodesics as unparameterised curves,
5. Now find all lightlike geodesics as parameterised curved, and deduce that $\mathcal{H}^{p,q}$ is not geodesically complete.
6. If you are motivated in computations, compute the Christoffel symbols, write down the geodesic equation and try to find solutions explicitly.
7. If you are *particularly* motivated in computations, show directly that the Riemann tensor of $\mathcal{H}^{p,q}$ can be expressed as $R(X,Y)Z = g(X,Z)Y - g(Y,Z)X$, and thus $\mathcal{H}^{p,q}$ has constant sectional curvature -1.
8. If you are *extremely* motivated in computations, show that the map

$$(x_0, \dots, x_{p+q}) \mapsto \left(\frac{1 - h(x) - x_1^2}{2x_1}, \frac{x_2}{x_1}, \dots, \frac{x_{p+q}}{x_1}, \frac{1 + h(x) - x_1^2}{2x_1} \right)$$

   where $h(x) = x_2^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q+1}^2$, defines an isometric embedding of $\mathcal{H}^{p,q}$ into $\widehat{\mathbb{H}}^{p,q} \subset \mathbb{R}^{p,q+1}$.

Further reading : Seppi-Trebeschi, The half-space model of pseudo-hyperbolic space, Developments in Lorentzian Geometry 2022, https ://arxiv.org/abs/2106.11122

Exercise 4 : Proximal elements and limit sets

Recall that a proximal element of $\mathrm{PO}(p, q+1)$ is an isometry of $\mathbb{H}^{p,q}$ with a unique attracting fixed point in $\mathbb{RP}^{p+q}$, and that the proximal limit set of a discrete subgroup $\Gamma < \mathrm{PO}(p, q+1)$ is the closure of the set of attracting fixed point of its proximal elements.

1. Show that, if $\gamma \in \mathrm{PO}(p, q+1)$ is a proximal element, then its attracting fixed point is in $\partial_\infty \mathbb{H}^{p,q}$.
2. Show that, if Γ acts properly discontinuously on an open convex subset Ω, then its proximal limit set is contained in $\partial_\infty \Omega = \overline{\Omega} \cap \partial_\infty \mathbb{H}^{p,q}$.
3. Show that, if Γ acts properly discontinuously on an open convex subset Ω, then the set of accumulation points of any orbit is contained in $\partial_\infty \Omega$ and does not depend on the orbit. (Hint : use the Hilbert distance on Ω.)

Further reading : Danciger-Guéritaud-Kassel, Convex cocompactness in pseudo-Riemannian hyperbolic spaces, section 2, Geom. Dedicata 192, 87-126, 2018.