Mémoire d'habilitation à diriger des recherches

Problèmes de Plateau asymptotiques, leurs généralisations, et applications aux structures géométriques

Asymptotic Plateau problems, their generalizations, and applications to geometric structures

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Introduction

It has been for me a challenging exercise to look back at the research that I have developed since obtention of my PhD in 2015, to “connect the dots”, and to describe the trajectory that my interests, my work, and my theorems, have followed since them. The answer that I have found is that there have been two main *leitmotive*: one is the study of *geometric structures*, or (G, X)-*structures*, on manifolds, mostly of low dimensions; the second — this time with the exception of two “digressions” of a more topological flavour, which I will discuss later — is the theory of submanifolds, and in particular the investigation of so-called *asymptotic Plateau problems.* This introduction mostly tries to describe how the latter, namely the study of submanifolds with special properties (hence with a differential geometric spirit) can have very interesting applications, in various directions, for the former, namely for geometric structures (typically a subject largely studied in geometric topology). Then, at the end of the introduction, I will say a few more words on the aforementioned “digressions”, that is, some results that I have obtained in a more topological, instead of analytical, spirit.

(G, X)-structures

As said, the first *leitmotiv* of this memoir is the notion of *geometric structure* on manifolds, which dates back to Felix Klein’s *Erlangen program* from 1872. Klein, inspired by the pioneering work of Sophus Lie and himself on continuous groups of symmetries, promoted the idea that a *geometry* is essentially determined by the *group of symmetries* which act on that geometric space. More precisely, the geometric objects or quantities which can be considered well-defined in a certain ambient space, such as lengths and angles in Euclidean space or in hyperbolic space, are those which are preserved by the corresponding symmetries (the isometries for Euclidean and hyperbolic space, in the examples).

This concept has been modernised and formalised in the setting of (G, X)-*structures* by the work of many mathematicians, among whom the most prominent are Charles Ehresmann and William Thurston. The work of Ehresmann in the 1930s led to the modern definition of (G, X)-structure on a manifold M, which is the data of an atlas for M where the charts take values in the model manifold X and the transition functions are restrictions of transformations in the Lie group G acting on X.

From the 1970s, Thurston gave new life to this subject and phrased his famous Geometrization Program in the language of (G, X)-structures — by studying the so-called *eight three-dimensional geometries*, namely $\mathbb{H}^3$, $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Nil, Sol and $\widetilde{SL_2}$ — which eventually led, among many spectacular achievements, to the proof of the Poincaré Conjecture. Among those geometries, a fundamental role in his program has been played by hyperbolic three-manifolds, namely manifolds endowed with a $(\mathrm{Isom}(\mathbb{H}^3), \mathbb{H}^3)$-structure (i.e. a hyperbolic structure). Quoting Thurston [Thu82], among three-manifolds, those are "by far the most interesting, the most complex, and the most useful".

In recent years, there has been a constantly growing interest in other types of geometric structures, that leave the Riemannian setting. This memoir is organized in three parts, corresponding to different types of geometric structures. Part I will focus on (some of) the three-dimensional Thurston's geometries, starting from hyperbolic geometry, including Euclidean geometry and, in the first topological "interlude", spherical and $\mathbb{S}^2 \times \mathbb{R}$ geometries. Part II will focus on pseudo-Riemannian geometries of mixed signature and of constant sectional curvature, more precisely: for negative sectional curvature, the study of Anti-de Sitter geometry and, more generally, pseudo-hyperbolic geometry of any signature (p, q) (Anti-de Sitter being the Lorentzian case, corresponding to $q = 1$), and, for vanishing sectional curvature, of Minkowski geometry. Part III will treat certain geometric structures that even leave the pseudo-Riemannian world. These include real projective structures, that can be seen as generalizations of hyperbolic structures and represent also the right setting for the second topological "interlude" about geometric transitions, and equi-affine structures, that instead generalize both Euclidean and Minkowski geometry.

I have been tempted to call some of these (non-Riemannian) geometric structures "exotic", borrowing the name from a workshop that was held at ICERM in 2013 and had been very influential for my research. But I believe that these "exotic" structures have drawn a lot of interest from the geometric topology community during the past 10 years, and important progress is still on-going, thus making them become much less "exotic" than they were at that time.

(Generalized) asymptotic Plateau problems

Let us now turn our attention to the second *leitmotiv*, namely, as said above, the study of asymptotic Plateau problems and their generalizations. To provide some context, the classical Plateau problem asks whether there exists a minimal surface (i.e. having vanishing mean curvature) in the Euclidean three-space whose boundary is a prescribed simple closed curve. This question was first asked by Joseph Lagrange in 1760, and was solved independently by Jesse Douglas and Tibor Radó in the 1930s. Clearly the question can be asked for a (pseudo-)Riemannian manifold M instead of the Euclidean space and, when M possesses an asymptotic boundary $\partial_\infty M$, it makes sense to study the *asymptotic Plateau problem.* For example, when M is the hyperbolic space, the latter asks whether, given a subset Λ in $\partial_\infty \mathbb{H}^3$, there exists a properly embedded minimal surface in $\mathbb{H}^3$ such that its asymptotic boundary (that is, its set of accumulation points in $\partial_\infty \mathbb{H}^3$) coincides with Λ.

Hyperbolic geometry In the hyperbolic space, it is in general a difficult problem to determine which subsets of $\partial_\infty \mathbb{H}^3$ are realized as the asymptotic boundaries of a minimal surface in $\mathbb{H}^3$. A natural class to study, however, is the class of Jordan curves in $\partial_\infty \mathbb{H}^3 \cong S^2$. Indeed, this class includes the quasicircles, which in turn include the limit sets of quasi-Fuchsian groups. In this context, the lift to the universal cover of a minimal surface in a quasi-Fuchsian hyperbolic three-manifold therefore represents a solution to the asymptotic Plateau problem, where the Jordan curve Λ is the limit set of the corresponding group acting on $\mathbb{H}^3$.

The asymptotic Plateau problem in $\mathbb{H}^3$ was first studied by Anderson, who proved in [And83] the existence for any given Jordan curve Λ. Using geometric measure theory, Anderson also obtained existence results in higher dimensions, in the class of volume-minimizing k-dimensional currents in $\mathbb{H}^{n+1}$; however, the solutions may fail to be smoothly embedded hypersurfaces on a singular set of dimension $n-7$.

The solution of the asymptotic Plateau problem is, however, not unique, as was already observed by Anderson. In the context of quasi-Fuchsian manifolds, Huang and Wang in [HW15] constructed a quasi-Fuchsian group whose limit set Λ is the asymptotic boundary of an arbitrarily large number of invariant stable minimal disks. In my joint work with Ben Lowe and Zheng Huang, we constructed a more pathological example, without any group action:

Theorem ([HLS23])**.** *There exists a quasicircle in $\partial_\infty \mathbb{H}^3$ that is the asymptotic boundary of uncountably many pairwise distinct stable minimal disks.*

On the other hand, in the same article we provide a criterion that ensures uniqueness. In the context of quasi-Fuchsian manifolds, Uhlenbeck was the first to observe the importance of the so-called *almost-Fuchsian* condition, namely the condition that the principal curvatures of the minimal surface are in $(-1, 1)$, and she observed that this condition implies the uniqueness of the minimal surface in a given quasi-Fuchsian manifold. In [HLS23] we improved this result, getting rid of any group action, and we showed that if a quasicircle spans a minimal surface with principal curvatures in $[-1, 1]$, then it is unique. A similar result has been recently achieved by independent methods, and improved to an ambient space of sectional curvature pinched between $-C$ and -1, in [Bro23].

Pseudo-hyperbolic geometry The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ is the pseudo-Riemannian analogue of $\mathbb{H}^n$, in any signature (p, q). When $q = 1$, $\mathbb{H}^{p,1}$ is a Lorentzian manifold also known as the $(p+1)$-dimensional Anti-de Sitter space. Similarly to the hyperbolic space, $\mathbb{H}^{p,q}$ possesses an asymptotic boundary $\partial_\infty \mathbb{H}^{p,q}$, so that $\mathbb{H}^{p,q} \cup \partial_\infty \mathbb{H}^{p,q}$ is compact, and $\partial_\infty \mathbb{H}^{p,q}$ is naturally endowed with a conformal pseudo-Riemannian structure.

One can therefore formulate the asymptotic Plateau problem in $\mathbb{H}^{p,q}$, and ask whether one can find a *maximal* submanifold (i.e. of vanishing mean curvature, the analogue of minimal submanifolds) with a prescribed asymptotic boundary. When $q \geq 1$, there are several main differences with respect to the case of the hyperbolic space, which makes the problem more treatable, at least when one considers spacelike submanifolds of dimension p:

- First, there is a class of subsets of $\partial_\infty \mathbb{H}^{p,q}$, namely the *non-negative* $(p-1)$*-spheres*, which are the only possible candidates for the asymptotic Plateau problem, because the asymptotic boundary of any complete spacelike p-dimensional submanifold — maximal or not — is necessarily in this class.

- Second, any complete spacelike p-dimensional submanifold — again, maximal or not — can be represented as the graph of a 1-Lipschitz map from a hemisphere in S^p to S^q; when it is moreover maximal, a direct consequence of elliptic regularity is that the solution has all the possible regularity, and therefore presence of singularities in high dimensions does not occur as in the hyperbolic setting.

- Finally, the structure of the pseudo-hyperbolic space allows to apply the maximum principle to, essentially, the *timelike* distance between two maximal p-submanifolds, and to prove uniqueness of the solution, unlike the hyperbolic case.

In my joint work with Graham Smith and Jérémy Toulisse, we obtained a complete solution to the asymptotic Plateau problem in $\mathbb{H}^{p,q}$, for p-dimensional spacelike submanifolds.

Theorem ([SST23])**.** *For every non-negative* $(p-1)$*-sphere* Λ *in* $\partial_\infty \mathbb{H}^{p,q}$*, there exists a unique complete maximal* p*-submanifold of* $\mathbb{H}^{p,q}$ *with asymptotic boundary* Λ*.*

This result is an improvement of several previous works that treated the case $q = 1$ ([ABBZ12, BS10]) or $p = 2$ ([CTT19, LTW20]). We remark here that, in codimension one, the asymptotic Plateau problem can be posed for other classes of submanifolds, for instance hypersurfaces of constant mean curvature or, when the ambient space has dimension 3, surfaces of constant Gaussian curvature, called k-surfaces — the latter can also be extended to higher dimensions in several ways, for instance constant Gauss-Kronecker curvature or constant scalar curvature. These problems have been studied in the hyperbolic case (for instance in [Cos16, Cos19] for CMC hypersurfaces, [RS94, Smi22a] for k-surfaces). When the ambient space is the Anti-de Sitter space, the (generalized) asymptotic Plateau problem for CMC hypersurfaces has been solved by my PhD student Enrico Trebeschi in [Tre23], while for k-surfaces in ambient dimension three in my work [BS18] with Francesco Bonsante.

Minkowski and equiaffine geometry To conclude this discussion of (generalized) asymptotic Plateau problems, let us focus on the $(n+1)$-dimensional Minkowski space $\mathbb{R}^{n,1}$, which is the geodesically complete Lorentzian manifold of vanishing sectional curvature, hence the flat version of Anti-de Sitter space, and the Lorentzian analogue of Euclidean space. There are here two major differences with respect to the Anti-de Sitter setting: the first is that $\mathbb{R}^{n,1}$ does not quite possess a natural asymptotic boundary; the second is that there is much more rigidity for maximal hypersurfaces in $\mathbb{R}^{n,1}$. Indeed, Cheng and Yau proved in [CY76] a Bernstein type result, namely that every properly embedded (spacelike) maximal hypersurface in $\mathbb{R}^{n,1}$ is an affine hyperplane — which makes the asymptotic Plateau problem for maximal hypersurfaces essentially empty.

Nonetheless, the first difference does not represent a real issue, because one can still define a notion of "compactification" of the Minkowski space, which is known in the physics literature as the *Penrose boundary*. At least for convex hypersurfaces, the asymptotic boundary of a spacelike hypersurface can then be interpreted as the datum of a lower-semicontinuous function on $\mathbb{S}^{n-1}$ (the *null support function*), which, roughly speaking, encodes the height of asymptotic planes of lightlike type. As for the second difference, the (generalized) asymptotic Plateau problem actually becomes perfectly meaningful when one consider, as discussed above, CMC hypersurfaces (which, when properly embedded, are always convex by another result of Cheng and Yau), or k-surfaces in $\mathbb{R}^{2,1}$, which are automatically convex when $k > 0$, or hypersurfaces of constant scalar curvature, and so on.

This led me, in a collaboration with Francesco Bonsante and Peter Smillie, to a number of results: we solved the asymptotic Plateau problem for CMC hypersurfaces in [BSS23] and for k-surfaces in [BSS19], improving several previous results [CT90, Tre82, Li95, BS17]. While properly embedded CMC hypersurfaces are always complete, k-surfaces are not, and we have studied the completeness problem in [BSS22]. Hypersurfaces of constant scalar curvature are the topic of a joint work (in progress) with Pierre Bayard ([BS]).

The results about k-surfaces in $\mathbb{R}^{2,1}$ have an interesting generalization in the context of affine differential geometry. Given a convex hypersurface in affine n-space, one can construct a transverse vector field, called the *affine normal*, which is invariant under the group of volume-preserving affine transformations (called *equiaffine transformations*). Affine differential geometry is then the study of the equiaffine invariants of convex hypersurface. For example, using the affine normal, one can define the affine shape operator, and its determinant is the *affine Gauss-Kronecker curvature*. When $n = 2$, the notion of *constant affine Gaussian curvature* makes sense, and in my work [NS22] with Xin Nie we studied the asymptotic Plateau problem, formulated in a similar way as it is done for Minkowski geometry. It is worth observing that, in general, the affine normal does not coincide with the Euclidean or Minkowski normal. Nonetheless, when a convex hypersurface in $\mathbb{R}^{2,1}$ has constant Gaussian curvature in the sense of Minkowski geometry, than the affine and Minkowski normals coincide, and therefore it is also a k-surface in the affine sense.

Hence, our results actually are a generalization of the picture that comes from Minkowski geometry.

All the above results on the (generalized) asymptotic Plateau problems will be discussed in details in the text: in Section 1.3 for hyperbolic geometry; in Sections 4.1 and 4.2 for pseudo-hyperbolic geometry; in Section 5.3 for k-surfaces in Anti-de Sitter three-space; in Sections 6.2 and 6.3 for various notions of curvature (CMC, Gaussian curvature, scalar curvature) in Minkowski space; finally, in Sections 8.1 and 8.3 for k-surfaces in affine differential geometry. Let us now describe their applications to the study of geometric structures in various directions: minimal Lagrangian maps, convex cocompactness in $\mathbb{H}^3$ and $\mathbb{H}^{p,q}$, affine deformations of Hitchin representations, and the structure of certain deformation spaces of geometric structures.

Minimal Lagrangian maps in Teichmüller theory

Submanifolds with special curvature properties in geometric three-manifolds are often associated to maps between surfaces. For example, a well-known fact in differential geometry is that a hypersurface in Euclidean or Minkowski space has constant *mean* curvature if and only if its Gauss map is *harmonic*. This observation has been largely used during the 1980s to construct examples of harmonic maps into $\mathbb{H}^n$, see for example [CT88, CT90, CT93].

Closely related to harmonic maps, *minimal Lagrangian maps* also played an important role in the study of hyperbolic surfaces. They are also related to surfaces in geometric three-manifolds, in several ways. In analogy with the previous characterization of harmonic maps, one can show that a surface in Euclidean or Minkowski three-space has constant *Gaussian* curvature if and only if its Gauss map is *minimal Lagrangian*. However, minimal Lagrangian maps are also obtained from maximal surfaces in Anti-de Sitter three-space, via a construction of Gauss maps type.

Now, Labourie [Lab92] and Schoen [Sch93] independently observed that, given two closed hyperbolic surfaces (Σ_1, h_1) and (Σ_2, h_2), there exists a unique minimal Lagrangian diffeomorphism in the homotopy class of every diffeomorphism $\Sigma_1 \to \Sigma_2$. Bonsante and Schlenker in [BS10] used the asymptotic Plateau problem in Anti-de Sitter three-space (a particular case of the result of [SST23] in pseudo-hyperbolic space of the previous section) to prove the existence of a minimal Lagrangian extension $\Phi_{ML} : \mathbb{H}^2 \to \mathbb{H}^2$ of any orientation-preserving circle homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$, thus generalizing the result of Labourie and Schoen. Moreover, they showed that, if ϕ is quasisymmetric — that is, its *cross-ratio norm* $||\phi||_{cr}$ is finite, then Φ_{ML} is quasiconformal — that is, the supremum of its *quasiconformal dilatation* $K(\Phi_{ML})$ is finite. This relates minimal Lagrangian maps to universal Teichmüller space, and in particular to the problem of finding *conformally natural* extensions of quasisymmetric circle homeomorphisms, by providing a new class of geometrically rich extensions.

By studying the geometry of maximal surfaces in Anti-de Sitter space, I have obtained some results on the *optimality* of the minimal Lagrangian extension. In the direction of the *quantitative* optimality, in [Sep19b], I proved:

Theorem. *There exists a universal constant C such that, for any quasisymmetric homeomorphism ϕ of $\mathbb{R}\mathrm{P}^1$, $\ln K(\Phi_{ML}) \leq C||\phi||_{cr}$.*

Moreover, as part of the method to solve the asymptotic Plateau problem in $\mathbb{H}^{p,q}$, in [SST23] we showed that if a complete maximal p-submanifold in $\mathbb{H}^{p,q}$ has $C^{3,\alpha}$ asymptotic boundary, then the norm of its second fundamental form is in L^s (with respect to the induced volume form) for all $s > p-1$. In particular, in $\mathbb{H}^{2,q}$, the norm of the second fundamental form is in L^2, or equivalently, the maximal surface has finite *renormalized area*. As a consequence, we obtained the following result on the *qualitative* optimality of minimal Lagrangian extensions:

Theorem. *If ϕ is a $C^{3,\alpha}$ circle diffeomorphism, then the Beltrami differential of its unique quasiconformal minimal Lagrangian extension is in $L^2(\mathbb{H}^2, \mathrm{dVol}_{\mathbb{H}^2})$.*

It is known that *Weil-Petersson* circle homeomorphisms are precisely those that admit *some* quasiconformal extension with Beltrami differential in L^2, and it seems therefore natural to conjecture that the *minimal Lagrangian* extension has this property.

Convex cocompactness in $\mathbb{H}^3$: quasi-Fuchsian manifolds

In the study of geometric structures, *convex cocompactness* is a very ubiquitous and important condition, which often allows to relate dynamical properties of a group action with geometric properties of the quotient manifold.

Among hyperbolic three-manifolds, *quasi-Fuchsian manifolds* are precisely those hyperbolic manifolds homeomorphic to $S \times \mathbb{R}$, for S a closed oriented surface, obtained as the quotient of $\mathbb{H}^3$ by a convex cocompact group of isometries. Many results on these structures have been obtained via minimal surfaces, which, as mentioned above, are related to the asymptotic Plateau problem, since their lifts to $\mathbb{H}^3$ are group-invariant solutions of the asymptotic Plateau problem with asymptotic boundary the limit set of the group.

Now, the importance of curvature conditions for minimal surfaces in $\mathbb{H}^3$ is highlighted by the class of *almost-Fuchsian manifolds*, whose definition was inspired by the visionary work of Uhlenbeck [Uhl83]. Almost-Fuchsian manifolds are defined as quasi-Fuchsian manifolds containing a closed minimal surface whose principal curvatures are *small*, which means smaller than 1 in absolute value — where geometrically this bound should be though of as “less curved than a horosphere”. A remarkable consequence of this definition is the fact that the geometry of the entire three-manifold is reconstructed from the minimal surface — indeed, the manifold is smoothly foliated by the equidistant surfaces to the minimal surface, which satisfy the condition (similar to CMC, but not the same) that the sum of the inverse hyperbolic tangents of the principal curvatures is constant. Moreover, the minimal surface is actually unique.

In the context of almost-Fuchsian manifolds, there are two conjectures that are open since the beginning of the 2000s, despite the attempts of several researchers. The first conjecture goes back to Thurston and asserts that every almost-Fuchsian manifold admits a (unique) foliation by CMC surfaces — that is, roughly speaking, the foliation equidistant from the minimal surface can be improved so as to actually satisfy the CMC condition. The second has been first formulated by Ben Andrews and other authors (see [Rub05]) and asserts that if a quasi-Fuchsian manifold contains a (not necessarily minimal) closed surface with principal curvatures in $(-1, 1)$, then it is almost-Fuchsian.

In my work [CMS23] with my PhD student Diptaishik Choudhury and Filippo Mazzoli, we made a (very) partial progress towards Thurston's conjecture on CMC foliations:

Theorem. *Let S be a closed oriented surface of genus ≥ 2. There exists a neighbourhood U of the Fuchsian locus in quasi-Fuchsian space $\mathcal{QF}(S)$ such that every quasi-Fuchsian manifold in U is smoothly monotonically foliated by CMC surfaces, with mean curvature ranging in $(-1, 1)$.*

The recent preprint [HLZ23] by Huang, Li and Zhang achieved a similar result, but by different methods, relying on a modified version of the mean curvature flow. Andrew's conjecture does not seem to be close to being solved at the present time, but my joint work [EES22a] with Christian El Emam, which studied a construction of Gauss map type for almost-Fuchsian manifolds, was motivated by a possible approach to Andrew's conjecture via the geometry of the space of geodesics of $\mathbb{H}^3$.

Convex cocompactness in $\mathbb{H}^{p,q}$: Anosov representations

We now discuss some applications of the asymptotic Plateau problem in $\mathbb{H}^{p,q}$ to the study of Anosov representations in $\mathrm{PO}(p, q+1)$, which is identified with the isometry group of $\mathbb{H}^{p,q}$. Let Γ be a word hyperbolic group whose Gromov boundary is homeomorphic to S^{p-1}. Following [DGK18], a representation ρ of Γ in $\mathrm{PO}(p, q+1)$, is positive P_1-Anosov (i.e., in short, it admits a proximal limit set Λ_ρ which is a positive, hence in particular non-negative, sphere of dimension $p-1$) if and only if it has finite kernel and $\rho(\Gamma)$ acts convex cocompactly on $\mathbb{H}^{p,q}$ (i.e. there exists a convex region, which can be taken to be the convex hull of Λ_ρ, on which $\rho(\Gamma)$ acts properly discontinuosly and cocompactly). An immediate consequence of the existence and uniqueness of the solution to the asymptotic Plateau problem in $\mathbb{H}^{p,q}$ is the following:

Corollary. *If Γ is a hyperbolic group with $\partial\Gamma \cong S^{p-1}$ and $\rho : \Gamma \to \mathrm{PO}(p, q+1)$ is a positive P_1-Anosov representation, then $\rho(\Gamma)$ preserves a unique complete maximal submanifold Σ in $\mathbb{H}^{p,q}$, and the action of $\rho(\Gamma)$ on Σ is properly discontinuous and cocompact.*

This corollary has been one of the ingredients in the recent work [BK23], which exhibited new examples of "higher higher Teichmüller spaces", that is, connected components of the space of representations of Γ into a Lie group G which consist entirely of discrete and faithful representations. Indeed, using the existence of ρ-invariant complete spacelike submanifolds, [BK23] showed that the space of positive P_1-Anosov representations is a union of connected components in the space of representations of Γ in $\mathrm{PO}(p, q+1)$. A posteriori, combining [BK23] and [SST23], the above corollary is actually an "if and only if".

The corollary above also permits to make some partial progress on the question of which (torsion-free) hyperbolic groups Γ as above do admit Anosov representations.

Corollary. *If Γ is a torsion-free hyperbolic group with $\partial\Gamma \cong S^{p-1}$ that admits a positive P_1-Anosov representation $\rho : \Gamma \to \mathrm{PO}(p, q+1)$, then Γ is isomorphic to the fundamental group of a closed smooth p-dimensional manifold M whose universal cover is diffeomorphic to $\mathbb{R}^p$.*

Observe that when $p \geq 6$, from [BLW10], any torsion-free hyperbolic group Γ with Gromov boundary homeomorphic to S^{p-1} is isomorphic to the fundamental group of a closed *topological* p-dimensional manifold M whose universal cover is homeomorphic to $\mathbb{R}^p$. However, for $p = 4k$, $k \geq 2$, there are examples where M cannot be made a *smooth* manifold. The following corollary follows.

Corollary. *For every $k \geq 2$, there exists a torsion-free hyperbolic group Γ with $\partial\Gamma \cong S^{4k-1}$ which does not admit any positive P_1-Anosov representation into $\mathrm{PO}(4k, q+1)$.*

In [SST23] we have also obtained an application to the topology of the quotient of Guichard-Wienhard's domain of discontinuity Ω_ρ, introduced in [GW12], in the space of maximally isotopic subspaces of $\mathbb{R}^{p,q+1}$. Indeed, it can be proved that $\Omega_\rho/\rho(\Gamma)$ has a fiber bundle structure over the manifold M as above.

Hitchin representations and affine deformations

In the setting of higher Teichmüller theory, given a closed oriented surface S of genus at least 2, the *Hitchin component* for $\mathrm{SL}(3, \mathbb{R})$ is the connected component of the space of representations of $\pi_1(S)$ into $\mathrm{SL}(3, \mathbb{R})$ that contains the compositions of Fuchsian representations in $\mathrm{SL}(2, \mathbb{R})$ with the unique (up to conjugacy) irreducible representation of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{SL}(3, \mathbb{R})$. One of the first instances of higher Teichmüller theory have been precisely the understanding of the $\mathrm{SL}(3, \mathbb{R})$-Hitchin component in terms of geometric structures: from [Gol90, CG93], Hitchin representations are precisely the holonomies of properly convex real projective structures on S. These convex real projective structures are obtained as the quotient of the projectivization of a proper convex cone $\mathcal{C}$ in $\mathbb{R}^3$ which is invariant by the action of the Hitchin representation. Moreover, tools from affine differential geometry permit to understand the action of a Hitchin representation ρ on $\mathcal{C}$, since ρ preserves a foliation of $\mathcal{C}$ by *affine spheres*, namely convex surfaces whose affine shape operator is a multiple of the identity.

In [Lab07], Labourie studied affine deformations of Hitchin representations, namely representations of $\pi_1(S)$ in $\mathrm{SL}(3,\mathbb{R}) \rtimes \mathbb{R}^3$ (the equiaffine group of $\mathbb{R}^3$), showing that such representations ρ preserve a convex surface of constant affine Gaussian curvature k, for every $k > 0$. On the other hand, affine deformations of *Fuchsian* representations had been largely studied, since the work of Mess [Mes07], in relation with Minkowski geometry: in fact, they preserve a so-called *regular domain*, whose quotient is a maximal globally hyperbolic (MGH) flat Lorentzian manifold, and this construction led to the classification of MGH flat Lorentzian manifolds of dimension $2 + 1$. Labourie's result thus recovers the existence result for k-surfaces in MGH flat Lorentzian manifold, originally proved in [BBZ11].

In my work [NS23a] with Xin Nie, we took an approach for the study of k-surfaces that derives from the (generalized) asymptotic Plateau problem, as in our previous work [NS22]. We improved the result of Labourie by completely independent methods. First, we proved the existence of an invariant $\mathcal{C}$-regular domain $\mathcal{D}$, a notion that generalizes regular domains in Minkowski space, for any affine deformation of a Hitchin representation. Second, using Monge-Ampère equations to solve an asymptotic Plateau problem, we managed to prove existence of the corresponding (group invariant) affine k-surface in $\mathcal{D}$. When k varies in $(0, +\infty)$, these k-surface provide a foliation of $\mathcal{D}$: this is a novel result, that generalizes both the foliations of $\mathcal{C}$ by affine spheres, and the foliations of MGH manifolds by constant Gaussian curvature from [BBZ11]. Finally, our method permit to extend those results also to surfaces S with punctures, replacing Hitchin representations by the holonomies of *finite volume* convex real projective structures.

Geometry of deformation spaces

Most results described above in the context of geometric structures are of a rather *direct* type, in the sense that they provide information on a given manifold endowed with a (G, X)-structure. Another very fruitful approach to the understanding of geometric structures rests instead on the study of their *deformation space*, namely a topological space consisting of equivalence classes of (G, X)-structures on a given manifold M, where two structures are in the same equivalence class if and only if they admit an isomorphism of (G, X)-structures which is in the identity component of the diffeomorphism group of M. A fundamental example is the Teichmüller space of a closed surface S, which can be interpreted as the deformation space of hyperbolic structures on S. Generalizations of the rich structures with which the Teichmüller space is endowed, such as the Weil-Petersson metric (which is a mapping class group invariant Kähler metric), have been an important subject with many remarkable achievements obtained in various directions and for different types of (G, X)-structures ([McM00, LSY04, BT08, Lou15, Li16, KZ17]).

In the pioneering paper [Don03], Donaldson exhibited surprising applications in differential geometry of infinite-dimensional *symplectic reductions*. The most relevant one in our context is the construction of a mapping class group invariant hyperKähler metric in the deformation space of almost-Fuchsian manifolds homeomorphic to $S \times \mathbb{R}$, for S a fixed closed oriented surface. Remarkably, this construction highly relies on the presence of (unique) closed minimal surfaces, with principal curvatures in $(-1, 1)$, in every almost-Fuchsian manifold. Donaldson's hyperKähler metric extends the Weil-Petersson metric on the Fuchsian locus, which is identified with the Teichmüller space via the totally geodesic surface.

Now, in my article [MST23a] with Filippo Mazzoli and Andrea Tamburelli we performed a construction that parallels the one of Donaldson, now in the context of MGH Anti-de Sitter manifolds, which are the Lorentzian counterpart of quasi-Fuchsian hyperbolic three-manifolds. Our results demonstrate that the natural structure in this setting is instead a so-called *para-hyperKähler metric* extending the Weil-Petersson metric. This para-hyperKähler metric recovers, and unifies, essentially all the structures that have been previously known on the Anti-de Sitter deformation space — see for instance [KS07, BMS13, SS18]. My work in preparation with Christian El Emam, Filippo Mazzoli and Andrea Tamburelli investigates a phenomenon of *transition* (see below for more details) between Donaldson's hyperKähler metric and our para-hyperKähler metric.

Importantly, as in the work of Donaldson, our techniques strongly rely on the existence and uniqueness of *maximal* surfaces in MGH Anti-de Sitter manifolds. These have been a fundamental tool also in my article [BST17] with Francesco Bonsante and Andrea Tamburelli, where we studied the large-scale behaviour of the volume functional on the deformation space of MGH Anti-de Sitter manifolds.

Two digressions of topological flavour

My interest in the study of (G, X)-structures led me, over the years, to a couple of "digressions" that have a more topological flavour, instead of relying on differential geometric tools such as the theory of submanifolds.

The first digression started when I was a Master student, and led to several papers with Mattia Mecchia and, recently, with Olivero Malech, who is now a PhD student at SISSA. We have been interested in the interplay, in the context of three-dimensional closed orbifolds, between some of the eight Thurston's geometries and the topological data of Seifert fibrations for orbifolds. Indeed, it is known that closed manifolds (and orbifolds) endowed with six of the eight Thurston's geometries admit Seifert fibrations, and the geometry is determined by some *global* topological invariants, namely the base orbifold and the Euler number. In order to obtain a complete list of invariants, however, one needs the additional information of some *local* invariants for each non-generic fiber.

Now, the classification of closed orbifolds endowed with one of those six geometries *would* boil down to the topological classification of Seifert orbifolds, which is completely understood in terms of the global and local invariants, *except* for the following issue: a given geometric orbifold may admit several Seifert fibrations, pairwise topologically inequivalent. It turns out that this may only occur when the geometry is one among Euclidean, spherical or $\mathbb{S}^2 \times \mathbb{R}$. In the papers [MS20, MMS23] we gave a complete answer to what we called the *multiple fibration problem.* For Euclidean geometry, the solution had already been presented in the work [CDFHT01] of Conway, Delgado, Huson and Thurston, where it is called the *alias problem*, meaning that a given orbifold may admit several inequivalent lists of Seifert invariants, i.e. several "names" . However, the method of [CDFHT01] is computer assisted, while in [MMS23] we developed a direct approach also for the Euclidean case, not relying on computer proofs.

The second digression concerns a phenomenon of *geometric transition*, introduced in [Dan11, Dan13] and further studied in [Dan14, DGK16, CDW18] and others. This phenomenon roughly consists in a continuous deformation of geometric structures on a fixed manifold M, that at a given time abruptly "transitions" from hyperbolic structures to Anti-de Sitter structures, going through an intermediate (degenerate) geometry called *half-pipe* geometry. This intermediate geometry is actually closely related, via a "spacelike duality" to Minkowski geometry, and it had indeed been known under the name of *co-Minkowski geometry* since much before. This geometric transition is related to the study of non pseudo-Riemannian (G, X)-structures, since, as explained by Danciger, the right setting to describe the transition is within the world of real projective structures.

Now, it is in general a difficult problem to determine which manifolds (together with a singular locus, which must necessarily appear) admit such a transition of geometric structures. The initial work of Danciger provided a sufficient criterion in order to produce examples of geometric transition on closed three-dimensional manifolds (with singularities on a knot). Recently, in [Dia23b] my PhD student Farid Diaf proved that a very large class of examples (not only in the closed case, but also for finite volume, with singularities on a link) can be obtained by doubling the convex cores of quasi-Fuchsian manifolds and MGH Anti-de Sitter manifolds. In my works with Stefano Riolo [RS22b, RS22a] we constructed and studied the first *four-dimensional examples*, giving a geometric transition from hyperbolic to Anti-de Sitter structures of finite volume. These two topological "digressions" are the topics described in Chapters 3 and 7 respectively.

List of publications

I list here all my research publications since the obtention of my PhD. They are all mentioned, at least partially, in this memoir. However, I have tried to give more emphasis to the results that I consider the most remarkable, and most relevant in light of the research trajectory — that this memoir aims to present — I have followed over these years. These are listed below, in chronological order (main published articles followed by preprints).

Main publications

[BST17] Francesco Bonsante, Andrea Seppi, and Andrea Tamburelli. On the volume of Anti-de Sitter maximal globally hyperbolic three-manifolds. *Geom. Funct. Anal. (GAFA)*, 27(5):1106–1160, 2017

[BS18] Francesco Bonsante and Andrea Seppi. Area-preserving diffeomorphisms of the hyperbolic plane and K-surfaces in anti-de Sitter space. *J. Topol.*, 11(2):420–468, 2018

[BSS19] Francesco Bonsante, Andrea Seppi, and Peter Smillie. Entire surfaces of constant curvature in Minkowski 3-space. *Math. Ann.*, 374(3-4):1261–1309, 2019

[MS19] Mattia Mecchia and Andrea Seppi. Isometry groups and mapping class groups of spherical 3-orbifolds. *Math. Zeitschrift*, 292(1291-1314), 2019

[Sep19b] Andrea Seppi. Maximal surfaces in Anti-de Sitter space, width of convex hulls and quasiconformal extensions of quasisymmetric homeomorphisms. *Jour. Eur. Math. Soc. (JEMS)*, 21(6):1855–1913, 2019

[MS20] Mattia Mecchia and Andrea Seppi. On the diffeomorphism type of Seifert fibred spherical 3-orbifolds. *Rend. Istit. Mat. Univ. Trieste*, 52:1–39, 2020

[EES22a] Christian El Emam and Andrea Seppi. On the Gauss map of equivariant immersions in hyperbolic space. *Journal of Topology*, 15(1):238–301, 2022

[EES22b] Christian El Emam and Andrea Seppi. Rigidity of minimal Lagrangian diffeomorphisms between spherical cone surfaces. *Journal de l'École polytechnique – Mathématiques*, 9:581–600, 2022

[NS22] Xin Nie and Andrea Seppi. Regular domains and surfaces of constant Gaussian curvature in three-dimensional affine space. *Analysis and PDE*, 15(3):643–697, 2022

[RS22b] Stefano Riolo and Andrea Seppi. Geometric transition from hyperbolic to Anti-de Sitter geometry in dimension four. *Annali della Scuola Normale Superiore – Classe di Scienze*, XXIII(1):115–176, 2022

[RS22a] Stefano Riolo and Andrea Seppi. Character varieties of a transitioning Coxeter 4-orbifold. *Groups Geom. Dyn.*, 16(3):779–842, 2022

[NS23a] Xin Nie and Andrea Seppi. Affine deformations of quasi-divisible convex cones. *Proceedings of the London Math. Society*, DOI 10.1112/plms.12537, 2023

[CMS23] Diptaishik Choudhury, Filippo Mazzoli, and Andrea Seppi. Quasi-Fuchsian manifolds close to the Fuchsian locus are foliated by CMC surfaces. *Math. Annalen*, DOI 10.1007/s00208-023-02625-7, 2023

[BSS23] Francesco Bonsante, Andrea Seppi, and Peter Smillie. Complete CMC hypersurfaces in Minkowski $(n+1)$-space. *ArXiv 1912.05477. To appear, Communications in Analysis and Geometry, 31 pages*, 2023+

[MST23a] Filippo Mazzoli, Andrea Seppi, and Andrea Tamburelli. Para-hyperKähler geometry of the deformation space of maximal globally hyperbolic Anti-de Sitter three-manifolds. *ArXiv 2107.10363. To appear, Memoirs of the American Mathematical Society, 111 pages*, 2023+

Preprints

[BSS22] Francesco Bonsante, Andrea Seppi, and Peter Smillie. Completeness of convex entire surfaces in Minkowski 3-space. *Preprint, ArXiv 2207.10019, 53 pages*, 2022

[MMS23] Oliviero Malech, Mattia Mecchia, and Andrea Seppi. The multiple fibration problem for Seifert 3-orbifolds. *Preprint, ArXiv 2302.06443, 51 pages*, 2023

[SST23] Graham Smith, Andrea Seppi, and Jérémy Toulisse. On complete maximal submanifolds in pseudo-hyperbolic space. *Preprint, ArXiv 2305.15103, 60 pages*, 2023

[HLS23] Zheng Huang, Ben Lowe, and Andrea Seppi. Uniqueness and non-uniqueness for the asymptotic Plateau problem in hyperbolic space. *Preprint, ArXiv 2309.00599, 38 pages*, 2023

In preparation

The following drafts are the most relevant works in preparation, and their results are partially described in the text.

[BS] Pierre Bayard and Andrea Seppi. Constant scalar curvature hypersurfaces in minkowski n-space. *In preparation*

[EMST] Christian El Emam, Filippo Mazzoli, Andrea Seppi, and Andrea Tamburelli. Transitions of (para-)hyperKähler structures between almost-Fuchsian and GHMC AdS deformation spaces. *In preparation*

Other works

The works below are considered of minor importance. They are mentioned briefly in the text, much less in details than the main publications and preprints above.

[Sep18] Andrea Seppi. The flux homomorphism on closed hyperbolic surfaces and Anti-de Sitter three-dimensional geometry. *Complex Manifolds*, 4(1):183–199, 2018

[BS19] Francesco Bonsante and Andrea Seppi. Equivariant maps into Anti-de Sitter space and the symplectic geometry of $\mathbb{H}^2 \times \mathbb{H}^2$. *Trans. Amer. Math. Soc.*, 371(8):5433–5459, 2019

[Sep19c] Andrea Seppi. On the maximal dilatation of quasiconformal minimal Lagrangian extensions. *Geometriae Dedicata*, 203:25–52, 2019

[Sep19a] Andrea Seppi. Examples of geometric transition from dimension two to four. *Actes du séminaire Théorie Spectrale et Géométrie*, 35:163–196, 2017-2019

[FS20] François Fillastre and Andrea Seppi. Generalization of a formula of Wolpert for balanced geodesic graphs on closed hyperbolic surfaces. *Annales Henri Lebesgue*, 3:873–899, 2020

[FS21] François Fillastre and Andrea Seppi. A remark on one-harmonic maps from a Hadamard surface of pinched negative curvature to the hyperbolic plane. *Josai Mathematical Monographs*, 13:163–171, 2021

[DS22] Farid Diaf and Andrea Seppi. The Anti-de Sitter proof of Thurston's earthquake theorem. "In the tradition of Thurston" vol. 3 (To appear, V. Alberge, K. Ohshika and A. Papadopoulos ed.), Springer Verlag, 2022

[ST22] Andrea Seppi and Enrico Trebeschi. The half-space model for pseudo-hyperbolic space. In *Developments in Lorentzian Geometry*, pages 285–313. Springer Proceedings in Mathematics and Statistics, 2022

[NS23b] Xin Nie and Andrea Seppi. Hypersurfaces of constant Gauss-Kronecker curvature with Li-normalization in affine space. *Calculus of Variations and PDE*, 62(4):1–31, 2023

Part I

Thurston's Riemannian geometries

Chapter 1

Hyperbolic geometry

In dimension three, hyperbolic structures represent a class of (G, X)-structures of fundamental importance in geometric topology for many reasons. For instance, they played an essential role in the celebrated Geometrization Program that was formulated by William Thurston in the 1970s and led to many spectacular achievements, among which the solution of the longstanding Poincaré Conjecture.

Quasi-Fuchsian manifolds are hyperbolic three-manifolds homeomorphic to $S \times \mathbb{R}$ where S is a closed oriented surface, which are obtained as quotients of the hyperbolic space $\mathbb{H}^3$ by a group of isometries whose set of accumulation points in the sphere at infinity $\partial_\infty \mathbb{H}^3$ (called *limit set*) is a Jordan curve. They represent an extremely important and ubiquitous class of hyperbolic manifolds: indeed, the Surface Subgroup Conjecture, solved in 2012 by Kahn and Markovic ([KM12]), asserts that every *closed* hyperbolic manifold admits (many) coverings which, endowed with the pull-back metric, are quasi-Fuchsian manifolds.

1.1 Minimal and CMC surfaces: conjectures and open questions

The story goes that William Thurston, when developing the study of quasi-Fuchsian manifolds, had initially planned to use an analytic approach through minimal surfaces, as a natural three-dimensional extension of the role played by closed geodesics on surfaces. However, a number of technical issues, not least the fact that, although quasi-Fuchsian manifolds always contain a closed minimal surface, it may not be unique [And83, HW15], led him to change approach and rely on more combinatorial objects, namely *pleated surfaces.* The latter gave rise to many spectacular developments for three-dimensional hyperbolic manifolds and nearby topics ([Bon86, KS92, BO05, Ser05, Lec06, Ser06, BS12, LS14]), but are extremely hard to generalize to other situations such as in a non-metric setting, in higher dimensions, and for manifolds of variable curvature.

Elucidating the difficulties that led Thurston to change perspective, by developing the study of minimal surfaces in quasi-Fuchsian manifolds, remains a guiding challenge in three-dimensional hyperbolic geometry. Among quasi-Fuchsian manifolds, a distinguished class is represented by *almost-Fuchsian manifolds*, whose definition was inspired by the visionary work of Uhlenbeck [Uhl83], see [KS07, GHW10, HW13, Sep16, San17, HL21]. Almost-Fuchsian manifolds are defined as quasi-Fuchsian manifolds containing a closed minimal surface whose principal curvatures are *small*, which means smaller than 1 in absolute value — where geometrically this bound should be though of as "less curved than a horosphere". The most remarkable consequence of this definition is the fact that the geometry of the entire three-manifold is reconstructed from the minimal surface, and the latter is actually unique.

Let us mention here three longstanding conjectures in the context of minimal and, more generally, constant mean curvature surfaces in hyperbolic manifolds, that are still open despite the attempts of many researchers.

The first conjecture goes back to Thurston and asserts the existence of foliations by *constant mean curvature* (CMC) surfaces. Indeed, it is known that an almost-Fuchsian manifold is foliated by the equidistant surfaces from its (unique) closed minimal surface. Such equidistant foliation has the property that the sum of the inverse hyperbolic tangent of the principal curvature is constant on any leaf, and in particular the sign of the mean curvature is constant. Moreover, like almost-Fuchsian manifolds, if a quasi-Fuchsian manifolds admits a CMC foliation which is in addition monotone (meaning that the mean curvature varies monotonically among the leafs), then the minimal surface is unique. This led to the following natural conjecture:

Conjecture (Thurston)**.** *Every almost-Fuchsian manifold is uniquely (monotonically) foliated by CMC surfaces.*

The second conjecture concerns (local) foliations by minimal surfaces. Hass-Thurston conjectured that no closed hyperbolic 3-manifold admits a foliation by minimal surfaces. We state here this conjecture in a stronger form, which appears in [And83]:

Conjecture (Hass-Thurston)**.** *No hyperbolic 3-manifold admits a local 1-parameter family of closed minimal surfaces.*

It has been a folklore conjecture that no Jordan curve in $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$ asymptotically bounds a 1-parameter family of minimal surfaces. The full extent of these conjectures remain as major open questions in the field.

What has been proven up to this point tends to support Hass-Thurston conjecture, and the above version for Jordan curves. Anderson [And83] proved it for quasi-Fuchsian hyperbolic 3-manifolds. Huang-Wang [HW19] and Hass [Has15] made progress on the Hass-Thurston conjecture for certain fibered closed hyperbolic 3-manifolds containing short geodesics; Wolf-Wu ([WW20]) ruled out so-called geometric local 1-parameter families of closed minimal surfaces; it follows from the work of Alexakis-Mazzeo [AM10] that a generic $C^{3,\alpha}$ simple closed curve in the boundary at infinity of $\mathbb{H}^3$ bounds only finitely many minimal surfaces of any given finite genus; Coskunuzer proved that generic simple closed curves in $\partial_\infty \mathbb{H}^3$ bound a unique area-minimizing surface [Cos11].

Finally, the third conjecture has been first formulated by Ben Andrews and other authors in the beginning of 2000s (see [Rub05]) and concerns an important *criterion* for a quasi-Fuchsian manifold to be almost-Fuchsian:

Conjecture (Andrews)**.** *If a quasi-Fuchsian manifold contains a closed surface with principal curvatures in* $(-1, 1)$*, then it is almost-Fuchsian.*

In fact, in the literature the name *nearly-Fuchsian* has been adopted for the class of quasi-Fuchsian manifolds containing a closed surface with principal curvatures in $(-1, 1)$. The class of nearly-Fuchsian manifolds is clearly larger than almost-Fuchsian manifolds, and Andrew's conjecture asserts that the two classes are equivalent.

In the following sections, we present (very) partial progress made towards these three conjectures, in the works [CMS23], [HLS23] and [EES22a] respectively.

1.2 CMC foliations

Let us start by discussing progress on the conjecture by Thurston on the existence of CMC foliations for almost-Fuchsian manifolds. Very few is known in this direction. First, clearly any Fuchsian manifold (namely, that is obtained as the quotient of $\mathbb{H}^3$ by a Fuchsian group in $\mathrm{PSL}(2, \mathbb{R}) < \mathrm{PSL}(2, \mathbb{C})$) admits a CMC foliation, which is given by the surfaces equidistant from the totally geodesic surface; this is the only known explicit example. By a special case of the results of Mazzeo and Pacard in [MP11], each end of any quasi-Fuchsian manifold (namely, each connected component of the complement of a compact set homeomorphic to $\Sigma \times I$ for I a closed interval) is smoothly monotonically foliated by CMC surfaces, with mean curvature ranging in $(-1, -1 + \epsilon)$ and $(1 - \epsilon, 1)$. This result has been reproved by Quinn in [Qui20], using an alternative approach. Moreover, the recent work of Guaraco-Lima-Pallete [GPL21] showed that every quasi-Fuchsian manifold admits a global foliation in which every leaf has constant sign of the mean curvature, meaning that it is either minimal or the mean curvature is nowhere vanishing on the entire leaf.

We also remark that existence results for CMC surfaces in the hyperbolic three-space with a given boundary curve at infinity, and in quasi-Fuchsian manifolds, have been obtained in [Cos16, Cos17, Cos19]. In [CMS23], together with my PhD student Diptaishik Choudhury and with Filippo Mazzoli, we proved the following result, which is a partial progress in the direction of Thurston's conjecture. We denote by $\mathcal{QF}(\Sigma)$ the space of quasi-Fuchsian manifolds homeomorphic to $S \times \mathbb{R}$, for S a given closed orientable surface of genus ≥ 2.

Theorem 1.2.1. *Let* Σ *be a closed oriented surface of genus* ≥ 2*. Then there exists a neighbourhood* U *of the Fuchsian locus in quasi-Fuchsian space* $\mathcal{QF}(\Sigma)$ *such that every* *quasi-Fuchsian manifold in U is smoothly monotonically foliated by CMC surfaces, with mean curvature ranging in* $(-1,1)$.

The monotone CMC foliation of a quasi-Fuchsian manifold $M \cong \Sigma \times \mathbb{R}$, when it exists, is automatically unique by a standard application of the geometric maximum principle. More precisely, the leaf of the foliation with mean curvature H is the unique closed surface homotopic to $\Sigma \times \{*\}$ in M having mean curvature identically equal to H.

Observe that, if a quasi-Fuchsian manifold admits a monotone CMC foliation, then the mean curvature necessarily ranges in $(-1,1)$. Indeed, any leaf of the foliation must necessarily have mean curvature in $(-1,1)$, see [Cos06, Lemma 2.2]. Moreover, by the aforementioned result of Mazzeo-Pacard, the mean curvature converges to -1 and 1 as the foliation approaches the ends.

The methods of the proof of Theorem 1.2.1, which are outlined below, also provide a direct proof of the *existence* of closed embedded CMC surfaces of mean curvature $H \in (-1,1)$ in the quasi-Fuchsian manifolds M within the neighbourhood U. Our proof is independent of previous result in the literature, and does not rely on geometric measure theory techniques.

The rough idea of the proof of Theorem 1.2.1 is to "combine" foliations of the ends, which have been provided in the works of Mazzeo-Pacard and Quinn for every quasi-Fuchsian manifold, with foliations of the compact part that we obtain by a "deformation" from Fuchsian manifolds. For the foliations of the ends, we adapted the proof given by Quinn in [Qui20], which relies on the Epstein map construction ([Eps84, Dum17]), that associates to a conformal metric defined in (a subset of) the boundary at infinity of $\mathbb{H}^3$ an immersed surface in $\mathbb{H}^3$ by "envelope of horospheres". One can then translate the condition of constant mean curvature into a PDE on the conformal factor, to which we apply an implicit function theorem method in an infinite-dimensional setting. The fact that the obtained solutions provide a smooth monotone foliation of the complement of a large compact set in the quasi-Fuchsian manifold M follows from another application of the implicit function theorem. The main difference with respect to Quinn's proof is that we refined his method in order to achieve the existence of monotone foliations by CMC surfaces of mean curvature $(-1,-1+\epsilon) \cup (1-\epsilon,1)$ for *any* quasi-Fuchsian manifold in a neighbourhood U_M of a given $M \in \mathcal{QF}(\Sigma)$, where the constant ϵ is *uniform* over U_M.

For the compact part, we again obtained the existence of CMC surfaces, for $H \in (-1,1)$, with an implicit function theorem method in infinite-dimensional spaces, using the Epstein construction. In this case, however, the initial solution to which we apply the implicit function theorem is not "at infinity"; it is instead the umbilical CMC surface in a Fuchsian manifold. In other words, we "deform" CMC surfaces in a Fuchsian manifold M' to nearby quasi-Fuchsian manifolds in a neighbourhood $U_{M'}$. Similarly as above, the main technical difficulty is to have a uniform control of the constants, which must not depend on the quasi-Fuchsian manifold as long as we remain in the neighbourhood $U_{M'}$.

The proof of Theorem 1.2.1 is then concluded by showing that these surfaces "patch" together to a global smooth monotone foliation. It is worth mentioning that the recent work [HLZ23] by Huang, Li and Zhang showed the convergence of modified version of the mean curvature flow under some hypothesis on the initial surface, thus obtaining a CMC foliation in certain almost-Fuchsian manifolds which are "close" to the Fuchsian locus in a suitable, quantitative sense.

1.3 Uniqueness and non-uniqueness of minimal surfaces

The classical "asymptotic Plateau problem" asks, given a Jordan curve Λ on $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$, how to count the number of (properly embedded) minimal surfaces Σ in $\mathbb{H}^3$, if any, that are asymptotic to Λ, in the sense that the closure of Σ in $\mathbb{S}^2_\infty \cup \mathbb{H}^3$ is equal to $\Lambda \cup \Sigma$. The existence of minimal disk solutions to the asymptotic Plateau problem was obtained by Anderson ([And83]). Using geometric measure theory, Anderson also obtained existence results in higher dimensions: he proved the existence of volume-minimizing k-dimensional currents in $\mathbb{H}^{n+1}$ bounded by any given closed, embedded $(k-1)$-dimensional submanifold of $\partial_\infty \mathbb{H}^{n+1}$. While for $n = 2$ Anderson showed that every Jordan curve Λ in $\partial_\infty \mathbb{H}^3$ bounds a smoothly embedded minimal disk, in higher dimension and codimension the questions of regularity, topological type, and uniqueness of such minimizing currents present hard and subtle problems (see, for example, [Cos14]). Indeed, the solution lies in the class of locally integral n-currents, and may fail to be smoothly embedded hypersurfaces on a singular set of dimension $n - 7$.

The uniqueness does not hold in general: as shown in [And83, HW15], taking advantage of group actions, one can construct a Jordan curve Λ in $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$ which is the limit set of some quasi-Fuchsian group such that Λ spans multiple minimal disks (even an arbitrarily large, but finite, number). Anderson ([And83]) even constructed a curve Λ which spans infinitely many minimal surfaces (the surfaces he constructs have positive genus). On the other hand, when Λ is a round circle, the unique minimal surface it spans is a totally geodesic disk. To look for unique solutions, it is therefore natural to consider the class of minimal surfaces that are "close" to totally geodesic, for which Λ is "close" to a round circle. Related questions with conditions on natural invariants of Λ were studied in [Sep16] (for the quasi-conformal constant of Λ), and [HW13, San18] (for the Hausdorff dimension of Λ). Some properness questions for the asymptotic Plateau problem solutions for various classes of curves were addressed, for example, in [GS00, AM10].

1.3.1 Strong non-uniqueness

While compatible with the folklore conjecture that no Jordan curve is the asymptotic boundary of a 1-parameter family of minimal surfaces, in [HLS23] we proved a result in the other direction. One might be tempted to strengthen the folklore conjecture to the statement that any Jordan curve in $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$ bounds at most countably many minimal surfaces. We show that this stronger statement is false:

Theorem 1.3.1. *There exists a quasicircle in* $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$ *spanning uncountably many pairwise distinct stable minimal disks.*

Let us emphasize some important features of the construction of this extreme curve Λ. In ([And83]), Anderson constructed a Jordan curve which is the limit set of a quasi-Fuchsian groups (hence continuous but almost nowhere differentiable) such that it spans infinitely many minimal surfaces, one of which is a minimal disk. In [HW15], for each integer $N > 1$, also using the limit set of a quasi-Fuchsian group, an extreme curve spanning at least 2^N distinct minimal disks, invariant under the quasi-Fuchsian group, was constructed. However, Anderson ([And83]) has shown that any quasi-Fuchsian manifolds only admits finitely many least area closed minimal surfaces diffeomorphic to the fiber, which poses a possible limitation on how much one can improve the aforementioned constructions to find infinitely many minimal disks if one insists on using the limit set of some quasi-Fuchsian group as the curve at infinity. The starting point of our construction is similar to the ideas in [HW15], but the Jordan curve is constructed in such a way to allow an improvement of the argument, leading to $2^{\mathbb{N}}$ pairwise distinct minimal disk. Moreover, since this Jordan curve is not invariant under any quasi-Fuchsian group, we must adopt a different approach in order to produce the minimal disks, namely, in a spirit similar to the proofs of Theorems 1.3.2 and 1.3.3 below, we take the limit of a sequence of solutions of the finite Plateau problems inside $\mathbb{H}^3$.

1.3.2 Characterizing uniqueness

Let us now turn to conditions that ensure the uniqueness of the solutions of the asymptotic Plateau problem. Our first theorem shows that it suffices to check uniqueness in the class of *stable minimal disks.*

Theorem 1.3.2. *Let* Λ *be a Jordan curve on* $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$*. Then* Λ *spans a unique minimal surface if and only if it spans a unique stable minimal disk.*

A statement similar to Theorem 1.3.2, but in the context of the *finite* Plateau problem, was proved in [MY19]. When Λ is invariant under the action of a Kleinian group (i.e. a discrete subgroup of isometries of $\mathbb{H}^3$), we can prove a stronger statement.

Theorem 1.3.3. *Let* Λ *be a Jordan curve on* $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$*, and let* Γ *be any Kleinian group preserving* Λ*. Then* Λ *spans a unique minimal surface if and only if it spans a unique* Γ*-invariant stable minimal disk.*

Theorem 1.3.3 applies, for instance, to Γ a quasi-Fuchsian group and Λ its limit set, but also more generally to any discrete group of isometries preserving a Jordan curve Λ.

The main idea in the proof of Theorem 1.3.2 is an adaptation of an argument in [And83, Theorem 3.1]: we show that if there is a minimal surface, which is not stable or is not topologically a disk, with asymptotic boundary Λ, then we can construct two distinct — actually, disjoint — stable minimal disks with the same asymptotic boundary Λ. The proof of Theorem 1.3.3 then relies on a further improvement of the arguments of [And83, Theorem 3.1], showing that if there is a non-invariant stable minimal disk, then we can construct two disjoint invariant stable minimal disks.

1.3.3 Uniqueness criteria via curvature conditions

Next, we turn our attention to sufficient conditions for uniqueness. For an immersed hypersurface in $\mathbb{H}^{n+1}$, or more generally in a hyperbolic $(n+1)$-manifold, we say it has *strongly small curvature* if its principal curvaturess $\{\lambda_i\}$ satisfy that

$$|\lambda_i| \leq 1-\epsilon,\ i=1,\ldots,n, \quad \text{for some small } \ \epsilon>0. \tag{1.1}$$

Similarly we say it has *small curvature* if $|\lambda_i|<1$, and it has *weakly small curvature* if $|\lambda_i|\leq 1$. This definition has some immediate consequences: for instance, a complete immersion of weakly small curvature is in fact a properly embedded topological disk ([Eps84, Eps86].

Surfaces of small curvature are very special in Teichmüller theory: Thurston observed that a closed surface of small principal curvatures in a complete hyperbolic three-manifold is incompressible ([Thu86, Lei06]); they are abundant in closed hyperbolic three-manifolds ([KM12]); many results have been extended to the study of complete noncompact hyperbolic three-manifold of finite volume ([Rub05, CF19, KW21]).

It is known ([Uhl83]) that any almost-Fuchsian manifolds admits a unique closed minimal surface — in other words, identifying the almost-Fuchsian manifold with a quotient $\mathbb{H}^3/\Gamma$, the limit set Λ of the group Γ bounds a unique Γ-invariant minimal disk asymptotic to Λ. Inspired by this fact, we prove (Corollary 1.3.5 below) that if a Jordan curve Λ (not necessarily group equivariant) spans a minimal disk Σ of strongly small principal curvatures in $\mathbb{H}^3$, then Σ is the unique minimal surface asymptotic to Λ. Our results, however, are more general. The main result we proved in this direction is the following:

Theorem 1.3.4. *Let Λ be a topologically embedded $(n-1)$-sphere on $\mathbb{S}^n_\infty=\partial_\infty\mathbb{H}^{n+1}$ of finite width, and let Σ be a minimal hypersurface in $\mathbb{H}^{n+1}$ of weakly small principal curvatures asymptotic to Λ. Then Σ is the unique minimal hypersurface in $\mathbb{H}^{n+1}$ asymptotic to Λ. Moreover, Σ is area-minimizing.*

Let us explain the terminology of the statement. First, recall that a hypersurface Σ is *area-minimizing* if any compact codimension-zero submanifold with boundary has smaller area than any rectifiable hypersurface with the same boundary in the ambient space. This implies that Σ is a stable minimal hypersurface. Second, the *width* of a Jordan curve Λ in $\mathbb{S}^2_\infty=\partial_\infty\mathbb{H}^3$ has been introduced in [BDMS21] — and the definition is immediately extended to higher dimensions — as the supremum over all points in the convex hull of Λ of the sum of the distances from each boundary component of the convex hull.

A similar result has been recently achieved (with an independent approach) by Bronstein in [Bro23] and improved to an ambient space of sectional curvature pinched between $-C$ and -1 and to minimal submanifolds of any codimension, but requiring the small curvature condition (instead of the weakly small curvature condition).

Theorem 1.3.4 has several corollaries. Firstly, if a properly embedded hypersurface Σ has strongly small curvatures, then its asymptotic boundary has finite width. Hence we obtained:

Corollary 1.3.5. *Let Λ be a topologically embedded $(n-1)$-sphere on $\mathbb{S}^n_\infty = \partial_\infty \mathbb{H}^{n+1}$, and let Σ be a minimal hypersurface in $\mathbb{H}^{n+1}$ of strongly small principal curvatures asymptotic to Λ. Then Σ is the unique minimal hypersurface in $\mathbb{H}^{n+1}$ asymptotic to Λ. Moreover, Σ is area-minimizing.*

Secondly, in dimension $n = 2$, quasicircles are an important class of Jordan curves, which are known to have finite width. (On the other hand, there exists Jordan curves of finite width which are not quasicircles, as constructed in [BDMS21].) We thus obtain immediately the following.

Corollary 1.3.6. *Let Λ be a quasicircle on $\mathbb{S}^2_\infty = \partial_\infty \mathbb{H}^3$, and let Σ be a minimal surface in $\mathbb{H}^3$ of weakly small principal curvatures asymptotic to Λ. Then Σ is the unique minimal surface in $\mathbb{H}^3$ asymptotic to Λ. Moreover, Σ is area-minimizing.*

We remark that the setting of Theorems 1.3.2 and 1.3.3 is more general than Theorem 1.3.4 and Corollary 1.3.6. Indeed, it follows from [HL21, Theorem 5.2] that there are examples of quasi-Fuchsian groups Γ whose limit set Λ bounds a unique Γ-invariant stable minimal disk Σ (hence, by Theorem 1.3.3, a unique minimal surface), but Σ does *not* have weakly small curvature.

1.3.4 Quasiconformal constant

Corollary 1.3.6 led to an improvement of the curvature estimates that I had obtained in [Sep16], during my PhD thesis. Indeed, [Sep16, Theorem A] showed that there exist universal constants $C > 0$ and $K_0 > 1$ such that any stable minimal disk in $\mathbb{H}^3$ with asymptotic boundary a K-quasicircle, for $K < K_0$, has principal curvatures bounded in absolute value by $C \log K$. This result has been recently applied in several directions, see [Bis21, Low21, CMN20, KMS23].

The proof, however, relied on the application of compactness for minimal surfaces, and therefore requires stability. However, when K is sufficiently small, the principal curvatures of the area-minimizing (hence stable) disk whose existence is guaranteed by [And83] are less than $1 - \epsilon$ in absolute value, and therefore, as a consequence of Corollary 1.3.6, Σ is the unique minimal surface. Up to taking a smaller constant K_0, one can therefore remove the stability assumption:

Corollary 1.3.7. *There exist universal constants $C > 0$ and $K_0 > 1$ such that the principal curvatures λ_i of any minimal surface Σ in $\mathbb{H}^3$ with asymptotic boundary a K-quasicircle with $K \leq K_0$ satisfy*

$$|\lambda_i| \leq C \log K, \ i = 1, 2.$$

In particular, this also improves [Sep16, Theorem B] (up to choosing a smaller constant) by removing the stability assumption.

Corollary 1.3.8. *There exists a universal constant $K_0' > 1$ such that any K-quasicircle with $K \leq K_0'$ is the asymptotic boundary of a unique minimal surface, which is an area-minimizing disk of strongly small curvature.*

1.4 Gauss map in hyperbolic geometry

The purpose of the work [EES22a] consists in presenting a framework in which it appears natural to study the Andrew's conjecture on the equivalence between the classes of almost-Fuchsian and nearly-Fuchsian manifolds. The idea is to study immersions of (hyper)surfaces in $\mathbb{H}^3$ (and more in general, in the hyperbolic space $\mathbb{H}^{n+1}$ of any dimension), in relation with the geometry of their Gauss maps in the space of oriented geodesics of $\mathbb{H}^{n+1}$. We considered in particular *nearly-Fuchsian* immersions, namely with principal curvatures in $(-1, 1)$, and immersions of $\widetilde{M}$ which are equivariant with respect to some group representation $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$, for M a n-manifold.

1.4.1 Spaces of geodesics

In the groundbreaking paper [Hit82], Hitchin observed the existence of a natural complex structure on the space of oriented geodesics in Euclidean three-space. A large interest has then grown on the geometry of the space of oriented (maximal unparametrized) geodesics of Euclidean space of any dimension (see [GK05, Sal05, Sal09, GG14]) and of several other Riemannian and pseudo-Riemannian manifolds (see [AGK11, Anc14, Bar18]). Here we are interested in the case of hyperbolic n-space $\mathbb{H}^n$, whose space of oriented geodesics is denoted here by $\mathcal{G}(\mathbb{H}^n)$. The geometry of $\mathcal{G}(\mathbb{H}^n)$ has been addressed in [Sal07] and, for $n = 3$, in [GG10a, GG10b, Geo12, GS15]. The most relevant geometric structure on $\mathcal{G}(\mathbb{H}^n)$ in our context is a natural *para-Kähler structure* $(\mathbb{G}, \mathbb{J}, \Omega)$ first introduced in [AGK11, Anc14]. A particularly relevant feature of such para-Kähler structure is the fact that the Gauss map of an oriented immersion $\sigma : M \to \mathbb{H}^n$, which is defined as the map that associates to a point of M the orthogonal geodesic of σ endowed with the compatible orientation, is a Lagrangian immersion in $\mathcal{G}(\mathbb{H}^n)$. As a consequence of the geometry of the hyperbolic space $\mathbb{H}^n$, an oriented geodesic in $\mathbb{H}^n$ is characterized, up to orientation preserving reparametrization, by the ordered pair of its "endpoints" in the visual boundary $\partial\mathbb{H}^n$: this gives an identification $\mathcal{G}(\mathbb{H}^n) \cong \partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \Delta$. Under this identification the

Gauss map G_σ of an immersion $\sigma : M \to \mathbb{H}^n$ corresponds to a pair of *hyperbolic Gauss maps* $G^\pm_\sigma : M \to \partial\mathbb{H}^n$.

As already mentioned before, the works of Uhlenbeck [Uhl83] and Epstein [Eps84, Eps86, Eps87] highlighted the relevance of hypersurfaces satisfying the geometric condition for which principal curvatures are everywhere different from ± 1, sometimes called *horospherically convexity*: this is the condition that ensures that the hyperbolic Gauss maps $G^\pm_\sigma$ are locally invertible. Epstein' description "from infinity" of horospherically convex hypersurfaces as envelopes of horospheres has been later developed by many authors by means of analytic techniques, see for instance [Sch02, IdCR06, KS08], and permitted to obtain remarkable classification results often under the assumption that the principal curvatures are larger than 1 in absolute value ([Cur89, AC90, AC93, EGM09, BEQ10, BEQ15, BQZ17, BMQ18]).

1.4.2 Integrability of immersions in $\mathcal{G}(\mathbb{H}^n)$

The first goal of [EES22a] was to discuss when an immersion $G\colon M^n \to \mathcal{G}(\mathbb{H}^{n+1})$ is *integrable*, namely when it is the Gauss map of an immersion $M \to \mathbb{H}^{n+1}$, in terms of the geometry of $\mathcal{G}(\mathbb{H}^{n+1})$. We distinguished three types of integrability conditions, which we list from the weakest to the strongest:

- An immersion $G\colon M \to \mathcal{G}(\mathbb{H}^{n+1})$ is *locally integrable* if for all $p \in M$ there exists a neighbourhood U of p such that $G|_U$ is the Gauss map of an immersion $U \to \mathbb{H}^{n+1}$;
- An immersion $G\colon M \to \mathcal{G}(\mathbb{H}^{n+1})$ is *globally integrable* if it is the Gauss map of an immersion $M \to \mathbb{H}^{n+1}$;
- Given a representation $\rho\colon \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$, a ρ-equivariant immersion $G\colon \widetilde{M} \to \mathcal{G}(\mathbb{H}^{n+1})$ is *ρ-integrable* if it is the Gauss map of a ρ-equivariant immersion $\widetilde{M} \to \mathbb{H}^{n+1}$.

Since the definition of Gauss map requires to fix an orientation on M, the above three definitions of integrability have to be interpreted as: "there exists an orientation on U (in the first case) or M (in the other two) such that G is the Gauss map of an immersion in $\mathbb{H}^{n+1}$ with respect to that orientation". We restrict to immersions σ with small principal curvatures, which is equivalent to the condition that the Gauss map G_σ is Riemannian, meaning that the pull-back by G_σ of the ambient pseudo-Riemannian metric of $\mathcal{G}(\mathbb{H}^{n+1})$ is positive definite, hence a Riemannian metric.

Local integrability As it was essentially observed in [Anc14, Theorem 2.10], local integrability admits a very simple characterization in terms of the symplectic geometry of $\mathcal{G}(\mathbb{H}^{n+1})$.

Theorem 1.4.1. *Let M^n be a manifold and $G\colon M \to \mathcal{G}(\mathbb{H}^{n+1})$ be an immersion. Then G is locally integrable if and only if it is Lagrangian.*

Theorem 1.4.1 can be reinterpreted in a different set-up with respect to that of [Anc14], as follows. Let us denote by $T^1\mathbb{H}^{n+1}$ the unit tangent bundle of $\mathbb{H}^{n+1}$ and by

$$\mathrm{p}\colon T^1\mathbb{H}^{n+1} \to \mathcal{G}(\mathbb{H}^{n+1}) \ , \tag{1.2}$$

the map such that $\mathrm{p}(x, v)$ is the oriented geodesic of $\mathbb{H}^{n+1}$ tangent to v at x. Then, if G is Lagrangian, we proved that one can locally construct maps $\zeta : U \to T^1\mathbb{H}^{n+1}$ (for U a simply connected open set) such that $\mathrm{p} \circ \zeta = G$. Up to restricting the domain again, one can find such a ζ so that it projects to an immersion σ in $\mathbb{H}^{n+1}$, and the Gauss map of σ is G by construction. The next results provide characterizations of global integrability and ρ-integrability under the assumption of small principal curvatures.

Global integrability The problem of global integrability is in general more subtle than local integrability. As a matter of fact, one can construct an example of a locally integrable immersion $G : (-T, T) \to \mathcal{G}(\mathbb{H}^2)$ that is not globally integrable. By taking a cylinder on this curve, one easily sees that the same phenomenon occurs in any dimension. In such example, $M = (-T, T)$ (or the product $(-T, T)\times\mathbb{R}^{n-1}$ for $n > 2$) is simply connected: the key point is that one can find globally defined maps $\zeta : M \to T^1\mathbb{H}^{n+1}$ such that $G = \mathrm{p}\circ\zeta$, but no such ζ projects to an immersion in $\mathbb{H}^{n+1}$.

Nevertheless, we show that this issue does not occur for Riemannian immersions G. In this case any immersion σ whose Gauss map is G (if it exists) necessarily has small principal curvatures. We will always restrict to this setting hereafter. In summary, we obtained the following characterization of global integrability for M simply connected:

Theorem 1.4.2. *Let M^n be a simply connected manifold and $G\colon M \to \mathcal{G}(\mathbb{H}^{n+1})$ be a Riemannian immersion. Then G is globally integrable if and only if it is Lagrangian.*

We give a characterization of global integrability for $\pi_1(M) \neq \{1\}$ in Corollary 1.4.5, which is a direct consequence of our first characterization of ρ-integrability (Theorem 1.4.4). Before that, we remark that if a Riemannian and Lagrangian immersion $G\colon M \to \mathcal{G}(\mathbb{H}^{n+1})$ is also complete (i.e. has complete first fundamental form), then M is necessarily simply connected:

Theorem 1.4.3. *Let M^n be a manifold. If $G\colon M \to \mathcal{G}(\mathbb{H}^{n+1})$ is a complete Riemannian and Lagrangian immersion, then M is diffeomorphic to $\mathbb{R}^n$ and G is the Gauss map of a proper embedding $\sigma : M \to \mathbb{H}^{n+1}$.*

ρ-integrability Let us first observe that the problem of ρ-integrability presents some additional difficulties than global integrability. Assume $G\colon \widetilde{M} \to \mathcal{G}(\mathbb{H}^{n+1})$ is a Lagrangian, Riemannian and ρ-equivariant immersion for some representation $\rho\colon \pi_1(M^n) \to \mathrm{Isom}(\mathbb{H}^{n+1})$. Then, by Theorem 1.4.2, there exists $\sigma\colon \widetilde{M} \to \mathbb{H}^{n+1}$ with Gauss map G, but the main issue is that such a σ will not be ρ-equivariant in general.

Nevertheless, ρ-integrability of Riemannian immersions into $\mathcal{G}(\mathbb{H}^{n+1})$ can still be characterized in terms of their extrinsic geometry. Let $\overline{\mathrm{H}}$ be the mean curvature vector of G, defined as the trace of the second fundamental form, and Ω the symplectic form of $\mathcal{G}(\mathbb{H}^{n+1})$. Since G is ρ-equivariant, the 1-form $G^*(\Omega(\overline{\mathrm{H}}, \cdot))$ on $\widetilde{M}$ is invariant under the action of $\pi_1(M)$, so it descends to a 1-form on M. One can prove that such 1-form on M is closed: we will denote its cohomology class in $H^1_{dR}(M, \mathbb{R})$ with μ_G and we will call it the *Maslov class* of G, in accordance with some related interpretations of the Maslov class in other geometric contexts (see among others [Mor81, Oh94, Ars00, Smo02]). The Maslov class encodes the existence of equivariantly integrating immersions, in the sense stated in the following theorem.

Theorem 1.4.4. *Let M^n be an orientable manifold, $\rho\colon \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$ be a representation and $G\colon \widetilde{M} \to \mathcal{G}(\mathbb{H}^{n+1})$ be a ρ-equivariant Riemannian and Lagrangian immersion. Then G is ρ-integrable if and only if the Maslov class μ_G vanishes.*

Applying Theorem 1.4.4 to a trivial representation, we immediately obtain a characterization of global integrability for Riemannian immersions, thus extending Theorem 1.4.2 to the case $\pi_1(M) \neq \{1\}$.

Corollary 1.4.5. *Let M^n be an orientable manifold and $G\colon M \to \mathcal{G}(\mathbb{H}^{n+1})$ be a Riemannian and Lagrangian immersion. Then G is globally integrable if and only if its Maslov class μ_G vanishes.*

1.4.3 Nearly-Fuchsian representations

Let us now focus on the case of M a closed oriented manifold. We say that a representation $\rho\colon \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$ is *nearly-Fuchsian* if there exists a ρ-equivariant immersion $\sigma\colon \widetilde{M} \to \mathbb{H}^{n+1}$ with small principal curvatures. We showed that the action of a nearly-Fuchsian representation on $\mathbb{H}^{n+1}$ is free, properly discontinuously and convex cocompact; the quotient of $\mathbb{H}^{n+1}$ by $\rho(\pi_1(M))$ is called (extending the classical terminology to any dimensions) *nearly-Fuchsian manifold.*

Moreover, the action of $\rho(\pi_1(M))$ extends to a free and properly discontinuous action on the complement of a topological $(n-1)$-sphere Λ_ρ (the *limit set* of ρ) in the visual boundary $\partial\mathbb{H}^{n+1}$. Such complement is the disjoint union of two topological n-discs which we denote by Ω_+ and Ω_-. It follows that there exists a maximal open region of $\mathcal{G}(\mathbb{H}^{n+1})$ over which a nearly-Fuchsian representation ρ acts freely and properly discontinuously. This region is defined as the subset of $\mathcal{G}(\mathbb{H}^{n+1})$ consisting of oriented geodesics having either final endpoint in Ω_+ or initial endpoint in Ω_-. The quotient of this open region via the action of ρ, that we denote with $\mathcal{G}_\rho$, inherits a para-Kähler structure.

Let us first state a uniqueness result concerning nearly-Fuchsian representations. A consequence of Theorem 1.4.4 and the definition of Maslov class is that if G is a ρ-equivariant, Riemannian and Lagrangian immersion which is furthermore *minimal*, i.e.

with $\overline{\mathrm{H}} = 0$, then it is ρ-integrable. Together with an application of a maximum principle in the corresponding nearly-Fuchsian manifold, we proved:

Corollary 1.4.6. *Given a closed orientable manifold M^n and a representation $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$, there exists at most one ρ-equivariant Riemannian minimal Lagrangian immersion $G : \widetilde{M} \to \mathcal{G}(\mathbb{H}^{n+1})$ up to reparametrization. If such a G exists, then ρ is nearly-Fuchsian and G induces a minimal Lagrangian embedding of M in $\mathcal{G}_\rho$.*

In fact, for any ρ-equivariant immersion $\sigma : \widetilde{M} \to \mathbb{H}^{n+1}$ with small principal curvatures, the hyperbolic Gauss maps $G^{\pm}_\sigma$ are equivariant diffeomorphisms between $\widetilde{M}$ and $\Omega_\pm$. Hence up to changing the orientation of M, which corresponds to swapping the two factors $\partial\mathbb{H}^{n+1}$ in the identification $\mathcal{G}(\mathbb{H}^{n+1}) \cong \partial\mathbb{H}^{n+1} \times \partial\mathbb{H}^{n+1} \setminus \Delta$, the Gauss map of σ takes values in the maximal open region defined above, and induces an embedding of M in $\mathcal{G}_\rho$.

This observation permits to deal (in the cocompact case) with embeddings in $\mathcal{G}_\rho$ instead of ρ-equivariant embeddings in $\mathcal{G}(\mathbb{H}^{n+1})$. In analogy with the definition of ρ-integrability defined above, we will say that a n-dimensional submanifold $\mathcal{L} \subset \mathcal{G}_\rho$ is *ρ-integrable* if it is the image in the quotient of a ρ-integrable embedding in $\mathcal{G}(\mathbb{H}^{n+1})$. Clearly such $\mathcal{L}$ is necessarily Lagrangian by Theorem 1.4.1. We are now ready to state our second characterization result for ρ-integrability .

Theorem 1.4.7. *Let M be a closed orientable n-manifold, $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$ be a nearly-Fuchsian representation and $\mathcal{L} \subset \mathcal{G}_\rho$ a Riemannian ρ-integrable submanifold. Then a Riemannian submanifold $\mathcal{L}'$ is ρ-integrable if and only if there exists $\Phi \in \mathrm{Ham}_c(\mathcal{G}_\rho, \Omega)$ such that $\Phi(\mathcal{L}) = \mathcal{L}'$.*

In Theorem 1.4.7 we denoted by $\mathrm{Ham}_c(\mathcal{G}_\rho, \Omega)$ the group of compactly-supported *Hamiltonian symplectomorphisms* of $\mathcal{G}_\rho$ with respect to its symplectic form Ω. The proof of Theorem 1.4.7 in fact shows that if $\mathcal{L}$ is ρ-integrable and $\mathcal{L}' = \Phi(\mathcal{L})$ for $\Phi \in \mathrm{Ham}_c(\mathcal{G}_\rho, \Omega)$, then $\mathcal{L}'$ is integrable as well, even without the hypothesis that $\mathcal{L}$ and $\mathcal{L}'$ are Riemannian submanifolds.

If ρ admits an equivariant Riemannian minimal Lagrangian immersion, then Theorem 1.4.7 can be restated by saying that a Riemannian and Lagrangian submanifold $\mathcal{L}'$ is ρ-integrable if and only if it is in the $\mathrm{Ham}_c(\mathcal{G}_\rho, \Omega)$-orbit of *the* minimal Lagrangian submanifold $\mathcal{L} \subset \mathcal{G}_\rho$, which is unique by Theorem 1.4.6.

1.4.4 Relation with geometric flows and Andrew's conjecture

Finally, this approach highlights the relation between evolutions by geometric flows in $\mathbb{H}^{n+1}$ and in $\mathcal{G}(\mathbb{H}^{n+1})$. More precisely, suppose that $\sigma_\bullet : M \times (-\epsilon, \epsilon) \to \mathbb{H}^{n+1}$ is a smoothly varying family of Riemannian immersions that satisfy:

$$\frac{d}{dt}\sigma_t = f_t \nu_t$$

where ν_t is the normal vector of σ_t and $f_\bullet : M \times (-\epsilon, \epsilon) \to \mathbb{R}$ is a smooth function. Then the variation of the Gauss map G_t of σ_t is given, up to a tangential term, by the normal term $-\mathbb{J}(dG_t(\overline{\nabla}^t f_t))$, where $\overline{\nabla}^t f_t$ denotes the gradient with respect to the first fundamental form of G_t, that is, the Riemannian metric $G_t^*\mathbb{G}$.

Let us now consider the special case of the function $f_t := f_{\sigma_t}$ defined as the sum of $\operatorname{arctanh}(\lambda_i)$, where λ_i are the principal curvatures of σ_t. The study of the associated flow has been suggested in dimension three in [And02b], by analogy of a similar flow on surfaces in the three-sphere. By the results of [EES22a], we obtain that such flow in $\mathbb{H}^{n+1}$ induces the Lagrangian mean curvature flow in $\mathcal{G}(\mathbb{H}^{n+1})$ up to tangential diffeomorphisms. A similar approach has been developed in Anti-de Sitter space (in dimension three) in [Smo13].

In conclusion, the framework of the geometry of the space of geodesics of the hyperbolic space $\mathbb{H}^{n+1}$, when applied to $n = 2$, suggests a possible strategy towards the Andrew's conjecture as follows. Starting from a nearly-Fuchsian immersion in a quasi-Fuchsian manifold $\mathbb{H}^3/\rho(\pi_1(S))$ homeomorphic to $S \times \mathbb{R}$, one can consider the associated ρ-equivariant immersion of $\widetilde{S}$ via the Gauss map construction, into the space $\mathcal{G}(\mathbb{H}^3)$ of geodesics of $\mathbb{H}^3$. Such immersion is Riemannian and Lagrangian, and its Maslov class vanishes (Theorem 1.4.4). One could then try to apply the Lagrangian mean curvature flow. In this context, the main goal is to show the long-term convergence of such Lagrangian mean curvature flow, with the remarkable difficulty of ensuring that the evolution by Lagrangian mean curvature flow consists, at every time, of Riemannian submanifolds. If this is the case, then one would obtain in the limit a Riemannian and Lagrangian ρ-equivariant *minimal* immersion, whose Maslov class, by definition, vanishes automatically. Hence, by Theorem 1.4.4 again, such minimal Lagrangian immersion would automatically correspond to a minimal immersion in $\mathbb{H}^3/\rho(\pi_1(S))$ with small principal curvature, thus providing a positive answer to Andrew's conjecture.

Chapter 2

Euclidean geometry

The aim of this section is to introduce minimal Lagrangian maps between Riemannian surfaces of constant curvature, which play an important role in several chapters of this manuscript. They are deeply related with many three-dimensional geometries. In [EES22b] we explored their connection with surfaces of constant Gaussian curvature in Euclidean three-space, and we will outline how this relation can be exploited in two directions: on the one hand, one can use Euclidean geometry in order to obtain results on the study of minimal Lagrangian maps (in this case, in constant *positive* curvature, see Section 2.2); on the other, minimal Lagrangian maps serve to provide information on classical problems on the differential geometry of surfaces in $\mathbb{R}^3$ (Section 2.3). In Chapters 5 and 6 we will see several other instances of both directions, in that case by using the connection between minimal Lagrangian maps on surfaces of constant *negative* curvature, and surfaces in Lorentzian geometries of constant sectional curvature.

2.1 The ubiquitous minimal Lagrangian maps

Let us start with an interlude on hyperbolic surfaces and Teichmüller theory, which will appear again in Chapters 5 and 6. Minimal Lagrangian maps have played an important role in the study of hyperbolic surfaces. As observed independently by Labourie [Lab92] and Schoen [Sch93], given two closed hyperbolic surfaces (Σ_1, h_1) and (Σ_2, h_2), there exists a unique minimal Lagrangian diffeomorphism in the homotopy class of every diffeomorphism $\Sigma_1 \to \Sigma_2$. See also [Lee94] and [TV95, Smi20] for extensions of this result. Alternative proofs have been provided later, in the context of Anti-de Sitter three-dimensional geometry (see [BBZ07] and [BS20, §7]), and using higher codimension mean curvature flow (see [Wan01] and [LS11]). Using Anti-de Sitter geometry, the result of Labourie and Schoen has been generalized in various directions: in [BS10] in the setting of universal Teichmüller space; in [Tou16] for closed hyperbolic surfaces with cone singularities of angles in $(0, \pi)$, provided the diffeomorphism $\Sigma_1 \to \Sigma_2$ maps cone points to cone points of the same angles. Toulisse then proved in [Tou19] the existence of minimal maps between closed hyperbolic surface of different cone angles, by purely analytic methods. We remark that interesting results in a similar spirit have been obtained for minimal Lagrangian diffeomorphisms between bounded domains in the Euclidean plane ([Del91, Wol97]) and in a complete non-positively curved Riemannian surface ([Bre08]).

Minimal Lagrangian maps are very often associated to immersions of surfaces in three-manifolds. The main observation used in [EES22b] is that minimal Lagrangian maps between two spherical surfaces (i.e. of constant Gaussian curvature $+1$) are precisely those that can be realized locally as the Gauss map of a surface of constant Gaussian curvature $+1$ in Euclidean space, with values in the unit sphere. The version of this fact in negative curvature is the realizability of minimal Lagrangian maps between hyperbolic surfaces as the Gauss maps of surfaces of curvature -1 in Minkowski space, with values in the unit hyperboloid (namely, a copy of the hyperbolic plane $\mathbb{H}^2$) — a fact which will be largely used in Chapter 6. Minimal Lagrangian maps between spherical surfaces (resp. hyperbolic surfaces) can also be associated to minimal surfaces in $\mathbb{S}^3$ (resp. maximal surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, see Chapter 5). This is essentially a manifestation of the so-called Lawson correspondence that associates to a CMC surface in $\mathbb{R}^3$ to a minimal surface in $\mathbb{S}^3$, and an equidistance surface construction that provides a constant Gaussian curvature surface from a CMC surface in $\mathbb{R}^3$ — and the analogous versions of these correspondences in Lorentzian geometry, connecting surfaces in Minkowski and Anti-de Sitter geometry.

2.2 The "classical" Gauss map and spherical surfaces

Spherical metrics with cone singularities on a closed surfaces have been studied in [Tro86, McO88, Tro89, Tro91, LT92]. Very recently the works [MP16, MP19, EMP20], by geometric methods, and [MW17, MZ20, MZ19], by analytic methods, developed the study of the deformations spaces of spherical cone metric, highlighting their complexity. It thus seems a natural question to ask whether one can find a minimal Lagrangian diffeomorphism between two spherical cone surfaces. In [EES22b] we answered negatively to this question, without any assumption on the cone angles. We showed that two spherical cone surfaces do not admit any minimal Lagrangian diffeomorphism unless they are isometric. When they are isometric, the only minimal Lagrangian diffeomorphisms are isometries. We summarize these statements as follows:

Theorem 2.2.1. *Given two closed spherical cone surfaces $(\Sigma_1, \mathfrak{p}_1, g_1)$ and $(\Sigma_2, \mathfrak{p}_2, g_2)$, any minimal Lagrangian diffeomorphism $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$ is an isometry.*

We remark that a minimal Lagrangian diffeomorphism φ is defined as a smooth diffeomorphism between $\Sigma_1 \setminus \mathfrak{p}_1$ and $\Sigma_2 \setminus \mathfrak{p}_2$ that extends *continuously* to the sets of cone points, denoted $\mathfrak{p}_1$ and $\mathfrak{p}_2$. *A priori*, we do not assume that such a smooth map extends *smoothly* to the cone points.

The idea behind the proof of Theorem 2.2.1 is quite simple, and we provide an outline here. A map $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$ is minimal Lagrangian if it is area-preserving and its graph (restricted to the nonsingular locus) is minimal in the product $\Sigma_1 \times \Sigma_2$. A useful characterization is that one can express (on the nonsingular locus) $\varphi^* g_2 = g_1(b\cdot, b\cdot)$ for b a (1,1) tensor which is self-adjoint with respect to g_1, positive definite, and satisfies the conditions $d^{\nabla^{g_1}} b = 0$ and $\det b = 1$. From this characterization, one can prove, as mentioned above, that minimal Lagrangian maps are those that can be *locally* realized as the Gauss maps of surfaces of constant Gaussian curvature one in Euclidean three-space, as a consequence of the Gauss-Codazzi equations.

Starting by this characterization, using the spherical metric g_1 and the (1,1) tensor b, we produce a pair (G, B) where G is a Riemannian metric on Σ_1 and B a (1,1) G-self-adjoint traceless tensor, satisfying the equations

$$d^{\nabla^G} B = 0 \qquad \text{and} \qquad K_G = 1 + \det B \ .$$

Although we will not use spherical three-dimensional geometry in this paper, we remark that these are precisely the Gauss-Codazzi equations for a surface in $\mathbb{S}^3$, which is minimal since B is traceless. Equivalently, by the Lawson correspondence, the pair $(G, \mathbb{1} + B)$ satisfies the Gauss-Codazzi equations for a surface of constant mean curvature one in $\mathbb{R}^3$. Such constant mean curvature surface is realized (at least locally) as the parallel surface from the surface of constant Gaussian curvature mentioned above, which is determined by the pair (g_1, b). Assuming $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$ is a minimal Lagrangian diffeomorphism, the goal of the proof is to show that B vanishes identically, which is equivalent to φ being an isometry.

For this purpose, assuming by contradiction that B does not vanish identically, the next step consists in computing the Laplace-Beltrami operator of the function χ defined, in the complement of the zeros of B, as the logarithm of the positive eigenvalue of B (up to a certain constant). It turns out that $\Delta^G \chi$ equals the curvature of the metric G, which is positive, hence χ is subharmonic and negative and the contradiction is then obtained by an application of the maximum principle.

However, it is essential to prove that the metric G has the conformal type of a punctured disc in a neighbourhood of every cone point of Σ_1. This would be automatically satisfied assuming some additional regularity on the minimal Lagrangian map φ: for instance, if φ is supposed quasiconformal, which is equivalent to boundedness of the (1,1) tensor b, then g_1 and G are quasiconformal, and therefore both g_1 and G have the conformal type of a punctured disc near the cone points. But, as we mentioned above, in our Theorem 2.2.1 we assume a weaker regularity on φ at the cone points, namely we only suppose that φ is continuous.

To prove that G has the conformal type of a punctured disc around the cone points, we apply the interpretation in terms of surfaces in Euclidean space, and we show that G

can be realized in a punctured neighbourhood U^* of any cone point as the metric induced by the first fundamental form of an equivariant immersion of $\widetilde{U^*}$ in $\mathbb{R}^3$. We also prove that the normal vector of the equivariant immersion admits a limit, and the vertical projection induced a bi-Lipschitz equivalence between G and a flat metric on U^*. A complex analytic argument, based on Schwarz Reflection Principle, shows that this flat metric has the conformal type of $\mathbb{D}^*$ at the puncture, and this implies that G has the conformal type of $\mathbb{D}^*$ as well.

A possible generalization of Theorem 2.2.1 would concern the rigidity of the so-called *θ-landslides*, a larger class of maps (inspired by their hyperbolic counterpart, discussed in Chapter 5) depending on the value of a real parameter θ, for which minimal Lagrangian maps represent a special case. In this direction, instead of surfaces of constant Gaussian curvature in Euclidean space, one would need to use the alternative interpretation of minimal Lagrangian maps as associated to minimal surfaces in $\mathbb{S}^3$. The θ-landslides then correspond to surfaces of constant mean curvature H in $\mathbb{S}^3$, for a given value of H (which depends on the parameter θ). My PhD students Farid Diaf and Enrico Trebeschi are currently working jointly on this problem.

2.3 A Liebmann type theorem

Going back to Euclidean geometry, Theorem 2.2.1 has a direct application for branched immersions of constant Gaussian curvature in Euclidean three-space, generalizing the classical Liebmann's theorem which states that every closed immersed surface of positive constant Gaussian curvature in Euclidean space is a round sphere.

In [GHM13], Gálvez, Hauswirth and Mira classified the *isolated singularities* of surfaces of constant Gaussian curvature. According to their definition, isolated singularities of an immersion $\sigma : U \setminus \{p\} \to \mathbb{R}^3$, for U a disc, are those that extend continuously on U. Among these, they considered *extendable singularities*, namely those for which the normal vector extends smoothly at p, and showed that they are either *removable*, meaning that they extend to an immersion of U, or *branch points*, meaning that the Gauss map is locally expressed as the map $z \mapsto z^k$ with respect to some coordinates on U and on $\mathbb{S}^2$. Here we show a rigidity result for branched immersions of closed surfaces:

Corollary 2.3.1. *Every branched immersion in Euclidean three-space of a closed surface of constant positive Gaussian curvature is a branched covering onto a round sphere.*

As we mentioned, Corollary 2.3.1 can be regarded as a generalization of Liebmann's theorem, which we indeed recovered by an independent proof when the immersion has no branch points. Roughly speaking, we proved Corollary 2.3.1 by applying Theorem 2.2.1 to the Gauss map of a branched immersion $\sigma : \Sigma \to \mathbb{R}^3$, which induces a minimal Lagrangian self-diffeomorphism of Σ with respect to the first and third fundamental form, both of which are spherical cone metrics.

Finally, we remark that the hypothesis that the surface Σ is closed is essential in Corollary 2.3.1, as well as the closedness of Σ_1 and Σ_2 in Theorem 2.2.1. Indeed one can find *local* deformations of spheres of constant Gaussian curvature, with branch points (see [Bra16] for many examples) or without (for instance by surfaces of revolution); their Gauss maps provide non-isometric minimal Lagrangian diffeomorphisms between open spherical surfaces (with or without cone points).

Chapter 3

First topological interlude: Seifert fibrations

This chapter will focus on geometric structures on three-dimensional manifolds and orbifolds, with a more topological flavour. Smooth orbifolds are topological spaces that are locally homeomorphic to quotients of $\mathbb{R}^n$ by the action of a finite group G, and they are therefore a natural generalization of smooth manifolds, allowing the presence of *singular points* corresponding to the fixed points of the action of G. Smooth orbifolds can be naturally constructed as quotients of a smooth manifold by a properly discontinuous action. An orbifold is called *good* if it can be obtained as such a global quotient, and *bad* otherwise.

In dimension three, a smooth orbifold is called *geometric* if it is locally modelled on one of the eight Thurston's geometries. Closed geometric orbifolds are always good orbifolds. They played a fundamental role, among many things, in the Orbifold Geometrization Theorem proved in [BLP05] (see also [CHK00] and [BMP03]).

3.1 The multiple fibration problem

The topology of closed geometric orbifolds, whose geometry is one among $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Nil or $\widetilde{SL_2}$, is studied very effectively via the notion of *Seifert fibration* for orbifold, a generalization of the classical definition of Seifert fibration for manifolds, where the fibres are allowed to be either circles or intervals. Fibres homeomorphic to circles are entirely contained either in the singular locus of the orbifold or in its complement, the regular locus. Those homeomorphic to intervals, instead, are in the regular locus except for the endpoints, which always lie in the singular locus.

Seifert fibred orbifolds are uniquely determined, up to diffeomorphisms that preserve the orientation and the fibration, by a collection of *invariants*, which consists of a *base* (a 2-dimensional orbifold $\mathcal{B}$, whose underlying topological space is a 2-manifold with boundary) and several rational numbers: the *local invariants* associated to every cone and corner point of $\mathcal{B}$, the *boundary invariants* associated to every boundary component of the underlying topological space of $\mathcal{B}$, and a global *Euler number*.

From this perspective, closed geometric orbifolds can be classified via the collection of invariants of their Seifert fibration, with a single major difficulty: the same smooth orbifold may admit several Seifert fibrations which are not equivalent, where two Seifert fibrations of a 3-orbifold $\mathcal{O}$ are *equivalent* if there exists an orientation-preserving diffeomorphism of $\mathcal{O}$ mapping one to the other (or equivalently, they have the same collection of invariants).

In the articles [MS20] and [MMS23] we provided a conclusion of what we call the *multiple fibration problem*, namely, determining which closed orientable geometric orbifolds admit several inequivalent Seifert fibrations, and what are those fibrations. A reduction of the problem is provided by a result proved in [BMP03]: given be a compact orientable Seifert fibred good 3-orbifold $\mathcal{O}$ (possibly with boundary) with infinite fundamental group, if $\mathcal{O}$ is not covered by $S^2 \times \mathbb{R}$, T^3 or $T^2 \times I$ then the Seifert fibration on $\mathcal{O}$ is unique up to isotopy.

A closed orientable good 3-orbifold has finite fundamental group if and only if it is geometric with geometry $\mathbb{S}^3$. Moreover, by Bieberbach Theorem, closed flat 3-orbifolds are covered by T^3. As a consequence, for closed orbifolds, the above fact implies that, if a closed orientable Seifert fibred 3-orbifold $\mathcal{O}$ admits several inequivalent Seifert fibrations, then it is either geometric with geometry $\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{S}^2 \times \mathbb{R}$, or bad.

3.2 Euclidean geometry continued: crystallographic groups

Let us first consider orbifolds endowed with Euclidean geometry. Closed orbifolds with geometry $\mathbb{R}^3$ (which we will call *flat* in the following) are obtained as the quotients $\mathbb{R}^3/\Gamma$, where Γ is a *space group*, that is, a crystallographic group of dimension three. The study of the Seifert fibrations of closed flat three-orbifolds has been tackled in [CDFHT01] (including the non-orientable case, which we did not consider).

In particular, for closed flat orbifolds Conway, Delgado-Friedrichs, Huson and Thurston solved the multiple fibration problem, which is called *alias problem* in their work, since a compact notation (a "name") is used to denote Seifert fibrations, and a given orbifold may have several "names". However, in [CDFHT01] a computer-assisted method is used to solve the problem. That method is based on a consequence of Bieberbach Theorem, namely the fact that two closed flat orbifolds are diffeomorphic if and only if their fundamental groups are isomorphic; hence an algorithm can be used to determine whether two orbifolds, expressed in terms of their Seifert fibrations, have isomorphic fundamental groups.

The first result that we obtained in [MMS23] is the following theorem, which recovers the results of [CDFHT01] by a direct proof, based on geometric and topological arguments. In the table, Seifert fibred orbifolds are denoted by a string containing their base orbifold, the local invariants associated to each cone point, the local invariants associated to each corner point, the boundary invariants, and the Euler number.

Theorem 3.2.1. *A closed orientable flat Seifert 3-orbifold has a unique Seifert fibration up to equivalence, with the exceptions contained in the following table:*

$(S^2(2,2,2,2);0/2,0/2,0/2,0/2;0)$	$(S^1 \times I;;;;0;0;0)$	
$(S^2(2,2,2,2);0/2,0/2,1/2,1/2;0)$	$(S^1 \times I;;;;0;1;1)$	$(Mb;;;0;0)$
$(S^2(2,2,2,2);1/2,1/2,1/2,1/2;0)$	$(Kb;;0)$	
$(D^2(2,2;);0/2,0/2;;0;0)$	$(D^2(;2,2,2,2);;1/2,1/2,1/2,1/2;0;0)$	
$(D^2(2,2;);0/2,1/2;;0;1)$	$(D^2(2;2,2);1/2;1/2,1/2;0;0)$	
$(D^2(2,2;);1/2,1/2;;0;0)$	$(\mathbb{R}P^2(2,2);0/2,0/2;0)$	
$(D^2(2;2,2);0/2;0/2,0/2;0;0)$	$(D^2(;2,2,2,2);;0/2,0/2,1/2,1/2;0;1)$	

Two Seifert fibred orbifolds in the table are orientation-preserving diffeomorphic if and only if they appear in the same line. In particular, seven flat Seifert 3-orbifolds admit several inequivalent fibrations; six of those have exactly two inequivalent fibrations and one has three.

In the table, Mb is the Möbius band and Kb is the Klein bottle.

Let us now outline discuss some of the ideas in the proofs of Theorem 3.2.1, which are partially similar to those employed to prove Theorem 3.3.1 below, concerning $\mathbb{S}^2 \times \mathbb{R}$ geometry.

A general method to construct Seifert fibration on an orbifold with geometry $\mathbb{R}^3$ (or $\mathbb{S}^2 \times \mathbb{R}$) is the following. Consider a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^3)$ (or $\mathrm{Isom}(\mathbb{S}^2) \times \mathrm{Isom}(\mathbb{R})$, with compact quotient $\mathcal{O}$. The fibration of $\mathbb{R}^3$ (or $\mathbb{S}^2 \times \mathbb{R}$) given by the parallel lines $\{pt\} \times \mathbb{R}$ then induces a Seifert fibration of $\mathcal{O}$. By construction, the base 2-orbifold of this fibration of $\mathcal{O}$ is a quotient of $\mathbb{R}^2$ (or $\mathbb{S}^2$), and the Euler number vanishes. These are actually known to be necessary condition: every Seifert fibration of a closed orientable orbifold with geometry $\mathbb{R}^3$ (resp. $\mathbb{S}^2 \times \mathbb{R}$) has flat (resp. spherical) base and vanishing Euler number. More importantly, we prove that the converse holds true: every Seifert fibration of a closed orientable orbifold with vanishing Euler number and flat or spherical base orbifold is equivalent to one obtained by the above construction.

From now one, the argument becomes peculiar of Euclidean geometry. Indeed, by Bieberbach Theorem, two flat orbifolds $\mathbb{R}^3/\Gamma_1$ and $\mathbb{R}^3/\Gamma_2$ are diffeomorphic if and only if the space groups Γ_1 and Γ_2 are conjugate by an affine transformation, which in particular sends families of parallel lines to families of parallel lines. Since by every Seifert fibration of a closed flat orientable orbifold $\mathcal{O}$ is induced by a family of parallel lines of $\mathbb{R}^3$, the proof of Theorem 3.2.1 essentially consists in a careful analysis of the different families of parallel lines of $\mathbb{R}^3$ that a space group $\Gamma < \mathrm{Isom}(\mathbb{R}^3)$ may preserve.

3.3 $\mathbb{S}^2 \times \mathbb{R}$ geometry

Let us now move on to the geometry $\mathbb{S}^2 \times \mathbb{R}$. The following theorem provides the classification of multiple fibrations of closed orientable orbifolds with geometry $\mathbb{S}^2 \times \mathbb{R}$:

Theorem 3.3.1. *A closed orientable Seifert 3-orbifold with geometry $\mathbb{S}^2 \times \mathbb{R}$ has a unique Seifert fibration up to equivalence, with the exceptions contained in the following table:*

		for
$(S^2(d,d); 0/d; 0/d; 0)$	$(S^2(n,n); m/n; (n-m)/n; 0)$	$n \geq 1$ *and* $1 \leq m \leq n-1$ *where* $d = \gcd(n,m)$
$(D^2(;d,d); 0/d; 0/d; ; 0; 0)$	$(D^2(;n,n); m/n; (n-m)/n; ; 1; 0)$	$n \geq 1$ *and* $1 \leq m \leq n-1$ *where* $d = \gcd(n,m)$

Two Seifert fibred orbifolds in the table are orientation-preserving diffeomorphic if and only if they appear in the same line, with $d = \gcd(n,m)$.

Let us explain how to visualize the two inequivalent fibrations of the first line in the following example. The orbifold $\mathcal{O}$ admitting multiple fibrations in the first line of the table in Theorem 3.3.1 has underlying space $S^2 \times S^1$. Seeing $S^2 \times S^1$ as the union of two solid tori T_1 and T_2 glued by a diffeomorphism of their boundaries sending a meridian of T_1 to a meridian of T_2, the singular locus consists of the two cores of T_1 and T_2, both with singularity index d. The orbifold $\mathcal{O}$ can be obtained as the quotient of $\mathbb{S}^2 \times \mathbb{R}$ by the group Γ of isometries generated by a pure translation in the $\mathbb{R}$ direction (with translation length, say, equal to one) and a pure rotation in $\mathbb{S}^2$ of order d. The fibration in vertical lines of $\mathbb{S}^2 \times \mathbb{R}$ then induces the first fibration $(S^2(d,d); 0/d; 0/d; 0)$ of the quotient. To obtain other fibrations, we can consider the group Γ' generated by Γ and by an isometry that acts simultaneously on S^2 as a rotation of order n, and on $\mathbb{R}$ as a translation of length $1/n$. The quotient $(\mathbb{S}^2 \times \mathbb{R})/\Gamma'$ has again the same diffeomorphism type, and the fibration in vertical lines of $\mathbb{S}^2 \times \mathbb{R}$ now induces the other fibration $(S^2(n,n); m/n; (n-m)/n; 0)$. The orbifolds in the second line have underlying topological space S^3, and are obtained as a double quotient of the ones described above.

Unlike in $\mathbb{R}^3$, in the geometry $\mathbb{S}^2 \times \mathbb{R}$ there is no notion of "affine transformation", and there is a "privileged" direction, namely the vertical direction, which is preserved by the isometry group. One might be tempted to conjecture, in analogy with Bieberbach Theorem, that $(\mathbb{S}^2 \times \mathbb{R})/\Gamma_1$ and $(\mathbb{S}^2 \times \mathbb{R})/\Gamma_2$ are diffeomorphic (for $\Gamma_1 < \mathrm{Isom}(\mathbb{S}^2) \times \mathrm{Isom}(\mathbb{R})$) if and only if Γ_1 and Γ_2 are conjugate by a transformation acting by isometries on $\mathbb{S}^2$ and by affine transformations on $\mathbb{R}$. This statement is false: indeed, it would imply the uniqueness of the Seifert fibration for geometry $\mathbb{S}^2 \times \mathbb{R}$. In a certain sense, Theorem 3.3.1 describes the failure of an analogue of Bieberbach Theorem for $\mathbb{S}^2 \times \mathbb{R}$. Its proof shows that the orbifold $\mathcal{O}$ in the example above, together with a 2-to-1 quotient of $\mathcal{O}$ itself, is the only situation where two discrete groups of isometries induce the same diffeomorphism type in the quotient, but the vertical fibration of $\mathbb{S}^2 \times \mathbb{R}$ gives rise to inequivalent fibrations in the quotient.

To conclude this section, the analysis of bad orbifolds has similarities to the situation of the geometry $\mathbb{S}^2 \times \mathbb{R}$.

Theorem 3.3.2. *A closed orientable Seifert bad 3-orbifold admits infinitely many non-equivalent Seifert fibrations. More precisely, two bad Seifert fibred orbifolds are orientation-preserving diffeomorphic if and only if they appear in the same line of the following table (for $c \neq d$):*

		for
$(S^2(c,d);0/c;0/d;0)$	$(S^2(c\nu,d\nu);c\mu/c\nu;d(\nu-\mu)/d\nu;0)$	$\nu \geq 1$ *and* $1 \leq \mu \leq \nu-1$ *where* $\gcd(\mu,\nu)=1,\ c\neq d$
$(D^2(;c,d);0/c;0/d;0;0)$	$(D^2(c\nu,d\nu);c\mu/c\nu;d(\nu-\mu)/d\nu;1;0)$	$\nu \geq 1$ *and* $1 \leq \mu \leq \nu-1$ *where* $\gcd(\mu,\nu)=1,\ c\neq d$

3.4 Spherical orbifolds

Finally, let us consider orbifolds with geometry $\mathbb{S}^3$. The presentation of all possibile diffeomorphisms, which are studied in [MS19], is rather technical and difficult to summarize it in a single statement. The following statement provides a description of the number of possible fibrations.

Theorem 3.4.1. *Let $\mathcal{O}$ be a closed spherical Seifert fibered 3-orbifold with base orbifold $\mathcal{B}$ and b an integer greater than one.*

1. *If $\mathcal{B} \cong S^2(2,2,b)$, $D^2(b)$, $\mathbb{R}P^2(b)$, $D^2(2;b)$ or $D^2(;2,2,b)$ then $\mathcal{O}$ admits two inequivalent fibrations with the following exceptions:*
   - $\left(S^2(2,2,b);\ \frac{0}{2},\frac{0}{2},\pm\frac{2}{b};\ \mp\frac{2}{b}\right)$, $\left(S^2(2,2,b);\ \frac{0}{2},\frac{1}{2},\pm\frac{1+b/2}{b};\ \mp\frac{1}{b}\right)$ *with b even,* $\left(D^2(2;);\pm\frac{b}{2};\ ;\mp\frac{b}{2};0\right)$, $\left(D^2(2;b);\ \frac{1}{2};\ \pm\frac{1}{b};\ \mp\frac{1}{2b};\ 1\right)$ *and* $\left(D^2(;2,2,b);;\ \frac{1}{2},\frac{1}{2},\pm\frac{1}{b};\ \mp\frac{1}{2b};1\right)$ *which admit three fibrations;*
   - $\left(S^2(2,2,b);\ \frac{0}{2},\frac{0}{2},\pm\frac{1}{b};\ \mp\frac{1}{b}\right)$, $\left(S^2(2,2,b);\ \frac{0}{2},\frac{1}{2},\pm\frac{(1+b)/2}{b};\ \mp\frac{1}{2b}\right)$ *with b odd,* $\left(D^2(b;);\pm\frac{1}{b};\ ;\mp\frac{1}{b};0\right)$, $\left(D^2(b;);\pm\frac{(1+b)/2}{b};\ ;\mp\frac{1}{2b};1\right)$ *with b odd,* $\left(\mathbb{R}P^2(b);\pm\frac{1}{b};\mp\frac{1}{b}\right)$, $\left(D^2(2;b);\ \frac{0}{2};\ \pm\frac{1}{b};\ \mp\frac{1}{2b};\ 1\right)$ *with b even,* $\left(D^2(;2,2,b);\ ;\ \frac{0}{2},\frac{0}{2},\pm\frac{1}{b};\ \mp\frac{1}{2b};0\right)$ *with b odd*
     and $\left(D^2(;2,2,b);\ ;\ \frac{0}{2},\frac{1}{2},\pm\frac{(b+1)/2}{b};\ \mp\frac{1}{4b};1\right)$ *with b odd which admit infinitely many fibrations.*

2. *If $\mathcal{B} \cong S^2(2,3,b)$ or $D^2(;2,3,b)$ with $b=3,4,5$ then $\mathcal{O}$ admits a unique fibration with the following exceptions:*

- $\left(S^2(2,3,3)\,;\frac{0}{2}\,,\pm\frac{2}{3}\,,\pm\frac{2}{3}\,;\mp\frac{1}{3}\right)$, $\left(S^2(2,3,4)\,;\frac{0}{2}\,,\pm\frac{2}{3}\,,\pm\frac{2}{4}\,;\mp\frac{1}{6}\right)$, $\left(S^2(2,3,4)\,;\frac{0}{2}\,,\pm\frac{1}{3}\,,\pm\frac{3}{4}\,;\mp\frac{1}{12}\right)$, $\left(S^2(2,3,5)\,;\frac{0}{2}\,,\pm\frac{2}{3}\,,\pm\frac{2}{5}\,;\mp\frac{1}{15}\right)$, $\left(D^2(;2,3,3)\,;;\frac{1}{2}\,,\pm\frac{1}{3}\,,\pm\frac{1}{3};\mp\frac{1}{12};1\right)$, $\left(D^2(;2,3,4)\,;;\frac{1}{2}\,,\pm\frac{1}{3}\,,\pm\frac{1}{4};\mp\frac{1}{24};1\right)$ *and* $\left(D^2(;2,3,5)\,;;\frac{1}{2}\,,\pm\frac{1}{3}\,,\pm\frac{1}{5};\mp\frac{1}{60};1\right)$ *which admit two fibrations.*

3. *If $\mathcal{B}$ is a* 2*-sphere with at most two cone points or a* 2*-disk with at most two corner points, then $\mathcal{O}$ admits infinitely many fibrations.*

Let us discuss some topological aspects relating to the theorem. A Seifert fibered 3-orbifold with base orbifold a 2-sphere with at most two cone points has a lens space as underlying topological space and the singular set is a subset of the union of the cores of the two tori giving the lens space; these orbifolds can be considered as the generalization of lens spaces in the setting of orbifolds. However, we remark that there are Seifert fibered 3-orbifolds with base 2-orbifold different than a sphere with at most two cone points, whose underlying topological space is still a lens space; when this happens, the singular set does not entirely consist of a union of fibers. For instance, the other orbifolds in the third case of Theorem 3.4.1 (whose base 2-orbifold is a 2-disk) can be obtained as a quotient of an “orbifold lens space” by an involution whose action is not free; in this case the underlying topological space is always S^3, see also [Dun88].

The case $\mathcal{B} \cong S^2(2,2,b)$ contains the classical family of prism manifolds. Prism manifolds are known to admit two inequivalent fibrations, the second one with $\mathcal{B} \cong \mathbb{R}P^2(b)$, see [Orl72] or [Hat07, Theorem 2.3]. As a result of this analysis, we also obtain the following statement in analogy with the situation for spherical Seifert 3-manifolds:

Theorem 3.4.2. *If a closed spherical Seifert fibered* 3*-orbifold admits several inequivalent fibrations, then its underlying topological space is either a lens space or a prism manifold.*

However, unlike the manifold case, this is not a complete characterization of non-uniqueness, since there are 3-orbifolds with underlying manifold a lens space, whose fibration is unique up to diffeomorphism.

To conclude the discussion of spherical orbifolds, the methods employed to analyse fibrations of spherical orbifolds, together with a study of their isometry groups from the algebraic viewpoint, led in [MS20] to a proof of what is called the π_0*-part of the Generalized Smale Conjecture* for spherical 3-orbifolds.

To provide the context, a widely studied problem concerning the isometry group of 3-manifolds is the *Smale Conjecture*, and its stronger version, called *Generalized Smale Conjecture*. The latter asserts that the natural inclusion of $\mathrm{Isom}(M)$, the group of isometries of a compact spherical 3-manifold M, into $\mathrm{Diff}(M)$ (its group of diffeomorphisms) is a homotopy equivalence. The original version was stated for $M = S^3$ by Smale. The π_0*-part* of the original conjecture, namely the fact that the natural inclusion induces a bijection on the sets of path components, was proved by Cerf in [Cer68]. The full conjecture was then proved by Hatcher in [Hat83]. The Generalized Smale Conjecture for spherical

3-manifolds was proven in many cases, but is still open in full generality [HKMR12]. The π_0-part was instead proved in [McC02]. In [MS20] we proved the π_0-part of the analogous statement for spherical 3-orbifolds, namely:

Theorem 3.4.3. *Let $\mathcal{O} = S^3/\Gamma$ be a compact spherical oriented orbifold. The inclusion* $\mathrm{Isom}(\mathcal{O}) \to \mathit{Diff}(\mathcal{O})$ *induces a group isomorphism* $\pi_0\mathrm{Isom}(\mathcal{O}) \cong \pi_0\mathit{Diff}(\mathcal{O})$*.*

3.5 Some consequences

Finally, let us discuss some consequences of the above results.

A particular consequence of our Theorem 3.2.1 is that closed orbifolds with geometry $\mathbb{R}^3$ admit at most three inequivalent fibrations. From Theorem 3.3.1 (and Theorem 3.3.2), the same statement does not hold for geometry $\mathbb{S}^2 \times \mathbb{R}$ (nor for bad orbifolds), since the closed orbifolds for which the fibration is not unique (namely, those in the table of Theorem 3.3.1), admit infinitely many fibrations. Finally, for spherical geometry, Theorem 3.4.1 shows that a closed spherical orbifold may admit either infinitely many fibrations, or up to three fibrations. By combining these results, an immediate corollary is the following.

Corollary 3.5.1. *If a closed Seifert 3-orbifold does not admit infinitely many inequivalent Seifert fibrations, then it admits at most three inequivalent fibrations.*

Second, there results provide a characterization of those closed 3-orbifolds admitting infinitely many inequivalent Seifert fibrations. Before that, we need to introduce some definitions.

A lens space is a 3-manifold obtained by glueing two solid tori along their boundaries by an orientation-reversing diffeomorphism. If we allow cores of tori to be singular curves, the glueing gives an orbifold whose underlying topological space is a lens space and the singular set is a clopen subset (possibly empty) of the union of the two cores. We call these orbifolds *lens space orbifolds.* Moreover, we call a *Montesinos graph* a trivalent graph in S^3, which consists of a Montesinos link labelled 2, plus possibly one "strut" for every rational tangle, namely an interval (with any possible label) whose endpoints lie on the two connected components of the rational tangle. See [Dun88, Section 4] for a detailed description.

Corollary 3.5.2. *Let $\mathcal{O}$ be a closed Seifert fibred 3-orbifold. Then $\mathcal{O}$ admits infinitely many inequivalent fibrations if and only if either it is a lens space orbifold or it has underlying topological S^3 and singular set a Montesinos graph with at most two rational tangles.*

Part II

Pseudo-Riemannian geometries

Chapter 4

Pseudo-hyperbolic geometry

In this chapter, we will focus on the study of the *pseudo-hyperbolic space* $\mathbb{H}^{p,q}$, which is the analogue of the hyperbolic space $\mathbb{H}^n$ for pseudo-Riemannian metrics of any signature (p,q). The space $\mathbb{H}^{p,q}$ is defined as the projective space of negative-definite lines inside the pseudo-Euclidean space $\mathbb{R}^{p,q+1}$ of signature $(p,q+1)$. It is a homogeneous pseudo-Riemannian space of constant sectional curvature equal to -1 and of signature (p,q). When q is equal to 0 or 1, $\mathbb{H}^{p,q}$ reduces to the hyperbolic space $\mathbb{H}^p$ (considered in Chapter 1) and the anti-de Sitter space $\mathbb{H}^{p,1} = \mathbb{A}\mathrm{d}\mathbb{S}^{p+1}$ (that will be treated in Chapter 5) respectively. (Incidentally, in the proceedings article [ST22] that I co-authored with Enrico Trebeschi, who was at that time a Master student under my supervision, we developed an alternative model for pseudo-hyperbolic geometry, generalizing to any dimension and signature the well-known upper half-space model of hyperbolic geometry.)

As in the hyperbolic and anti-de Sitter cases, $\mathbb{H}^{p,q}$ possesses an *asymptotic boundary* $\partial_\infty\mathbb{H}^{p,q}$, which we identify with the space of isotropic lines in $\mathbb{R}^{p,q+1}$. The union $\mathbb{H}^{p,q} \cup \partial_\infty\mathbb{H}^{p,q}$ is compact with respect to the topology that it inherits as a subset of projective space.

4.1 Maximal submanifolds in $\mathbb{H}^{p,q}$

In the article [SST23], we provided a full classification of complete maximal p-dimensional spacelike submanifolds in the pseudo-hyperbolic space $\mathbb{H}^{p,q}$. Maximal surfaces in pseudo-Riemannian symmetric spaces are an important subject, that has been studied for instance in [Ish88, BBD$^+$12, Col16, CTT19, LTW20, Nie22, LT20, CT23, TW23].

To formulate the classification result of [SST23], let us consider a special class of topologically embedded spheres in $\partial_\infty\mathbb{H}^{p,q}$ as follows. We say that a triple of pairwise distinct points in $\partial_\infty\mathbb{H}^{p,q}$ - which, we recall, represent lines in $\mathbb{R}^{p,q+1}$ - is *positive* whenever their span is 3-dimensional with signature $(2,1)$, and *non-negative* whenever their span contains no negative-definite 2-plane. When $p \geq 3$, we say that a topologically embedded $(p-1)$-sphere Λ in $\partial_\infty\mathbb{H}^{p,q}$ is *positive* (respectively *non-negative*) whenever every triple of pairwise distinct points that it contains is positive (respectively non-negative). When $p = 2$, for topological reasons, we require in addition that Λ contain at least 1 positive triple. This latter case is studied in detail by Labourie–Toulisse–Wolf in [LTW20], where they call non-negative 1-spheres *semi-positive loops*. We denote by $\mathcal{B}$ the space of non-negative $(p-1)$-spheres in $\partial_\infty \mathbb{H}^{p,q}$, furnished with the Hausdorff topology.

We define a *maximal p-submanifold* of $\mathbb{H}^{p,q}$ to be a connected, p-dimensional, smooth spacelike submanifold which is a critical point of the area functional with respect to compactly supported variations. Equivalently, the trace of the second fundamental form vanishes identically. We denote by $\mathcal{M}$ the space of complete maximal p-submanifolds of $\mathbb{H}^{p,q}$, furnished with the topology of smooth convergence over compact sets. Given any complete maximal p-submanifold M of $\mathbb{H}^{p,q}$, we denote by $\partial_\infty M$ the intersection of its closure with $\partial_\infty \mathbb{H}^{p,q}$, and we call this set the *asymptotic boundary* of M. It is relatively straightforward to show that $\partial_\infty M$ is always a non-negative $(p-1)$-sphere, and even that ∂_∞ maps $\mathcal{M}$ continuously into $\mathcal{B}$. We prove the converse of this fact.

Theorem 4.1.1. *The asymptotic boundary map $\partial_\infty : \mathcal{M} \to \mathcal{B}$ is a homeomorphism. In particular, for every non-negative $(p-1)$-sphere Λ in $\partial_\infty \mathbb{H}^{p,q}$, there exists a unique complete maximal p-submanifold of $\mathbb{H}^{p,q}$ with asymptotic boundary Λ.*

Recall that the asymptotic Plateau problem in $\mathbb{H}^3$ has been discussed in Chapter 1, in particular in relation with the existence results of Anderson in [And83] and the question of uniqueness and non-uniqueness of its solutions.

This contrasts sharply with the situation in anti-de Sitter space. Indeed, in [ABBZ12], Andersson–Barbot–Béguin–Zeghib showed that, given any representation $\rho : \Gamma = \pi_1(N) \to \mathrm{PO}(p, 2)$, which is the holonomy of a maximal, globally hyperbolic, anti-de Sitter manifold homeomorphic to $N \times \mathbb{R}$, for some closed manifold N, there exists a unique ρ-invariant maximal hypersurface in $\mathbb{H}^{p,1} = \mathbb{A}\mathrm{d}\mathbb{S}^{p+1}$ with asymptotic boundary equal to the limit set of ρ. In particular, ρ acts on this hypersurface freely and properly discontinuously, with quotient diffeomorphic to N. In [BS10], Bonsante–Schlenker showed that every positive $(p-1)$-sphere in $\partial_\infty \mathbb{H}^{p,1}$ is the asymptotic boundary of some complete maximal hypersurface and, furthermore, when $p = 2$ and the boundary curve is the graph of a quasi-symmetric homeomorphism, this hypersurface is also unique.

Analogous results have recently been obtained for maximal surfaces in the pseudo-hyperbolic space $\mathbb{H}^{2,q}$. Indeed, in [CTT19], Collier–Tholozan–Toulisse proved that, for any closed, orientable surface S of genus at least 2, and any maximal representation $\rho : \pi_1(S) \to \mathrm{PO}_0(2, q+1)$, the limit set of ρ is the asymptotic boundary of a unique complete ρ-invariant maximal surface in $\mathbb{H}^{2,q}$. As before, ρ then acts freely and properly-discontinuously on this surface, with quotient diffeomorphic to S. Finally, in [LTW20], Labourie–Toulisse–Wolf generalised this result to prove that every non-negative 1-sphere in $\partial_\infty \mathbb{H}^{2,q}$ is the asymptotic boundary of a unique complete maximal surface.

Theorem 4.1.1 thus unifies the known results for $\mathbb{H}^{p,1}$ and $\mathbb{H}^{2,q}$, and extends them to $\mathbb{H}^{p,q}$ for all (p,q), whilst addressing the most general hypotheses on the asymptotic boundary.

The proof of Theorem 4.1.1, which is outlined in Section 4.2 below, requires an in-depth study of the asymptotic structure of complete maximal p-submanifolds in $\mathbb{H}^{p,q}$. The techniques that we developed yield, in addition, the following new result concerning the total curvatures of complete maximal p-submanifolds with sufficiently regular asymptotic boundaries.

Theorem 4.1.2. *If M is a complete maximal p-submanifold in $\mathbb{H}^{p,q}$ with $C^{3,\alpha}$ asymptotic boundary, and if* II *denotes its second fundamental form, then, for all $s > p-1$,*

$$\|\mathrm{II}\| \in L^s(M, \mathrm{dVol}_M) \ . \tag{4.1}$$

Motivated by applications to the AdS-CFT correspondence, the *renormalized area* of a maximal surface M in $\mathbb{H}^{2,q}$ with second fundamental form II is defined by

$$A_{\mathrm{ren}}(M) := \int_M \|\mathrm{II}\|^2 \mathrm{dVol}_M \ . \tag{4.2}$$

The study of the renormalized area of maximal surfaces in $\mathbb{H}^{2,q}$ presents a number of interesting, as yet unstudied, problems. For example, following [AM10] it is of interest to determine its first and second variations. Likewise, in the spirit of [Bis21], it is also of interest to determine how finiteness of the renormalized area may be expressed in terms of properties of the asymptotic boundary. Substituting $p = s = 2$ in Theorem 4.1.2 yields a partial response to this latter problem for surfaces in pseudo-hyperbolic space.

Corollary 4.1.3. *Every complete maximal surface M in $\mathbb{H}^{2,q}$ with $C^{3,\alpha}$ asymptotic boundary has finite renormalized area.*

4.2 Techniques and approach

The approach of [SST23] is quite different from those used in the earlier works mentioned above. Indeed, Anderson used geometric measure theory to address the Plateau problem in $\mathbb{H}^p$, a technique which to date has no pseudo-Riemannian analogue. Likewise, Andersson–Barbot–Béguin–Zeghib and Bonsante–Schlenker used the works [Ger83a, Ger06b] of Gerhardt, [Bar88] of Bartnik, and [Eck03] of Ecker, which are peculiar to the Lorentzian setting. Finally Collier–Tholozan–Toulisse used Higgs bundles, while Labourie–Toulisse–Wolf used pseudo-holomorphic curves, both of which are peculiar to the case of surfaces.

In [SST23] we proved Theorem 4.1.1 by applying the continuity method in a global manner. This has the advantage over previous approaches of yielding detailed information concerning the asymptotic structures of complete maximal p-submanifolds with smooth asymptotic boundaries, yielding, as a byproduct, Theorem 4.1.2.

The continuity method decomposes into three main steps, namely compactness, uniqueness and perturbation (or, stability).

Our compactness result is a manifestation of the dichotomy first observed by Labourie in [Lab97] in the context of k-surfaces in hyperbolic 3-space (see also [Sch96a, Smi13, Smi18, Smi22b]).

Step 1 (Compactness). *If $\{M_m\}_{m\in\mathbb{N}}$ is a sequence of complete maximal p-submanifolds of $\mathbb{H}^{p,q}$ then, either*

1. *$\{M_m\}_{m\in\mathbb{N}}$ subconverges in the C^∞_{loc} topology to a complete maximal p-submanifold of $\mathbb{H}^{p,q}$, or*

2. *$\{M_m\}_{m\in\mathbb{N}}$ subconverges in the Hausdorff topology to a Lipschitz p-submanifold foliated by complete, lightlike geodesics, all having the same endpoint at infinity.*

We call submanifolds of the second type *degenerate*. Up to isometries of the ambient space, the space of degenerate submanifolds is itself homeomorphic to the space of 1-Lipschitz maps from a hemisphere $\mathbb{S}^{p-1}_+ \subseteq \mathbb{S}^{p-1}$ into $\mathbb{S}^{q-1}$. Indeed, degenerate submanifolds are simply the graphs of suspensions of such maps.

Recall now that the space $\mathcal{M}$ of complete maximal p-submanifolds carries the topology of smooth convergence over compact sets, whilst the space $\mathcal{B}$ of non-negative $(p-1)$-spheres carries the Hausdorff topology. Since no degenerate submanifold can have a non-negative sphere as its asymptotic boundary, one of the main consequences of Step 1 for the proof of Theorem 4.1.1 is that the asymptotic boundary map $\partial_\infty : \mathcal{M} \to \mathcal{B}$ is proper.

Uniqueness is proven using the maximum principle. Our proof is similar to that of [LTW20], although some care is required in the higher-dimensional setting.

Step 2 (Uniqueness). *A non-negative $(p-1)$-sphere in $\partial_\infty\mathbb{H}^{p,q}$ is the asymptotic boundary of at most one complete maximal p-submanifold of $\mathbb{H}^{p,q}$.*

The lengthiest and most technical part of the proof is the following stability result.

Step 3 (Stability). *Let $(\Lambda_t)_{t\in(-\epsilon,\epsilon)}$ be a smoothly varying family of smooth, spacelike spheres in $\partial_\infty\mathbb{H}^{p,q}$. If there exists a complete maximal p-submanifold M of $\mathbb{H}^{p,q}$ with asymptotic boundary Λ_0 then, upon reducing ϵ if necessary, there exists a family $(M_t)_{t\in(-\epsilon,\epsilon)}$ of complete maximal p-submanifolds such that, $M_0 = M$ and, for all t, M_t has asymptotic boundary Λ_t.*

The usual approach to proving stability results of this kind is to first represent maximal p-submanifolds near M as zeroes of some functional over some Banach space, and then to apply the implicit function theorem. In the present case, the non-compactness of the submanifolds in question presents an extra layer of difficulty. Having established Steps 1, 2 and 3, Theorem 4.1.1 readily follows by the continuity method. Indeed, let $\mathrm{Im}(\partial_\infty)$ denote the image of ∂_∞, let $\mathcal{B}^\infty$ denote the space of smooth, spacelike $(p-1)$-spheres in $\partial_\infty\mathbb{H}^{p,q}$, and note that this is a dense subset of $\mathcal{B}$. Since totally geodesic, spacelike p-subspaces of $\mathbb{H}^{p,q}$ are trivially maximal, $\mathrm{Im}(\partial_\infty)$ has non-trivial intersection with $\mathcal{B}^\infty$. Since $\mathcal{B}^\infty$ is path-connected, the continuity method then shows that $\mathrm{Im}(\partial_\infty)$ contains $\mathcal{B}^\infty$. Finally, by density of $\mathcal{B}^\infty$, and properness of ∂_∞, it follows that $\mathrm{Im}(\partial_\infty)$ contains $\mathcal{B}$, thus proving Theorem 4.1.1.

The asymptotic analysis developed to prove Step 3 involves the use of weighted function spaces over complete maximal p-submanifolds. Although weighted spaces are not actually required for the proof of Theorem 4.1.1, they provide detailed asymptotic information at little extra cost. In particular, they permit us to show that the norms of the second fundamental forms of suitably regular cones, on the one hand, and suitably regular, complete maximal p-submanifolds, on the other, both decay exponentially at an equal rate, whilst their volume forms both grow exponentially at an equal, slower rate. Theorem 4.1.2 is then a straightforward consequence of these two properties.

4.3 Anosov representations

Amongst the main motivations for the study of complete maximal p-submanifolds in $\mathbb{H}^{p,q}$ are their applications to the study of Anosov representations of word-hyperbolic groups in $\mathrm{PO}(p,q+1)$, see [Lab06, GW12, Kas18, DGK18, Zim21, Can]. In what follows, Γ will denote a word-hyperbolic group with connected Gromov boundary.

In [DGK18], Danciger–Guéritaud–Kassel introduced a notion of convex-cocompactness for representations in $\mathrm{PO}(p,q+1)$. We say that a representation $\rho:\Gamma\to\mathrm{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-*convex-cocompact* whenever it has finite kernel and acts properly-discontinuously and cocompactly on some closed, convex subset K of $\mathbb{H}^{p,q}$ whose interior $\mathrm{Int}(K)$ is non-trivial and whose ideal boundary $\partial_\infty K$ contains no non-trivial segment. We will be concerned here with the more specific case where the Gromov boundary of Γ is homeomorphic to a $(p-1)$-sphere. In this case, Danciger–Guéritaud–Kassel showed that $\Lambda=\partial_\infty K$ is a positive $(p-1)$-sphere in $\partial_\infty\mathbb{H}^{p,q}$. Since every positive sphere is, in particular, non-negative, Theorem 4.1.1 thus yields the following result.

Corollary 4.3.1. *Let Γ be a word-hyperbolic group with Gromov boundary homeomorphic to a $(p-1)$-sphere. Every $\mathbb{H}^{p,q}$-convex-cocompact representation $\rho:\Gamma\to\mathrm{PO}(p,q+1)$ preserves a unique complete maximal p-submanifold M in $\mathbb{H}^{p,q}$. Furthermore*

1. *ρ acts properly-discontinuously and cocompactly on M, and*
2. *M depends real analytically on ρ.*

Corollary 4.3.1 has applications to the study of higher-dimensional extensions of higher-rank Teichmüller theory. Recall (c.f. [Wie18]) that when Γ is a compact surface group, and G is a reductive Lie group of higher rank, a connected component of $\mathrm{Hom}(\Gamma,G)$ is said to be a *higher-rank Teichmüller space* whenever it consists entirely of discrete and faithful representations. The study of such spaces, which share a number of properties of classical Teichmüller space, has yielded over the last two decades a rich and fascinating theory. Although it is natural to generalize this concept to higher-dimensional word-hyperbolic groups, to date only a few higher-dimensional cases have been shown to exist. One such is given by Barbot in [Bar15], where he showed that, when Γ is the fundamental group of a compact, p-dimensional hyperbolic manifold, the quasi-fuchsian component of $\mathrm{Hom}(\Gamma, \mathrm{PO}(p,2))$ consists entirely of $\mathbb{H}^{p,1}$-convex-cocompact representations, which, in particular, are discrete and faithful. In [BK23] Beyrer–Kassel prove a converse of Corollary 4.3.1, namely that any representation acting properly-discontinuously and cocompactly on a weakly spacelike p-dimensional submanifold of $\mathbb{H}^{p,q}$ is $\mathbb{H}^{p,q}$-convex-cocompact. Together with Corollary 4.3.1, this allows them to show in [BK23, Theorem 1.3] that, for all $p \geq 2$ and $q \geq 1$, and for any word-hyperbolic group Γ with Gromov boundary homeomorphic to a $(p-1)$-sphere, the set of $\mathbb{H}^{p,q}$-convex-cocompact representations $\rho : \Gamma \to \mathsf{PO}(p, q+1)$ is a union of connected components of $\mathrm{Hom}(\Gamma, \mathrm{PO}(p, q+1))$. In this manner, they extend Barbot's result to yield a large family of new higher-dimensional higher-rank Teichmüller spaces, including many examples which are not quasi-fuchsian (c.f. [LM19, MST23b]).

Let us discuss the applications of Corollary 4.3.1 to the study of P_1-Anosov representations of word-hyperbolic groups in $\mathrm{PO}(p, q+1)$. Anosov representations were introduced by Labourie in [Lab06], and have since become a cornerstone of higher-rank Teichmüller theory (see Danciger–Guéritaud–Kassel's paper [DGK18] for a formal definition of the concept of P_1-Anosov representations). In addition Danciger–Guéritaud–Kassel described a direct relationship between such representations and the $\mathbb{H}^{p,q}$-convex-cocompact representations discussed above. To understand this, note first that the set of non-null lines in $\mathbb{R}^{p,q+1}$ consists of two connected components, namely $\mathbb{H}^{p,q}$ and $\mathbb{H}^{q+1,p-1}$, where the sign of the metric of the latter is inverted. Danciger–Guéritaud–Kassel showed that every $\mathbb{H}^{p,q}$-convex-cocompact representation is P_1-Anosov and, conversely, that every P_1-Anosov representation is either $\mathbb{H}^{p,q}$-convex-cocompact or $\mathbb{H}^{q+1,p-1}$-convex-cocompact. We say that the representation is *positive* in the former case, and *negative* in the latter, so that every $\mathbb{H}^{p,q}$-convex-cocompact representation is a positive P_1-Anosov representation, and vice versa. In particular, in this case, $\Lambda := \partial_\infty K$ coincides with the proximal limit set of the representation ρ.

The first application of Corollary 4.3.1 provides a new constraint on the hyperbolic groups which admit positive P_1-Anosov representations.

Corollary 4.3.2. *Let Γ be a torsion-free word-hyperbolic group with Gromov boundary homeomorphic to a $(p-1)$-sphere. If Γ admits a positive P_1-Anosov representation in $\mathrm{PO}(p, q+1)$, then it is isomorphic to the fundamental group of a smooth, closed p-dimensional manifold with universal cover diffeomorphic to $\mathbb{R}^p$.*

It is worth comparing Corollary 4.3.2 to the result [BLW10] of Bartels–Lück–Weinberger, which states that, for $p \geq 6$, any torsion-free hyperbolic group with Gromov boundary homeomorphic to a $(p-1)$-sphere is the fundamental group of a closed p-dimensional *topological* manifold with universal cover *homeomorphic* to $\mathbb{R}^p$. Note, in particular, that in Example 5.2 and Lemma 5.3 of that paper, the authors construct, for all $k \geq 2$, a torsion-free hyperbolic group Γ, with Gromov boundary homeomorphic to $\mathbb{S}^{4k-1}$, which is *not* isomorphic to the fundamental group of any closed *smooth* aspherical manifold. We consequently have the following corollary.

Corollary 4.3.3. *For any $k \geq 2$ and $q \geq 1$, there exists a torsion-free word-hyperbolic group Γ with Gromov boundary homeomorphic to a $(4k-1)$-sphere which does not admit any positive P_1-Anosov representation in $\mathrm{PO}(4k, q+1)$.*

The second application concerns the structure of a class of manifolds introduced by Guichard–Weinhard in [GW12]. Indeed, let $\mathrm{Isot}(E)$ denote the space of maximally isotropic subspaces of E. Given a positive P_1-Anosov representation ρ of Γ in $\mathrm{PO}(p, q+1)$, Guichard–Wienhard constructed a domain Ω_ρ in $\mathrm{Isot}(E)$ upon which $\rho(\Gamma)$ acts properly-discontinuously and cocompactly. The quotient $\Omega_\rho/\rho(\Gamma)$ is a closed manifold locally modelled on $\mathrm{Isot}(E)$. However, it is a difficult question to determine the topology of such quotient (see also [AMWT23]). By means of the construction of an equivariant "projection" from Ω_ρ to the invariant maximal submanifold provided by Corollary 4.3.1, in [SST23] we have been able to provide information in this direction, generalizing the results of [CTT19] for $p = 2$.

Before stating the result, recall that, given two natural numbers $k \leq n$, the *Stiefel manifold* $\mathcal{V}_{k,n}$ is the space of k-tuples of unit vectors in $\mathbb{R}^n$ that are pairwise orthogonal. For all such k and n, $\mathcal{V}_{k,n}$ is diffeomorphic to the homogeneous space $\mathrm{O}(n)/\mathrm{O}(n-k)$. In particular, it is connected unless $k = n$, in which case it has 2 connected components.

Corollary 4.3.4. *Let Γ be a torsion-free word-hyperbolic group with Gromov boundary homeomorphic to a $(p-1)$-sphere, let $\rho : \Gamma \to \mathrm{PO}(p, q+1)$ be a positive P_1-Anosov representation, and let Ω_ρ denote its Guichard–Wienhard domain.*

1. *If $q \geq p$, and if M denotes the complete maximal p-submanifold preserved by ρ, then $\Omega_\rho/\rho(\Gamma)$ is homeomorphic to a $\mathcal{V}_{p,q}$-bundle over $M/\rho(\Gamma)$.*
2. *If $q < p$, then Ω_ρ is empty.*

In particular, in the former case, Ω_ρ is connected, unless $p = q$ and the first Stiefel-Whitney class of $\mathrm{N}M/\rho(\Gamma)$ vanishes, in which case it has 2 connected components.

Chapter 5

Anti-de Sitter geometry

In his 1990 pioneering paper [Mes07], Geoffrey Mess has first highlighted the deep connections between the Teichmüller theory of hyperbolic surfaces, and three-dimensional Lorentzian geometries of constant sectional curvature. The purpose of this chapter is to focus on Anti-de Sitter geometry, namely the Lorentzian geometry of constant curvature -1, mostly in dimension three, and on its relations and applications in Teichmüller theory. Chapter 6 below will concern Minkowski geometry, which is instead the model of flat Lorentzian geometry.

5.1 Mess' work on Anti-de Sitter geometry

The root of the connection between Anti-de Sitter geometry of dimension three and Teichmüller theory rests in the observation that Anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^3 = \mathbb{H}^{2,1}$ admits a Lie group model, which identifies it with the group of orientation-preserving isometries of $\mathbb{H}^2$. This identification comes from the fact that $\mathbb{A}\mathrm{d}\mathbb{S}^3$, which is defined as the projectivization of the space of negative vectors in a four-dimensional vector space V endowed with a quadratic form q of signature $(2,2)$, can be realized as the space $\mathrm{PSL}(2,\mathbb{R})$ of projective matrices of positive determinant, since one can choose V to be the vector space of 2-by-2 matrices and the quadratic form to be $q = -\det$. In this incarnation, the Lorentzian metric of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ coincides (up to a constant) with the Lorentzian metric induced by the (bi-invariant) Killing form of $\mathrm{PSL}(2,\mathbb{R})$ on its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Moreover, the identity component of the isometry group of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ (namely, the group of orientation-preserving and time-preserving isometries) corresponds to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, acting by left and right multiplication on $\mathrm{PSL}(2,\mathbb{R})$.

Let us outline two ideas of Mess that played a fundamental – and visionary – role in many research results up to now, including those presented in this chapter. The first idea is the construction of a Gauss map for spacelike surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, the second is the study of maximal globally hyperbolic Anti-de Sitter manifolds.

5.1.1 Gauss maps

The first crucial idea of Mess arises from the following *Gauss map* construction, using the Lie group structure of $\mathbb{A}\mathrm{d}\mathbb{S}^3 = \mathrm{PSL}(2,\mathbb{R})$. Given an embedded spacelike surface Σ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, one can construct two maps $G_l, G_r : \Sigma \to \mathbb{H}^2$, as follows. Given a point $p \in \Sigma$, one can push forward the future unit normal vector to Σ at p by left and right multiplication by the inverse of p. One thus obtains two future unit vectors in $T_{\mathrm{id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$, which are precisely, after identifying the hyperboloid of future unit vectors with $\mathbb{H}^2$, $G_l(p)$ and $G_r(p)$. Under certain assumptions on Σ, ensuring that G_l and G_r are injective, one can thus associate to Σ the map $G_r \circ G_l^{-1}$, which is defined between (subsets of) $\mathbb{H}^2$ and itself.

Mess has then observed that, applying this construction to convex hulls in Anti-de Sitter space, one can prove earthquake theorems in hyperbolic geometry. (Although convex hulls are not smooth surfaces, the construction can still be pursued in this setting, with the only caveat that the earthquake map is not uniquely determined on its discontinuity locus). In [Mes07], Mess outlined the proof of the earthquake theorem between closed hyperbolic surfaces, providing an alternative proof of Kerckhoff's earthquake theorem [Ker83]. His groundbreaking ideas have been improved and implemented by several authors, leading to many results of existence of earthquake maps in various settings [BKS11, BS09, BS12], including a proof of Thurson's earthquake theorem in the *universal setting* – namely, the existence of an earthquake of $\mathbb{H}^2$ extending any orientation-preserving circle homeomorphism – which has been written in detail in my chapter [DS22] with my PhD student Farid Diaf, when Farid was a Master student under my supervision. The Gauss map construction les to many developments also for other interesting types of maps of the hyperbolic plane, for instance *minimal Lagrangian maps*, which are associated to *maximal surfaces* and are discussed in Section 5.2.

5.1.2 Deformation space of maximally globally hyperbolic manifolds

Let us recall the definition of *maximal globally hyperbolic Cauchy compact Anti-de Sitter manifolds* (in short, MGHC AdS). A *Cauchy surface* in a Lorentzian manifold is an embedded hypersurface that intersects every inextensible causal curve exactly in one point; a Lorentzian manifold admitting a Cauchy surface is called *globally hyperbolic*. It is moreover *maximal* if every isometric embedding in another globally hyperbolic manifold sending a Cauchy surface to a Cauchy surface is surjective. Finally, a MGHC AdS manifold is a maximal globally hyperbolic Lorentzian manifold of constant sectional curvature -1 admitting a closed Cauchy surface. A simple example of MGHC AdS manifolds are *Fuchsian manifolds*, whose metric G can be written globally as a warped product

$$G = -dt^2 + \cos^2(t)h \ , \tag{5.1}$$

for $t \in (-\pi/2, \pi/2)$ and h a hyperbolic metric on a closed manifold. In this case the Cauchy surface $t = 0$ is totally geodesic.

A classical fact in Lorentzian geometry (see [Ger70, BE81, BS03]) is that globally hyperbolic Lorentzian manifolds are diffeomorphic to $\Sigma \times \mathbb{R}$, where Σ is a Cauchy surface, and any two Cauchy surfaces are diffeomorphic. Let us now consider three-dimensional AdS manifolds whose Cauchy surfaces are closed. Once a closed oriented surface Σ is fixed, we define the *deformation space* of MGHC AdS manifolds as follows:

$$\mathcal{MGH}(\Sigma) := \{G \,|\, G \text{ is a MGHC AdS metric on } \Sigma \times \mathbb{R}\}/\mathrm{Diff}_0(\Sigma \times \mathbb{R}) ,$$

where the group $\mathrm{Diff}_0(\Sigma \times \mathbb{R})$ of diffeomorphisms isotopic to the identity acts by pull-back of G. It turns out that $\mathcal{MGH}(S^2)$ is empty, $\mathcal{MGH}(T^2)$ is a four-dimensional manifold, while if Σ is a surface of genus ≥ 2, then $\mathcal{MGH}(\Sigma)$ has dimension $6|\chi(\Sigma)|$. When Σ has genus ≥ 2, the deformation space $\mathcal{MGH}(\Sigma)$ contains the *Fuchsian locus* $\mathcal{F}(\Sigma)$, namely those manifolds whose metric is of the form (5.1), which is naturally identified to the Teichmüller space $\mathcal{T}(\Sigma)$.

Now, Mess observed that, when Σ is a closed surface of genus ≥ 2, the deformation space $\mathcal{MGH}(\Sigma)$ of maximal globally hyperbolic Anti-de Sitter manifolds is intimately related with the Teichmüller space $\mathcal{T}(\Sigma)$. More concretely, he provided a parametrization of $\mathcal{MGH}(\Sigma)$ by the product $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$; other parametrizations again by $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ or by $T^*\mathcal{T}(\Sigma)$ were introduced in [KS07] — the latter relying on existence and uniqueness results for maximal surfaces which will be further discussed below.

5.2 Maximal surfaces and minimal Lagrangian maps

In this section, we will focus on the construction, and study, of minimal Lagrangian self-diffeomorphisms of $\mathbb{H}^2$. A (local) diffeomorphism of $\mathbb{H}^2$ is called *minimal Lagrangian* if its graph is a minimal Lagrangian surface in $\mathbb{H}^2 \times \mathbb{H}^2$. In [BS10], Bonsante-Schlenker observed that minimal Lagrangian maps are precisely those that are associated, via the Gauss map construction described in Section 5.1, to maximal surfaces (which, we recall, where defined in the more general setting of $\mathbb{H}^{p,q}$ in Chapter 4).

Bonsante and Schlenker applied this observation, together with the solution of the asymptotic Plateau problem in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ (see Chapter 4 again) to prove that every quasisymmetric circle homeomorphism admits a unique quasiconformal minimal Lagrangian extension to $\mathbb{H}^2$. Based on their results, a natural line of investigation concerns the *optimality* of the minimal Lagrangian extension, in relation with the *complexity* of the boundary circle homeomorphism. Indeed, the quasiconformal dilatation of the minimal Lagrangian extension of a circle homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ is directly related to the norm of the second fundamental form of the unique maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ whose asymptotic boundary is the graph of φ, using the identification of $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$, namely the space of projective classes of rank one 2-by-2 matrices, with $\mathbb{RP}^1 \times \mathbb{RP}^1$.

5.2.1 Qualitative optimality

In this context, a direct consequence of Corollary 4.1.3 is the following:

Corollary 5.2.1. *If ϕ is a $C^{3,\alpha}$ circle diffeomorphism, then the Beltrami coefficient of its unique quasiconformal minimal Lagrangian extension is in $L^2(\mathbb{H}^2, \mathrm{dVol}_{\mathbb{H}^2})$.*

Indeed, Corollary 4.1.3 (applied to the case $q = 1$) implies that a complete maximal surface S in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ whose graph is a $C^{3,\alpha}$ circle diffeomorphism satisfies:

$$A_{\mathrm{ren}}(S) = \int_S \|\mathrm{II}\|^2 \mathrm{dArea}_S < +\infty \ .$$

Now, a direct computation shows that, if μ is the Beltrami differential of the minimal Lagrangian map of $\mathbb{H}^2$ obtained from S, then $|\mu| = O(\|\mathrm{II}\|)$, and the area form of S and of $\mathbb{H}^2$, compared using one of the two Gauss maps, are comparable. This gives immediately that

$$\int_{\mathbb{H}^2} |\mu|^2 \mathrm{dArea}_{\mathbb{H}^2} < +\infty \ .$$

It is known that quasiconformal maps of $\mathbb{H}^2$ with square integrable Beltrami differential extend to *Weil-Petersson* quasisymmetric circle homeomorphisms. Conversely, Weil-Petersson circle homeomorphisms are precisely those that admit *some* quasiconformal extension with square integrable Beltrami differential. The space of Weil-Petersson circle homeomorphisms is precisely the closure of the space of circle diffeomorphisms with respect to the topology induced on the universal Teichmüller space by the (infinite-dimensional) Kähler structure constructed in [TT06]. Motivated by the recent work of Bishop [Bis21], which characterized Weil-Petersson quasicircles in $\mathbb{H}^2$ as those Jordan curves that bound a complete minimal surface in $\mathbb{H}^3$ of finite renormalized area, it is natural to conjecture the following:

Conjecture. *A quasisymmetric circle homeomorphism is Weil-Petersson if and only if its quasiconformal minimal Lagrangian extension has square integrable Beltrami differential.*

I believe this will is an important question that deserves future investigation.

Next, another important class of quasisymmetric homeomorphisms are the *symmetric* ones. We omit here their intrinsic definition, but they are characterized by the condition of admitting *some* quasiconformal extension which is *asymptotically conformal*, meaning that its Beltrami differential tends to zero towards $\partial\mathbb{H}^2$. Conversely, any asymptotically conformal quasiconformal map of the disc extends to a symmetric homeomorphism of the circle. Together with Jérémy Toulisse, by showing that the second fundamental form of a complete maximal surface S in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ tends to zero if $\partial_\infty S$ is the graph of a symmetric homeomorphism, we obtained the following (currently unpublished):

Theorem 5.2.2. *A quasisymmetric circle homeomorphism is symmetric if and only if its quasiconformal minimal Lagrangian extension is asymptotically conformal.*

5.2.2 Quantitative optimality

Given an orientation-preserving homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$, an important invariant in Anti-de Sitter geometry, introduced in [BS10], is the *width* of the convex hull. This is defined as the supremum of the length of timelike paths contained in the convex hull of the curve $gr(\phi)$. By a simple application of the maximum principle, the maximal surface S with $\partial_\infty S = gr(\phi)$ is itself contained in the convex hull. Bonsante and Schlenker proved that for every orientation-preserving homeomorphism ϕ, the width is at most $\pi/2$, and it is strictly less than $\pi/2$ precisely when ϕ is quasisymmetric.

The first purpose of the paper [Sep19b] was to study the quantitative relations between the cross-ratio norm of ϕ (which is introduced below), the width w of its convex hull, and the supremum $||\lambda||_\infty$ of the principal curvatures of the maximal surface S of nonpositive curvature such that $\partial_\infty S = gr(\phi)$. By the above discussion, $||\phi||_{cr} < +\infty$ if and only if $w < \pi/2$ if and only if $||\lambda||_\infty < 1$, but it is not clear whether there is a direct relation between these quantities.

Width and principal curvatures The study of the relation between the principal curvatures of a maximal surface and the width of the convex hull is split into two parts. Observe that the principal curvatures of S vanish identically when S is a totally geodesic plane, in which case the width is zero since the convex hull consists of S itself. The first result of [Sep19b] describes the behaviour of maximal surfaces which are close to being a totally geodesic plane:

Theorem 5.2.3. *There exists a constant C_1 such that, for every maximal surface S with $||\lambda||_\infty < 1$ and width w,*

$$||\lambda||_\infty \leq C_1 \tan w \, .$$

This theorem provides interesting information only when w is in some neighborhood of zero, since for large w the already know bound $||\lambda||_\infty < 1$ is not improved. On the other hand, Bonsante and Schlenker showed that if a maximal surface of nonpositive curvature has a point where the principal curvatures are -1 and 1, then the principal curvatures are -1 and 1 everywhere, and therefore the induced metric is flat. Moreover, the surface is a so-called *Barbot surface*, which is described explicitly and has width $\pi/2$. The second theorem concerns surfaces which are close to this situation:

Theorem 5.2.4. *There exist universal constants $M > 0$ and $\delta \in (0,1)$ such that, if S is a maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $\delta \leq ||\lambda||_\infty < 1$ and width w, then*

$$\tan w \geq \left(\frac{1}{1 - ||\lambda||_\infty} \right)^{1/M} .$$

It is worth remarking here that an inequality going in the opposite direction can be obtained more easily, leading to the following:

Proposition 5.2.5. *Let S be a maximal surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $||\lambda||_\infty \leq 1$ and width w. Then*

$$\tan w \leq \frac{2||\lambda||_\infty}{1-||\lambda||_\infty^2} \, .$$

Since $2||\lambda||_\infty/(1-||\lambda||_\infty^2)$ behaves like $2||\lambda||_\infty$ as $||\lambda||_\infty \to 0$, one sees that Theorem 5.2.3 is optimal for small $||\lambda||_\infty$, up to determining the best possible value of the constant C_1. On the other hand, from Proposition 5.2.5 one obtains that $\tan w \leq 2/(1-||\lambda||_\infty)$, and it remains an open question whether Theorem 5.2.4 can be improved to an inequality of the form $\tan w \geq C_2^{-1}(1-||\lambda||_\infty)^{-1}$.

Quasiconformal dilatation A classical problem in Teichmüller theory concerns quasiconformal extensions to the disc of quasisymmetic homeomorphisms of the circle. Recall that quasisymmetric homeomorphisms $\phi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ are characterized by the finiteness of the *cross-ratio norm*, which is defined as:

$$||\phi||_{cr} = \sup_{cr(Q)=-1} |\ln |cr(\phi(Q))|| \, .$$

Given a quasisymmetric homeomorphism, classical quasiconformal extensions include, for instance, the Beurling-Ahlfors extension and the Douady-Earle extension. More recently, Markovic [Mar17] proved the existence of quasiconformal harmonic extensions, where the harmonicity is referred to the complete hyperbolic metric of $\mathbb{H}^2$.

Moreover, the maximal dilatation of the classical extensions has been widely studied. For instance, Beurling and Ahlfors in [BA56] proved that, if Φ_{BA} is the Beurling-Ahlfors extension of a quasisymmetric homeomorphism ϕ, then the maximal dilatation $K(\Phi_{BA})$ satisfies:

$$\ln K(\Phi_{BA}) \leq 2||\phi||_{cr} \, .$$

The asymptotic behaviour was later improved in [Leh83] by

$$\ln K(\Phi_{BA}) \leq ||\phi||_{cr} + \ln 2 \, .$$

For the Douady-Earle extension, [DE86] proved that there exist constants δ and C such that, for every quasisymmetric homeomorphism of the circle ϕ with $||\phi||_{cr} < \delta$, the Douady-Earle extension Φ_{DE} satisfies:

$$\ln K(\Phi_{DE}) \leq C||\phi||_{cr} \, .$$

More recently, Hu and Muzician proved in [HM12] that the following always holds:

$$\ln K(\Phi_{DE}) \leq C_1 ||\phi||_{cr} + C_2 \, .$$

One of the main achievements of [Sep19b] was to obtain analogous results for the minimal Lagrangian extension, whose existence was proved in [BS10] as already remarked. As application of Theorem 5.2.3 leads to the following inequality:

Theorem 5.2.6. *There exist universal constants δ and C_1 such that, for any quasisymmetric homeomorphism ϕ of $\mathbb{RP}^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian extension $\Phi_{ML} : \mathbb{H}^2 \to \mathbb{H}^2$ has maximal dilatation bounded by:*

$$\ln K(\Phi_{ML}) \leq C_1 ||\phi||_{cr} \, .$$

On the other hand, an application of Theorem 5.2.4 led to an asymptotic estimate of the maximal dilatation of Φ_{ML}:

Theorem 5.2.7. *There exist universal constants Δ and C_2 such that, for any quasisymmetric homeomorphism ϕ of $\mathbb{RP}^1$ with cross ratio norm $||\phi||_{cr} > \Delta$, the minimal Lagrangian extension $\Phi_{ML} : \mathbb{H}^2 \to \mathbb{H}^2$ has maximal dilatation bounded by:*

$$\ln K(\Phi_{ML}) \leq C_2 ||\phi||_{cr} \, .$$

Proposition 5.2.5 also provides an inequality in the converse direction, which holds for quasisymmetric homeomorphisms with small cross-ratio norm and shows that Theorem 5.2.6 is essentially not improvable.

Theorem 5.2.8. *There exist universal constants δ and C_0 such that, for any quasisymmetric homeomorphism ϕ of $\mathbb{RP}^1$ with cross ratio norm $||\phi||_{cr} < \delta$, the minimal Lagrangian extension $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ has maximal dilatation bounded by:*

$$C_0 ||\phi||_{cr} \leq \ln K(\Phi_{ML}) \, .$$

The constant C_0 can be taken arbitrarily close to $1/2$.

Putting together Theorem 5.2.6 and Theorem 5.2.7 led to the following:

Corollary 5.2.9. *There exists a universal constant C such that, for any quasisymmetric homeomorphism ϕ of $\mathbb{RP}^1$, the minimal Lagrangian extension $\Phi_{ML} : \mathbb{H}^2 \to \mathbb{H}^2$ has maximal dilatation $K(\Phi_{ML})$ bounded by:*

$$\ln K(\Phi_{ML}) \leq C ||\phi||_{cr} \, .$$

Corollary 5.2.9 is therefore a result for minimal Lagrangian extensions comparable to what has been proved for Beurling-Ahlfors and Douady-Earle extensions. In [Sep19c] I have moreover studied some explicit examples of minimal Lagrangian extensions in order to study the sharpness of these results, thus leading to constraints on the possible values of the optimal constant in the above inequalities.

Width and cross-ratio norm The bridge from Theorem 5.2.3 to Theorem 5.2.6, and from Theorem 5.2.4 to Theorem 5.2.7, is twofold. The first aspect is a direct relation between the supremum of the principal curvatures of a maximal surface S and the quasiconformal dilatation $K(\Phi_{ML})$ of the minimal Lagrangian extension, which is given by the following formula:

$$K(\Phi_{ML}) = \left(\frac{1+||\lambda||_\infty}{1-||\lambda||_\infty}\right)^2 .$$

On the other hand, the step from the width to the cross-ratio norm is more subtle. This is the content of the following proposition:

Proposition 5.2.10. *Given any quasisymmetric homeomorphism ϕ of $\mathbb{R}\mathrm{P}^1$, let w be the width of the convex hull of the graph of ϕ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$. Then*

$$\tan w \leq \sinh\left(\frac{||\phi||_{cr}}{2}\right).$$

By means of these two relations and some computations, Theorem 5.2.6 and Theorem 5.2.7 are proved on the base of Theorem 5.2.3 and Theorem 5.2.4.

To prove Proposition 5.2.10, assuming that the width is w, one can essentially find two support planes P_- and P_+ for the convex hull of $gr(\phi)$, on the two different sides of the convex hull, such that P_- and P_+ are connected by a timelike geodesic segment of length w. Using that the boundaries of the convex hull are pleated surfaces one can pick four points in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$ - two in the boundary at infinity $\partial_\infty P_-$ and the other two in $\partial_\infty P_+$ - and use such four points to show that the cross-ratio norm of ϕ is large. Turning this qualitative picture into quantitative estimates, leading to the proof of Proposition 5.2.10, involves careful and somehow technical constructions in Anti-de Sitter space.

Similar techniques also permit to prove an inequality in the converse direction, which is the content of the following proposition.

Proposition 5.2.11. *Given any quasisymmetric homeomorphism ϕ of $\mathbb{R}\mathrm{P}^1$, let w the width of the convex hull of the graph of ϕ in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$. Then*

$$\tanh\left(\frac{||\phi||_{cr}}{4}\right) \leq \tan w .$$

This inequality, however, is clearly not optimal, as the hyperbolic tangent tends to 1 as $||\phi||_{cr}$ tends to infinity. Hence the inequality is interesting only for $w < \pi/4$. Nevertheless, this inequality is used to obtain Theorem 5.2.8 from Proposition 5.2.5. To prove Proposition 5.2.11, one can assume the cross-ratio norm is $||\phi||_{cr}$ and - composing with

Möbius transformations in an appropriate way - construct a quadruple points in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$. Then we consider two spacelike lines connecting two pairs of points at infinity chosen in the above quadruple. By construction, those two lines are contained in the convex hull of $gr(\phi)$, hence the maximal length of a timelike geodesic segment between them provides a bound from below on the width.

5.3 K-surfaces, CMC surfaces, and landslides

In this section, we shall focus on other classes of (hyper)surfaces in Anti-de Sitter space, namely *constant mean curvature* hypersurfaces, for which the mean curvature is equal to a constant $H \in \mathbb{R}$ (and thus include maximal surfaces, which correspond to $H = 0$), and (for surfaces, namely two-dimensional submanifolds in $\mathbb{A}\mathrm{d}\mathbb{S}^3$) K-surfaces, defined by the condition that the *intrinsic curvature* is equal to a constant K which is supposed here to be in $(-\infty, -1)$. By Gauss' equation in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, $K = -1 - \kappa$ where κ is the determinant of the shape operator, also called *Gaussian curvature*. Hence the condition $K \in (-\infty, -1)$ corresponds to $\kappa > 0$, and thus to the convexity of the surface.

In dimension two, there two classes are related, at least locally, by a *normal flow* construction. Indeed, given an immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ of constant Gaussian curvature $\kappa > 0$, the normal evolution on the convex side of σ, for time $t_\kappa = \arctan(\kappa^{1/2})$ is an immersion of constant mean curvature $H = \kappa^{-1/2}(\kappa - 1)$.

5.3.1 Existence for K-surfaces

In [BS18], we studied existence and uniqueness results for K-surfaces with a given asymptotic behaviour. The first result concerns the case of boundaries at infinity which are graphs of an orientation-preserving homeomorphism of $\partial\mathbb{H}^2$:

Theorem 5.3.1. *Given any orientation-preserving homeomorphism $\phi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$, the two connected components of the complement of the convex hull of $\Lambda = graph(\phi)$ in the domain of dependence of Λ are both foliated by K-surfaces S_K, as $K \in (-\infty, -1)$, in such a way that if $K_1 < K_2$, then S_{K_2} is in the convex side of S_{K_1}.*

Analogously to the case of hyperbolic space $\mathbb{H}^3$ (as proved in [RS94]), there exists two K-surfaces with asymptotic boundary Λ, one of which is past-convex and the other future-convex. Theorem 5.3.1, in the case of ϕ a quasisymmetric homeomorphism, gives positive answer to the existence part of Question 8.3 in [BBD$^+$12].

Let us now turn to the more general case of asymptotic boundaries which are non-negative 1-spheres (which we call here *non-negative circles*), in the language of Chapter 4. There is a major difference with the formulation of the asymptotic Plateau problem for K-surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, as opposed to maximal surface. The main reason is that, in general, a non-negative circle Λ can contain null segments. In particular, if Λ contains a *sawtooth*, that is, the union of adjacent "horizontal" and "vertical" segments in $\mathbb{RP}^1 \times \mathbb{RP}^1 \cong \partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3$, then the convex hull of the sawtooth is a lightlike totally geodesic triangle, which is contained both in the boundary of the convex hull of Λ and in the boundary of the domain of dependence of Λ. Hence any (future or past) convex surface with boundary Λ must necessarily contain such lightlike triangle.

An example is a 2-*step curve* is the union of four segments, two horizontal and two vertical in an alternate way. It is not possible to have a convex surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with boundary a 2-step curve.

The proof of Theorem 5.3.1 actually extends to the case of a general non-negative circle Γ, except the degenerate case described above.

Theorem 5.3.2. *Given any non-negative circle Λ in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ which is not a 2-step curve, for every $K \in (-\infty, -1)$ there exists a past-convex (resp. future-convex) surface S_K^+ (resp. S_K^-) with $\partial S_K^\pm = \Lambda$, such that:*

- *Its lightlike part is union of lightlike triangles associated to sawteeth;*
- *Its spacelike part is a smooth K-surface.*

Moreover, the two connected components of the complement of the convex hull of Λ in the domain of dependence of Λ are both foliated by the spacelike part of surfaces $S_K^\pm$, as $K \in (-\infty, -1)$, in such a way that if $K_1 < K_2$, then $S_{K_2}^\pm$ is in the convex side of $S_{K_1}^\pm$.

In [BBZ11], the existence (and uniqueness) of a foliation by K-surfaces was proved in the complement of the convex core of any maximal globally hyperbolic Anti-de Sitter spacetime containing a compact Cauchy surface. Using results of [Mes07], this means that the statement of Theorem 5.3.2 holds for curves Λ which are the graph of an orientation-preserving homeomorphism which conjugates two Fuchsian representations of the fundamental group of a closed surface in $\mathrm{Isom}(\mathbb{H}^2)$. Moreover, the K-surfaces are invariant for the representation in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ given by the product of the two Fuchsian representations.

The proof of Theorem 5.3.1, and more generally Theorem 5.3.2, presented in [BS18], relies on an approximation from the case of [BBZ11]. Some technical tools are needed. First, we needed to show that it is possible to approximate any weakly spacelike curve Γ by curves invariant by a pair of Fuchsian representations. For this purpose, we adapted a technical lemma proved in [BS17].

Second, we used a theorem of Schlenker ([Sch96b]) which, in this particular case, essentially ensures that a sequence S_n of K-surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ converges C^∞ to a spacelike surface S_∞ (up to subsequences) unless they converge to a totally geodesic lightlike plane (whose boundary at infinity is not a non-negative sphere) or to the union of two totally geodesic lightlike half-planes, meeting along a spacelike geodesic (in this case the boundary is a 2-step curve).

To apply the theorem of Schlenker, and deduce that the limiting surface S_∞ is a K-surface with $\partial S_\infty = \Lambda$ (thus proving Theorem 5.3.1), one has to prove that S_∞ does not intersect the boundary of the domain of dependence of Γ. More in general, for the proof of Theorem 5.3.2, one must show that the *spacelike part* of S_∞ does not intersect the boundary of the domain of dependence of Λ. This is generally the most difficult step in this type of problems, and frequently requires the use of *barriers*. In fact, it is possible to compute (by means of a reduction to an ODE problem) a 1-parameter family of smooth, spacelike K-surfaces whose boundary coincides with the boundary of a totally geodesic spacelike half-plane in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. In other words, Theorem 5.3.2 is proved by a *hands-on* approach when the curve Λ is the union of two null segments and the boundary of a totally geodesic half-plane, in the boundary at infinity of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Such K-surfaces are then fruitfully used as *barriers* to conclude the proof of Theorem 5.3.1 — and the proof actually works under the more general hypothesis of Theorem 5.3.2.

In the case Λ is the graph of a quasisymmetric homeomorphism, in [BS18] we then proved that the K-surfaces with boundary Λ are unique. Moreover, it is not difficult to prove that if S is a convex surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $\partial S = \Lambda$ and with bounded principal curvatures, then Λ is the graph of a quasisymmetric homeomorphism. We gave a converse statement for K-surfaces, namely, a K-surface with boundary $\Lambda = gr(\phi)$, for ϕ quasisymmetric, necessarily has bounded principal curvatures.

Theorem 5.3.3. *Given any quasisymmetric homeomorphism $\phi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$, for every $K \in (-\infty, -1)$ there exists a unique future-convex K-surface S_K^+ and a unique past-convex K-surface S_K^- in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $\partial S_K^\pm = gr(\phi)$. Moreover, the principal curvatures of $S_K^\pm$ are bounded.*

To prove uniqueness, the standard arguments for these problems are applications of the *maximum principle*, by using the existence of a foliation $\{S_K\}$ by K-surfaces and showing that any other K-surface S'_K must coincide with a leaf of the given foliation. However, in this case, due to non-compactness of the surfaces, one would need a form of the maximum principle *at infinity*. This is achieved more easily in this case by applying isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ so as to bring a maximizing (or minimizing) sequence on S'_K to a compact region of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Then one applies two main tools: the first is again the convergence theorem of Schlenker, and the second is a compactness result for quasisymmetric homeomorphisms with uniformly bounded cross-ratio norm. Up to subsequences, both the isometric images of S'_K and the isometric images of the leaves $\{S_K\}$ of the foliation converge to an analogous configuration in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. But now it is possible to apply the classical maximum principle to conclude the argument.

5.3.2 Extensions by θ-landslides

Let us now introduce *θ-landslides*, a natural generalization of minimal Lagrangian maps introduced in [BMS13]. These maps turn out to be precisely the maps associated to CMC and constant Gaussian curvature surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Given two hyperbolic metrics h and h' on a surface S, and $\theta \in (0,\pi)$ a θ-landslide $f_\theta : (S,h) \to (S,h')$ is a smooth map which satisfies one of the equivalent conditions:

1. There exists a smooth $(1,1)$-tensor A such that (if $\mathcal{J}_h$ is the almost-complex structure of h):
$$f_\theta^* h' = h(((\cos\theta)\mathrm{id} + (\sin\theta)\mathcal{J}_h \circ A)\cdot, ((\cos\theta)\mathrm{id} + (\sin\theta)\mathcal{J}_h \circ A)\cdot)$$
which is positive-definite, h-symmetric, h-Codazzi and has unit determinant.

2. There exist harmonic maps $f : (S,X) \to (S,h)$ and $f' : (S,X) \to (S,h')$, where X is a conformal structure on S, such that $f_\theta = f' \circ f^{-1}$ whose Hopf differentials satisfy
$$\mathrm{Hopf}(f) = e^{2i\theta}\mathrm{Hopf}(f')\ .$$
Moreover, in the non-compact case, one has to further impose that f and f' have the same holomorphic energy density.

When $\theta = \pi/2$ we recover minimal Lagrangian maps, as the above two conditions are in fact equivalent to one of the possible characterizations of minimal Lagrangian maps – namely, being the composition of two harmonic maps with opposite Hopf differentials. It then turns out that θ-landslides are precisely the maps associated to surfaces of constant mean curvature $H = 2/\tan\theta$, and therefore also to the two equidistant surfaces of constant Gaussian curvature $\tan^2(\theta/2)$ and $1/\tan^2(\theta/2)$.

By interpreting Theorems 5.3.2 and 5.3.3 in this context, we can draw a direct consequence on the existence of landslide extensions, thus generalizing the result of [BS10] on minimal Lagrangian extensions:

Corollary 5.3.4. *Given any quasisymmetric homeomorphism $\phi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ and any $\theta \in (0,\pi)$, there exist a unique θ-landslide $\Phi_\theta : \mathbb{H}^2 \to \mathbb{H}^2$ which extends ϕ. Moreover, Φ_θ is quasiconformal.*

It is worth remarking that, when θ approaches 0, then one of the two associated surfaces of constant Gaussian curvature (namely the one having Gaussian curvature $\tan^2(\theta/2)$) approaches a boundary component of the convex core of the ambient manifold (M,g), while the other escapes at infinity in the other end of (M,g). When θ approaches π instead, the roles are switched. Hence the landslide maps f_θ converge to the left and right earthquake maps between (Σ,h) and (Σ,h') as θ diverges in its interval of definition $(0,\pi)$. Morally, θ-landslides are a natural one-parameter family of smooth extensions which interpolate between left earthquake, minimal Lagrangian maps, and right earthquakes.

From the above discussion, it is clear that several types of maps from $\mathbb{H}^2$ to itself, or (in the quotient) from a closed hyperbolic surface to another, can be constructed by means of special types of spacelike surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. The problem of characterizing the maps that can be obtained by means of this construction was studied in [BS19] and [Sep18].

5.3.3 CMC hypersurfaces

The PhD thesis of Enrico Trebeschi, whom I co-supervise with Francesco Bonsante, aims to study CMC hypersurfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^n$. In his first article [Tre23], Enrico proved the existence and uniqueness of the asymptotic Plateau problem for CMC hypersurfaces, having a prescribed non-negative $(n-1)$-sphere as the asymptotic boundary in $\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^n$. Moreover, Enrico proved that these hypersurfaces are always complete, which can be seen as the analogue in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ of the completeness of entire CMC hypersurfaces in Minkowski space by Cheng and Yau ([CY76]). In a second article, in preparation, Enrico is obtaining estimates on the curvature of these CMC hypersurfaces in terms of the width, in the spirit of (and generalizing) Theorem 5.2.3. When $n = 3$, these estimates will thus lead to inequalities on the maximal dilatation of the θ-landslides, generalizing Theorem 5.2.6 to any value of θ.

5.4 Volume of Anti-de Sitter manifolds

The volume is, in general, an extremely rich invariant of pseudo-Riemannian geometric manifolds. In the celebrated paper [Bro03], Brock proved that the volume of the convex core of a quasi-Fuchsian manifold M behaves coarsely like the Weil-Petersson distance between the two components in $\mathcal{T}(S) \times \mathcal{T}(S)$ provided by Bers' parameterization ([Bro03]). The main purpose of the article [BST17] is to study how the volume of the convex core of maximal globally hyperbolic manifolds is related to some analytic or geometric quantities only depending on the two parameters in Teichmüller space provided by Mess' parameterization.

A first main difference between the quasi-Fuchsian and the Anti-de Sitter setting consists in the fact that the volume of the whole maximal globally hyperbolic Anti-de Sitter manifold $M_{h,h'}$ is always finite. By considering the foliation by constant curvature surfaces ([BBZ11]) of the complement of the convex core, one can show that the volume of $M_{h,h'}$ and the volume of its convex core are coarsely equivalent. More precisely, in [BST17] we proved the following:

Proposition 5.4.1. *Given a maximal globally hyperbolic manifold M, let M_- and M_+ be the two connected components of the complement of $\mathcal{C}(M)$. Then*

$$\mathrm{Vol}(M_-) \leq \frac{\pi^2}{2}|\chi(S)| \qquad \textit{and} \qquad \mathrm{Vol}(M_+) \leq \frac{\pi^2}{2}|\chi(S)| ,$$

with equality if and only M is Fuchsian.

Using a foliation by equidistant surfaces from the boundary of the convex core, one can then prove the following formula (see also [BBD$^+$12] and [BB09, Subsection 8.2.3]) which connects the volume of the convex core, the volume of M_-, and the length of the left earthquake lamination λ:

$$\mathrm{Vol}(\mathcal{C}(M)) + \mathrm{Vol}(M_-) = \frac{1}{4}\ell_\lambda(h) + \frac{\pi^2}{2}|\chi(S)| \ . \tag{5.2}$$

5.4.1 L^1-energies and length of earthquake laminations

Let us use h to denote the class of a hyperbolic metric in $\mathcal{T}(S)$, and $M_{h,h'}$ will denote the maximal globally hyperbolic manifold corresponding to the point $(h,h') \in \mathcal{T}(S) \times \mathcal{T}(S)$ in Mess' parameterization. The main result of [BST17] is the fact that the volume of a maximal globally hyperbolic Anti-de Sitter manifold roughly behaves like the minima of certain types of L^1-energies of maps $f : (S,h) \to (S,h')$. In fact, in his groundbreaking preprint [Thu98] about the Lipschitz asymmetric distance, Thurston suggested the interest in studying other type of L^p-energies, in contrast to the case $p = \infty$ corresponding to the Lipschitz distance. In [BST17] we considered the functional which corresponds to $p = 1$:

$$C^1_{\mathrm{id}}(S) \ni f \mapsto \int_S ||df|| d\mathrm{A}_h \ ,$$

where $C^1_{\mathrm{id}}(S)$ denotes the space of C^1 self-maps of S homotopic to the identity, and $||df||$ is the norm of the differential of f, computed with respect to the metrics h and h' on S. This functional is usually called L^1-*energy*, or *total variation*, as it coincides with the total variation in the sense of BV maps. Our main result is the following:

Theorem 5.4.2. *Let $M_{h,h'}$ be a maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifold. Then*

$$\frac{1}{4}\inf_{f\in C^1_{\mathrm{id}}(S)}\int_S ||df|| d\mathrm{A}_h - \frac{\sqrt{2}}{2}\pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \frac{\sqrt{2}}{2}\inf_{f\in C^1_{\mathrm{id}}(S)}\int_S ||df|| d\mathrm{A}_h \ .$$

Observe that the volume of the convex core vanishes precisely for Fuchsian manifolds, that is, on those manifolds containing a totally geodesic spacelike surface. Those manifolds correspond to the diagonal in $\mathcal{T}(S) \times \mathcal{T}(S)$, that is, $h = h'$. In this case, a direct computation shows that the left-hand-side in the inequality of Theorem 5.4.2 vanishes. Indeed, the L^1-energy is minimized by the identity map $f = \mathrm{id} : (S,h) \to (S,h)$, and for this map $||df|| = \sqrt{2}$ at every point.

Theorem 5.4.2 will follow from two more precise statements about the behavior of the volume of the convex core, namely Theorem 5.4.3 and Theorem 5.4.4 below. The former uses the relation of Anti-de Sitter geometry, and in particular maximal surfaces (i.e. with vanishing mean curvature), with minimal Lagrangian maps between hyperbolic surfaces. The latter relies instead on the connection between pleated surfaces and earthquake maps.

Let us consider first the 1-*Schatten energy*. Given two hyperbolic surfaces (S,h) and

(S,h'), this functional, which we denote $E_{Sch}(\cdot,h,h')$, is defined as:

$$C^1_{\mathrm{id}}(S) \ni f \mapsto \int_S \mathrm{tr}\left(\sqrt{df^*df}\right) d\mathrm{A}_h \ ,$$

where df^* is the h-adjoint operator of the differential df, and $\sqrt{df^*df}$ denotes the unique positive, symmetric square root of the operator df^*df. When f is orientation-preserving, the functional $E_{Sch}(f,h,h')$ actually coincides with the *holomorphic L^1-energy*, which was already studied in [TV95], and is defined (on the space $\mathrm{Diff}_{\mathrm{id}}(S)$ of diffeomorphisms isotopic to the identity) by:

$$\mathrm{Diff}_{\mathrm{id}}(S) \ni f \mapsto \int_S ||\partial f|| d\mathrm{A}_h \ ,$$

where $||\partial f||$ is the norm of the $(1,0)$-part of the differential of f. In [TV95], Trapani and Valli proved that this functional admits a unique minimum, which coincides with the unique minimal Lagrangian diffeomorphism $m : (S,h) \to (S,h')$ isotopic to the identity (see also [Lab92] and [Sch93]). Using the Gauss map construction, described above in this chapter, which associates a minimal Lagrangian diffeomorphism from (S,h) to (S,h'), isotopic to the identity, to the unique maximal surface in $M_{h,h'}$, we obtained the following theorem which gives a precise description of the coarse behavior of the volume of the convex core in terms of the 1-Schatten energy.

Theorem 5.4.3. *Let $M_{h,h'}$ be a maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifold. Then*

$$\frac{1}{4}E_{Sch}(m,h,h') - \pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}E_{Sch}(m,h,h') \ ,$$

where $m : (S,h) \to (S,h')$ is the minimal Lagrangian map isotopic to the identity, that is, the minimum of the 1-Schatten energy functional $E_{Sch}(\cdot,h,h') : C^1_{\mathrm{id}}(S) \to \mathbb{R}$.

Again, the left-hand-side vanishes precisely on Fuchsian manifolds, that is, precisely when $\mathrm{Vol}(\mathcal{C}(M_{h,h'}))$ vanishes as well. We remark that in the proof of Theorem 5.4.3 we need the fact that the minimal Lagrangian map actually minimizes $E_{Sch}(\cdot,h,h')$ on $C^1_{\mathrm{id}}(S)$, which follows from the theorem of Trapani and Valli, and the convexity of the functional $E_{Sch}(\cdot,h,h')$. The upper bound in Theorem 5.4.2 then follows from Theorem 5.4.3, by using that, for any $f \in \mathrm{Diff}_{\mathrm{id}}(S)$ and every $x \in S$, $\mathrm{tr}(\sqrt{df_x^*df_x}) \leq \sqrt{2}||df_x||$.

A more combinatorial version of the relation between maximal surfaces and minimal Lagrangian maps is the association, already discovered by Mess, of (left and right) earthquake maps from (S,h) to (S,h') from the two pleated surfaces which form the boundary of the convex core of $M_{h,h'}$. The earthquake theorem, which provides the existence of a unique left (and a unique right) earthquake map from (S,h) to (S,h'), produces two *measured geodesic laminations*. The *length* of a measured geodesic lamination is then the unique continuous homogeneous function which extends the length of simple closed geodesics.

If we denote by $E^\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ the transformation which associates to $h \in \mathcal{T}(S)$ the metric $h' = E^\lambda(h)$ obtained by a (left or right) earthquake along λ, the following result — obtained by combining Equation (5.2) with Proposition 5.4.1 — gives a relation between the volume and the length of (any of the two) earthquake laminations:

Theorem 5.4.4. *Given a maximal globally hyperbolic manifold $M_{h,h'}$, let λ be the (left or right) earthquake lamination such that $E^\lambda(h) = h'$. Then*

$$\frac{1}{4}\ell_\lambda(h) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{1}{4}\ell_\lambda(h) + \frac{\pi^2}{2}|\chi(S)| \ .$$

The lower inequality of Theorem 5.4.2 is then a consequence of Theorem 5.4.4 and the basic observation that the total variation of the earthquake map along the lamination λ is at most $\ell_\lambda(h) + 2\sqrt{2}\pi|\chi(S)|$.

Another straightforward consequence of these results is the fact that the length of the left and right earthquake laminations, and the holomorphic energy of the minimal Lagrangian map (up to a factor), are comparable. Namely, their difference is bounded only in terms of the topology of S. For instance:

Corollary 5.4.5. *Given two hyperbolic metrics h and h' on S, if λ_l and λ_r are the measured laminations such that $E_l^{\lambda_l}(h) = h'$ and $E_r^{\lambda_r}(h) = h'$, then*

$$|\ell_{\lambda_l}(h) - \ell_{\lambda_r}(h)| \leq 2\pi^2|\chi(S)| \ .$$

Corollary 5.4.5 seems to be a non-trivial result to obtain using only techniques from hyperbolic geometry.

5.4.2 Metrics on Teichmüller space

A result like Brock's Theorem for maximal globally hyperbolic Anti-de Sitter manifolds, replacing Bers' parameterization by Mess' parameterization, turns out not to be true. The problem of relating the volume $\mathrm{Vol}(\mathcal{C}(M_{h,h'}))$ to the distance between h and h' for some metric structure on $\mathcal{T}(S)$ was mentioned in [BBD+12, Question 4.1]. We showed that the volume of the convex core of a maximal globally hyperbolic manifold $M = M_{h,h'}$ is bounded asymptotically from above by Thurston's asymmetric distance between (S, h) and (S, h') (actually by the minimum of the two asymmetric distances), from below by the Weil-Petersson distance. Neither of these bounds holds on both sides, hence this seems to be the best affirmative answer one can give to this question.

Recall that Thurston's distance $d_{\mathrm{Th}}(h, h')$ is the logarithm of the best Lipschitz constant of diffeomorphisms from (S, h) to (S, h'), isotopic to the identity. This definition satisfies the properties of a distance on $\mathcal{T}(S)$, except the symmetry. As the norm of the differential $||df||$, which appears in Theorem 5.4.2, is bounded pointwise by the Lipschitz constant of f, we derive the following bound from above of the volume with respect to the minimum of the two asymmetric distances:

Theorem 5.4.6. *Let $M_{h,h'}$ be a maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifold. Then*

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \pi|\chi(S)| \exp(\min\{d_{\mathrm{Th}}(h,h'), d_{\mathrm{Th}}(h',h)\}) \ .$$

However, the volume of the convex core is not coarsely equivalent to the minimum of Thurston asymmetric distances, as we can produce examples of manifolds M_{h_n,h'_n} in which the minimum $\min\{d_{\mathrm{Th}}(h_n,h'_n), d_{\mathrm{Th}}(h'_n,h_n)\}$ goes to infinity while $\mathrm{Vol}(\mathcal{C}(M_{h_n,h'_n}))$ stays bounded, thus showing that there cannot be a bound from below on the volume using any of Thurston's asymmetric distances. However, in these examples the systole of both h_n and h'_n go to 0, and this condition is necessary for this phenomenon to happen. More precisely, the volume $\mathrm{Vol}(\mathcal{C}(M_{h,h'}))$ is coarsely equivalent to the minimum of Thurston asymmetric distances if one of the two points h, h' lie in the ϵ-thick part of $\mathcal{T}(S)$.

In [BST17] we also provided a sequence of maximal globally hyperbolic manifolds M_{h_g,h'_g}, for the surface S_g of genus $g \geq 2$, in which the volume $\mathrm{Vol}(\mathcal{C}(M_{h_g,h'_g}))$ has roughly the same growth as $|\chi(S_g)| \exp(\min\{d_{\mathrm{Th}}(h_g,h'_g), d_{\mathrm{Th}}(h'_g,h_g)\})$. Hence the growth, with respect to the genus, of the multiplicative factor in Theorem 5.4.6 is essentially optimal.

On the other hand, we obtain a coarse bound from below on the volume of the convex core of $M_{h,h'}$ by using the Weil-Petersson distance $d_{\mathrm{WP}}(h,h')$.

Theorem 5.4.7. *Let $M_{h,h'}$ be a maximal globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifold. Then there exist some positive constants $a, b, c > 0$ such that*

$$\exp\left(\frac{a}{|\chi(S)|} d_{\mathrm{WP}}(h,h') - b|\chi(S)|\right) - c \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \ .$$

There are examples in which $d_{\mathrm{WP}}(h_n,h'_n)$ remains bounded, but $\mathrm{Vol}(\mathcal{C}(M_{h_n,h'_n}))$ diverges, thus the volume of the convex core of $M_{h,h'}$ cannot be bounded from above by the Weil-Petersson distance between h and h'.

5.5 The para-hyperKähler structure of deformation space

In [Don03], Donaldson highlighted the existence of a natural *hyperKähler* structure on a neighborhood of the Fuchsian locus in the deformation space of almost-Fuchsian manifolds, seen as a neighborhood of the zero section in the cotangent bundle $T^*\mathcal{T}(\Sigma)$. See also [Hod05, Tra18, Tra19]. In the article [MST23a], we developed a similar approach in the context of maximal globally hyperbolic Anti-de Sitter manifolds, and we demonstrated that the natural structure that appears in this setting is a *para-hyperKähler structure*.

5.5.1 Para-hyperKähler structures

Let us introduce the notion of para-hyperKähler structure. For more details on para-Kähler and para-hyperKähler geometry, see [CFG96, GMV01, AMT09, Vac12]. Recall that a *pseudo-Kähler structure* on a manifold M consists of a pair $(\mathbf{g}, \mathbf{I})$ where $\mathbf{g}$ is a pseudo-Riemannian metric and $\mathbf{I}$ is an integrable almost complex structure (i.e. $\mathbf{I}^2 = -\mathbb{1}$) such that $\mathbf{g}(\mathbf{I}v, w) = -\mathbf{g}(v, \mathbf{I}w)$ and the 2-form $\omega_{\mathbf{I}}(\cdot, \cdot) := \mathbf{g}(\cdot, \mathbf{I}\cdot)$ is closed (hence a symplectic form). Similarly, a *para-Kähler structure* consists of an integrable almost para-complex structure $\mathbf{P}$, which means that

- $\mathbf{P}^2 = \mathbb{1}$;
- the $\mathbf{P}$-eigenspaces of 1 and -1 have the same dimension;
- the distributions on M given by the 1 and -1 eigenspaces of $\mathbf{P}$ are integrable;

and $\mathbf{P}$ is such that $\mathbf{g}(\mathbf{P}v, w) = -\mathbf{g}(v, \mathbf{P}w)$ and the 2-form $\omega_{\mathbf{P}}(\cdot, \cdot) := \mathbf{g}(\cdot, \mathbf{P}\cdot)$ is closed.

Observe that a direct consequence of the existence of a para-Kähler structure is that $\mathbf{g}(\mathbf{P}\cdot, \mathbf{P}\cdot) = -\mathbf{g}(\cdot, \cdot)$, hence $\mathbf{g}$ is necessarily of neutral signature. Moreover the condition that $d\omega_{\mathbf{I}} = 0$ (resp. $d\omega_{\mathbf{P}} = 0$) is known to be equivalent to $\nabla \mathbf{I} = 0$ (resp. $\nabla \mathbf{P} = 0$), for ∇ the Levi-Civita connection of $\mathbf{g}$. Finally, a para-hyperKähler structure on a manifold M is then defined as the data $(\mathbf{g}, \mathbf{I}, \mathbf{J}, \mathbf{K})$, where $(\mathbf{g}, \mathbf{I})$ is a pseudo-Kähler structure, $(\mathbf{g}, \mathbf{J})$ and $(\mathbf{g}, \mathbf{K})$ are para-Kähler structures, and $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ satisfy the para-quaternionic relations.

By para-quaternionic relations we mean the identities $\mathbf{I}^2 = -\mathbb{1}$, $\mathbf{J}^2 = \mathbf{K}^2 = \mathbb{1}$ — which are implicitly assumed by the condition that $\mathbf{I}$ (resp. $\mathbf{J}$, $\mathbf{K}$) is a complex (resp. para-complex) structure — and moreover $\mathbf{IJ} = -\mathbf{JI} = \mathbf{K}$.

We remark that, given a para-hyperKähler structure $(\mathbf{g}, \mathbf{I}, \mathbf{J}, \mathbf{K})$, a *complex symplectic form* is defined by:

$$\omega_{\mathbf{I}}^{\mathbb{C}} := \omega_{\mathbf{J}} + i\omega_{\mathbf{K}} \ .$$

It is complex in the sense that it is a $\mathbb{C}$-valued symplectic form and satisfies $\omega_{\mathbf{I}}^{\mathbb{C}}(\mathbf{I}v, w) = \omega_{\mathbf{I}}^{\mathbb{C}}(v, \mathbf{I}w) = i\omega_{\mathbf{I}}^{\mathbb{C}}(v, w)$. Similarly, one has two *para-complex symplectic forms* defined by

$$\omega_{\mathbf{J}}^{\mathbb{B}} := \omega_{\mathbf{I}} + \tau\omega_{\mathbf{K}} \qquad \text{and} \qquad \omega_{\mathbf{K}}^{\mathbb{B}} := \omega_{\mathbf{I}} - \tau\omega_{\mathbf{J}} \ ,$$

where we denote by $\mathbb{B} = \mathbb{R} \oplus \tau\mathbb{R}$ the algebra of para-complex numbers, i.e. $\tau^2 = 1$. Again, these are para-complex in the sense that $\omega_{\mathbf{J}}^{\mathbb{B}}(\mathbf{J}v, w) = \omega_{\mathbf{J}}^{\mathbb{B}}(v, \mathbf{J}w) = \tau\omega_{\mathbf{J}}^{\mathbb{B}}(v, w)$ and $\omega_{\mathbf{K}}^{\mathbb{B}}(\mathbf{K}v, w) = \omega_{\mathbf{K}}^{\mathbb{B}}(v, \mathbf{K}w) = \tau\omega_{\mathbf{K}}^{\mathbb{B}}(v, w)$.

Only manifolds of dimension $4n$ can support a para-hyperKähler structure. The first result of [MST23a] is that $\mathcal{MGH}(\Sigma)$, whose dimension is four if Σ has genus one and $6|\chi(\Sigma)|$ otherwise, does support a very natural one.

Theorem 5.5.1. *Let Σ be a closed oriented surface of genus ≥ 1. Then $\mathcal{MGH}(\Sigma)$ admits a $MCG(\Sigma)$-invariant para-hyperKähler structure $(\mathbf{g}, \mathbf{I}, \mathbf{J}, \mathbf{K})$. When Σ has genus ≥ 2, the*

Fuchsian locus $\mathcal{F}(\Sigma)$ is totally geodesic and $(\mathbf{g}, \mathbf{I})$ restricts to (a multiple of) the Weil-Petersson Kähler structure of Teichmüller space.

The para-hyperKähler structure of $\mathcal{MGH}(\Sigma)$ is extremely natural from the point of view of AdS geometry, in the sense that all the elements that constitute the para-hyperKähler structure have (at least one) interpretation in terms of the geometry of MGHC AdS manifolds. Let us state and explain all these interpretations.

5.5.2 Parameterizations of $\mathcal{MGH}(\Sigma)$

The first interpretation is in terms of the cotangent bundle of Teichmüller space. There is a natural map

$$\mathcal{F} : \mathcal{MGH}(\Sigma) \to T^*\mathcal{T}(\Sigma) ,$$

which associates to a MGHC AdS manifold $(\Sigma \times \mathbb{R}, G)$ the pair (J, q), where J is the (almost-)complex structure of the first fundamental form of the unique maximal Cauchy surface in (M, G), and q is the holomorphic quadratic differential whose real part is the second fundamental form. The map $\mathcal{F}$ is a ($MCG(\Sigma)$-equivariant) diffeomorphism if Σ has genus ≥ 2; for genus one it is a diffeomorphism onto the complement of the zero section. The cotangent bundle $T^*\mathcal{T}(\Sigma)$ is naturally a complex symplectic manifold; our first geometric interpretation is the fact that the map $\mathcal{F}$ is anti-holomorphic and preserves the complex symplectic forms up to conjugation.

Theorem 5.5.2. *Let Σ be a closed oriented surface of genus ≥ 1. Then*

$$\mathcal{F}^*(\mathcal{I}_{T^*\mathcal{T}(\Sigma)}, \Omega^{\mathbb{C}}_{T^*\mathcal{T}(\Sigma)}) = \left(-\mathbf{I}, -\frac{i}{2}\overline{\omega}^{\mathbb{C}}_{\mathbf{I}}\right) ,$$

where $\mathcal{I}_{T^*\mathcal{T}(\Sigma)}$ denotes the complex structure of $T^*\mathcal{T}(\Sigma)$ and $\Omega^{\mathbb{C}}_{T^*\mathcal{T}(\Sigma)}$ its complex symplectic form.

Let us assume (until the end of this section) that Σ has genus ≥ 2. In [Mes07], Mess proved that $\mathcal{MGH}(\Sigma)$ is parameterized by the product of two copies of the Teichmüller space of Σ, by a map

$$\mathcal{M} : \mathcal{MGH}(\Sigma) \to \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) ,$$

that essentially gives (under the isomorphism between the isometry group of AdS space and $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$), the left and right components of the holonomy map of a MGHC AdS manifold (M, G). The manifold $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ is easily a para-complex manifold, where the para-complex structure $\mathcal{P}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}$ is the endomorphism of the tangent bundle for which the integral submanifolds of the distribution of 1-eigenspaces are the slices $\mathcal{T}(\Sigma) \times \{*\}$, and those for the (-1)-eigenspaces are the slices $\{*\} \times \mathcal{T}(\Sigma)$. It has moreover a para-complex symplectic form compatible with $\mathcal{P}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}$:

$$\Omega_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)} := \frac{1}{2}(\pi_l^*\Omega_{WP} + \pi_r^*\Omega_{WP}) + \frac{\tau}{2}(\pi_l^*\Omega_{WP} - \pi_r^*\Omega_{WP})$$

where Ω_{WP} is the Weil-Petersson symplectic form and π_l, π_r denote the projections on the left and right factor.

Theorem 5.5.3. *Let Σ be a closed oriented surface of genus ≥ 2. Then*

$$\mathcal{M}^*(\mathcal{P}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}, 4\Omega^{\mathbb{B}}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}) = (\mathbf{J}, \omega^{\mathbb{B}}_{\mathbf{J}}) \ ,$$

where $\mathcal{P}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}$ denotes the para-complex structure of $\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$ and $\Omega^{\mathbb{B}}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}$ its para-complex symplectic form.

Combining Theorems 5.5.2 and 5.5.3 in a particular case, we see that $(1/2)\omega_{\mathbf{K}}$ equals on the one hand the pull-back by $\mathcal{M}$ of the symplectic form $\pi_l^*\Omega_{WP} - \pi_r^*\Omega_{WP}$, and on the other hand the pull-back by $\mathcal{F}$ of minus the real part of $\Omega^{\mathbb{C}}_{T^*\mathcal{T}(\Sigma)}$ (i.e. the natural real symplectic form of the cotangent bundle). This identity has been proved in [SS18, Theorem 1.14], by completely different methods.

There is another parameterization of $\mathcal{MGH}(\Sigma)$ by the product of two copies of the Teichmüller space of Σ, which has been introduced in [KS07]. It is given by the map

$$\mathcal{C} : \mathcal{MGH}(\Sigma) \to \mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma) \ ,$$

which associates to (M, G) the first fundamental forms of the two Cauchy surfaces (one future-convex, one past-convex) of constant intrinsic curvature -2. These two Cauchy surfaces of constant curvature are unique ([BBZ11, BS18]), and we rescale their first fundamental forms by a factor so as to consider them as hyperbolic metrics. We showed:

Theorem 5.5.4. *Let Σ be a closed oriented surface of genus ≥ 2. Then*

$$\mathcal{C}^*(\mathcal{P}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}, 4\Omega^{\mathbb{B}}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}) = (\mathbf{K}, \omega^{\mathbb{B}}_{\mathbf{K}}) \ ,$$

where $\mathcal{P}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}$ denotes the para-complex structure of $\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$ and $\Omega^{\mathbb{B}}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}$ its para-complex symplectic form.

We remark that there are formal analogues of Theorem 5.5.3 and Theorem 5.5.4 in genus one, but the corresponding maps $\mathcal{M}, \mathcal{C} : \mathcal{MGH}(T^2) \to \mathcal{T}(T^2)\times\mathcal{T}(T^2)$ do not have the same geometric interpretation (namely, the holonomy map or the constant curvature surfaces) as in the higher genus case, which is why we restricted to genus ≥ 2 when stating these results here.

5.5.3 The circle action

We now move on to describing a *circle action* on $\mathcal{MGH}(\Sigma)$. Using the diffeomorphism $\mathcal{F} : \mathcal{MGH}(\Sigma) \to T^*\mathcal{T}(\Sigma)$, the circle action on $T^*\mathcal{T}(\Sigma)$ given by $e^{i\theta} \cdot (J, q) = (J, e^{i\theta} q)$ (where J is an almost-complex structure on Σ and q a holomorphic quadratic differential) induces an action of S^1 on $\mathcal{MGH}(\Sigma)$. Let us denote by $R_\theta : \mathcal{MGH}(\Sigma) \to \mathcal{MGH}(\Sigma)$ the corresponding self-diffeomorphism. For genus ≥ 2, this action of S^1 induces an action on $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ by means of the map $\mathcal{M}$. The so obtained S^1-action on $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ has been studied in [BMS13, BMS15] under the name of *landslide flow*.

It will be relevant to introduce the function

$$\mathcal{A} : \mathcal{MGH}(\Sigma) \to \mathbb{R}$$

which associates to a MGHC AdS manifold the area of its unique maximal Cauchy surface. It is easy to see that $\mathcal{A}$ is constant on the orbits of the circle action. We showed the following:

Theorem 5.5.5. *Let Σ be a closed oriented surface of genus ≥ 1. The circle action on $\mathcal{MGH}(\Sigma)$ is Hamiltonian with respect to $\omega_{\mathbf{I}}$, and satisfies*

$$R_\theta^* \mathbf{g} = \mathbf{g} \qquad R_\theta^* \omega_{\mathbf{I}} = \omega_{\mathbf{I}} \qquad R_\theta^* \omega_{\mathbf{I}}^{\mathbb{C}} = e^{-i\theta} \omega_{\mathbf{I}}^{\mathbb{C}} \ .$$

When Σ has genus ≥ 2, the function $\mathcal{A}$ is a Hamiltonian function.

We remark that, in terms of the (para-)complex structures $\mathbf{I}, \mathbf{J}, \mathbf{K}$, the pull-back relations of Theorem 5.5.5 read:

$$R_\theta^* \mathbf{I} = \mathbf{I} \qquad R_\theta^* \mathbf{J} = \cos(\theta)\mathbf{J} + \sin(\theta)\mathbf{K} \qquad R_\theta^* \mathbf{K} = -\sin(\theta)\mathbf{J} + \cos(\theta)\mathbf{K} \ . \tag{5.3}$$

In [BMS15], Bonsante, Mondello and Schlenker showed that the landslide flow is Hamiltonian with respect to the symplectic form $\pi_l^* \Omega_{WP} + \pi_r^* \Omega_{WP}$. As a consequence of Theorem 5.5.3 and the first part of Theorem 5.5.5, we thus recovered (by independent methods) their results and included it in a more general context.

The map $\mathcal{A} : \mathcal{MGH}(\Sigma) \to \mathbb{R}$ that encodes the area of the maximal Cauchy surface is also applied in the following context. Given a para-Kähler structure $(\mathbf{g}, \mathbf{P})$ on a manifold M, a *para-Kähler potential* is a smooth function $\rho : M \to \mathbb{R}$ such that $\omega_{\mathbf{P}} = (\tau/2)\overline{\partial}_{\mathbf{P}} \partial_{\mathbf{P}} \rho$. We then proved:

Theorem 5.5.6. *Let Σ be a closed oriented surface of genus ≥ 1. Then the para-Kähler structures $(\mathbf{g}, \mathbf{J})$ and $(\mathbf{g}, \mathbf{K})$ admit a para-Kähler potential, which coincides up to a constant with a Hamiltonian function for the circle action.*

Observe that, when the genus of the surface Σ is greater than or equal to 2, then the para-Kähler potential coincides (up to a multiplicative constant) with the area functional

$\mathcal{A}$, in direct analogy with the hyperKähler structure on the space of almost-Fuchsian representations described by Donaldson (see [Don03, Section 3.2]).

One could alternatively have used the map $\mathcal{C}$ to induce a circle action on $\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$. However, the obtained action is the same as when using $\mathcal{M}$ (i.e. the landslide flow), as a consequence of the observation that $\mathcal{M}=\mathcal{C}\circ R_{-\pi/2}$. By this relation, Theorem 5.5.4 immediately follows from Theorem 5.5.3 and Theorem 5.5.5.

In fact we can define a one-parameter family of maps

$$\mathcal{C}_\theta : \mathcal{MGH}(\Sigma)\to\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)\ ,$$

simply defined by $\mathcal{C}_\theta=\mathcal{C}\circ R_\theta$. An immediate consequence of our previous Theorems 5.5.4 and 5.5.5 is the following identity:

$$\mathcal{C}_\theta^*(\mathcal{P}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}, 4\Omega^{\mathbb{B}}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}) = (\cos\theta\mathbf{K}-\sin\theta\mathbf{J}, \omega_{\mathbf{I}}-\tau(\cos(\theta)\omega_{\mathbf{J}}+\sin(\theta)\omega_{\mathbf{K}}))\ . \tag{5.4}$$

The maps $\mathcal{C}_\theta$ have the following interpretation purely in terms of harmonic maps and Teichmüller theory. From the theory of harmonic maps between hyperbolic surfaces ([Sam78, Wol89, Wol91b, Wol91a, Min92]), Teichmüller space admits a parameterization by the vector space of holomorphic quadratic differentials $H^0(\Sigma,\mathcal{K}_J^2)$ with respect to a fixed complex structure J on Σ. The construction goes as follows. To a holomorphic quadratic differential q, we associate the hyperbolic metric $h_{(J,q)}$ on Σ (unique up to isotopy) such that the (unique) harmonic map $(\Sigma,J)\to(\Sigma,h)$ isotopic to the identity has Hopf differential q. We now let J vary over $\mathcal{T}(\Sigma)$. Then the map

$$\mathcal{H}_\theta := \mathcal{C}_\theta\circ\mathcal{F}^{-1} : T^*\mathcal{T}(\Sigma)\to\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$$

can be interpreted as follows:

$$\mathcal{H}_\theta(J,q) = (h_{(J,-e^{i\theta}q)}, h_{(J,e^{i\theta}q)})\ .$$

There is a completely analogous construction in genus one, by replacing hyperbolic surfaces by flat tori. As a consequence of Equation (5.4), we obtained:

Theorem 5.5.7. *Let Σ be a closed oriented surface of genus ≥ 1. Then*

$$\Im\mathcal{H}_\theta^*(2\Omega^{\mathbb{B}}_{\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)}) = -\Re(ie^{i\theta}\Omega^{\mathbb{C}}_{T^*\mathcal{T}(\Sigma)})\ .$$

We remark that the statement above is expressed purely in terms of Teichmüller theory, and is independent of Anti-de Sitter geometry.

5.5.4 The character variety

Let us now consider the character variety of the fundamental group $\pi_1(\Sigma)$ in the isometry group of AdS space. We have already observed that the isometry group is isomorphic to $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$; using the model of Hermitian matrices ([Dan13, Dan14]), it can be described as the Lie group $\mathrm{PSL}(2,\mathbb{B})$, where as usual $\mathbb{B}$ denotes the algebra of para-complex numbers. Using the (para-complex) Killing form, the character variety $\chi(\pi_1(S),\mathrm{PSL}(2,\mathbb{B}))$ is endowed with a para-complex symplectic form $\Omega^{\mathbb{B}}_{Gol}$, which is defined by adapting the work of Goldman ([Gol84]) to this context. It is para-complex with respect to the para-complex structure $\mathcal{T}$ induced by multiplication by τ. It can be checked that, under the isomorphism $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})\cong\mathrm{PSL}(2,\mathbb{B})$, the para-complex structure $\mathcal{T}$ corresponds to the para-complex structure $\mathcal{P}$ for which the integral distributions of the 1 and -1 eigenspaces are the horizontal and vertical slices.

Hence if we denote by

$$\mathcal{H}ol:\mathcal{MGH}(\Sigma)\to\chi(\pi_1(S),\mathrm{PSL}(2,\mathbb{B}))$$

the map that associates to a MGHC AdS manifold its holonomy representation, we obtain the following corollary of Theorem 5.5.3:

Corollary 5.5.8. *Let Σ be a closed oriented surface of genus ≥ 2. Then*

$$\mathcal{H}ol^*(\mathcal{T},4\Omega^{\mathbb{B}}_{Gol})=(\mathbf{J},\omega^{\mathbb{B}}_{\mathbf{J}})\ .$$

We conclude the overview of our results by a concrete description of the para-complex symplectic structure $\omega^{\mathbb{B}}_{\mathbf{J}}$. In [Tam20], Tamburelli introduced $\mathbb{B}$-valued Fenchel-Nielsen coordinates. Roughly speaking, these are defined as follows. Let $\rho=(\rho_+,\rho_-):\pi_1(\Sigma)\to\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ be the holonomy representation of a MGHC AdS manifold. Since both ρ_- and ρ_+ are Fuchsian representations, $\rho_\pm(\alpha)$ are loxodromic elements for any non-trivial $\alpha\in\pi_1(\Sigma)$. As a consequence, we can associate to α a *principal axis* $\tilde{\alpha}$ in AdS space, which is the spacelike geodesic with endpoints in $\mathbb{RP}^1\times\mathbb{RP}^1$ given by the pair of attracting and the pair of repelling fixed points of $\rho_\pm(\alpha)$. Then the Fenchel-Nielsen coordinates of ρ are $(\ell^{\mathbb{B},j}_\rho,\mathrm{tw}^{\mathbb{B},j}_\rho)$ (for γ_j a pants decomposition of Σ), where $\ell^{\mathbb{B},j}_\rho$ are para-complex numbers whose real part corresponds to the translation length and imaginary part to the bending angle of $\rho(\gamma_j)$ on the principal axis $\tilde{\gamma}_j$; a similar interpretation can be given for the (para-complex) twist coordinates $\mathrm{tw}^{\mathbb{B},j}_\rho$.

These coordinates are an analogue of the complex Fenchel-Nielsen coordinates on the space of quasi-Fuchsian manifolds, which are Darboux coordinates ([Wol83],[Pla01],[PP08]). In [MST23a] we showed that an analogous result holds for $\omega^{\mathbb{B}}_{\mathbf{J}}$, which we recall corresponds (up to a multiplicative constant) both to the para-complex sympletic form on $\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$ and to the Goldman form $\Omega^{\mathbb{B}}_{Gol}$.

Theorem 5.5.9. *The $\mathbb{B}$-valued Fenchel-Nielsen coordinates are para-holomorphic for* $\mathbf{J}$*, and are Darboux coordinates with respect to the para-complex symplectic form* $\omega^{\mathbb{B}}_{\mathbf{J}}$*.*

In other words, we expressed the symplectic form $\omega^{\mathbb{B}}_{\mathbf{J}}$, which coincides up to a multiplicative constant with the para-complex Goldman form $\Omega^{\mathbb{B}}_{Gol}$, as

$$\omega^{\mathbb{B}}_{\mathbf{J}} = \frac{1}{4}\sum_{j=1}^{n} d\ell^{\mathbb{B},j}_{\rho} \wedge d\mathrm{tw}^{\mathbb{B},j}_{\rho}$$

where $\ell^{\mathbb{B},j}_{\rho}$ and $\mathrm{tw}^{\mathbb{B},j}_{\rho}$ are the $\mathbb{B}$-valued length and twist parameters on the curve γ_j in a pants decomposition of Σ (where $n = (3/2)|\chi(\Sigma)|$ is the number of such curves).

Finally, we gave a formula for the value of the symplectic form $\omega^{\mathbb{B}}_{\mathbf{J}}$ along two twist deformations, generalizing Wolpert's cosine formula. For this purpose, given $\alpha, \beta \in \pi_1(\Sigma)$ two intersecting closed curves, the principal axes of $\rho(\alpha)$ and $\rho(\beta)$ admit a common orthogonal geodesic of timelike type. Then we define the $\mathbb{B}$-*valued* angle as the para-complex number whose imaginary part is the signed timelike distance between the two axes, and the real part is the angle between one principal axis and the parallel transport of the other, along the common orthogonal geodesic.

Theorem 5.5.10. *Let* $\rho = (\rho_+, \rho_-) : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ *be the holonomy of a MGHC AdS manifold, and let* α, β *be non-trivial simple closed curves. Then*

$$\omega^{\mathbb{B}}_{\mathbf{J}}\left(\frac{\partial}{\partial \mathrm{tw}^{\mathbb{B},\alpha}_{\rho}}, \frac{\partial}{\partial \mathrm{tw}^{\mathbb{B},\beta}_{\rho}}\right) = \frac{1}{4}\sum_{p \in \alpha\cap\beta} \cos(d^{\mathbb{B}}(\tilde{\alpha}_{\rho}, \tilde{\beta}_{\rho})) \ ,$$

where $\tilde{\alpha}_{\rho}$ *and* $\tilde{\beta}_{\rho}$ *are the principal axes of* $\rho(\alpha)$ *and* $\rho(\beta)$ *on AdS space, and* $d^{\mathbb{B}}(\tilde{\alpha}_{\rho}, \tilde{\beta}_{\rho})$ *is their* $\mathbb{B}$*-valued angle.*

Chapter 6

Minkowski geometry

Recall that Minkowski space of dimension $n+1$ (denoted $\mathbb{R}^{n,1}$) is the flat, geodesically complete Lorentzian manifold of signature $(n, 1)$. In this chapter we will outline several results of existence and uniqueness of spacelike hypersurfaces in $\mathbb{R}^{n,1}$ of constant mean curvature (Section 6.3) and constant Gaussian curvature (Section 6.2, for $n = 2$), on their geometric properties such as the completeness (Section 6.2), and mention some generalizations (work in progress) for constant scalar curvature hypersurfaces in any dimension (Section 6.3 again). Again, based on the pioneering work of Mess, the study of surfaces in $\mathbb{R}^{2,1}$ is intimately related to, and has applications for, the Teichmüller theory, as will be presented in Section 6.4.

6.1 Some notions from Lorentzian geometry

Let us provide some common background for the results of this chapter. In the following, we will focus our attention on *entire* spacelike hypersurfaces in $\mathbb{R}^{n,1}$, that is, graphs of functions $f : \mathbb{R}^n \to \mathbb{R}$ with $|Df| < 1$. Among spacelike hypersurfaces, entireness is equivalent to being properly embedded, and thus is invariant by the action of the isometry group of $\mathbb{R}^{n,1}$. The fundamental notion for all classification results is the *domain of dependence* $\mathcal{D}(\Sigma)$ of a spacelike hypersurfaces Σ. Namely, $\mathcal{D}(\Sigma)$ is the set of points $p \in \mathbb{R}^{n,1}$ such that every inextensible causal curve though p meets Σ. The domain of dependence of an entire spacelike hypersurface can be $\mathbb{R}^{n,1}$, a half-space bounded by a null hyperplane, or — the most interesting case in our context — a *regular domain*, a notion introduced in [Bon05], meaning an open domain obtained as the intersection of at least two half-spaces (that we will suppose to be *future* half-spaces) bounded by non-parallel null hyperplanes.

There is an equivalent, more analytic, way to the study of the asymptotics of entire hypersurfaces, under the assumption of convexity of Σ. To explain this, let us introduce a fundamental object for the present chapter, namely the *null support function.* Given an entire convex spacelike surface Σ in $\mathbb{R}^{2,1}$, expressed as the graph of a convex, 1-Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$, we define the function $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(\theta) = \sup_{(x,y)\in\mathbb{R}^2} (x\cos\theta + y\sin\theta - f(x,y)) = \sup_{p\in\Sigma}\langle p, \vec{\theta}\rangle \ ,$$

which is easily seen to be well-defined and lower-semicontinuous as a consequence of convexity of f. Here and in the following, we use the notation $\vec{\theta} := (\cos\theta, \sin\theta, 1)$.

The geometric interpretation of the null support function is the following. Given $\theta \in \mathbb{S}^1$, the null affine plane P defined by the equation $\langle\vec{\theta}, \cdot\rangle = \phi(\theta)$ is a support plane for Σ, meaning that every translate of P in the future intersects Σ, while every translate in the past is disjoint from Σ. The intersection of the future half-spaces bounded by the null support planes of equation $\langle\vec{\theta}, \cdot\rangle = \phi(\theta)$, as θ varies in $\mathbb{S}^1$, is exactly the domain of dependence $\mathcal{D}$ of Σ.

6.2 Constant Gaussian curvature and the completeness problem

Let us now describe surfaces of constant Gaussian curvature in $\mathbb{R}^{2,1}$. It is easily shown that every complete spacelike surface in $\mathbb{R}^{2,1}$ is entire, meaning that it is the graph of a function globally defined on the horizontal plane. Moreover, if the intrinsic curvature is non-positive, then (up to a reflection in a horizontal plane) such function is convex. The very first example is the well-known hyperboloid model of the hyperbolic plane $\mathbb{H}^2$, namely the future unit sphere in $\mathbb{R}^{2,1}$. A very natural question — still open as of now — is then a classification of all isometric embeddings of $\mathbb{H}^2$ in $\mathbb{R}^{2,1}$ — or equivalently, the classification of all *complete* spacelike surfaces of constant curvature -1 in $\mathbb{R}^{2,1}$. It turns out that the classification of *entire* spacelike surfaces of constant curvature -1, which form a larger class, is more easily achieved, and this is presented in Subsection 6.2.2, which contains the results of [BSS19]. Based on these results, the classification of isometric embeddings of $\mathbb{H}^2$ then boils down to characterizing completeness of the induced metric, which is a very difficult problem. Some progress has been made in [BSS22], whose results are presented in Subsection 6.2.3.

6.2.1 Previous results

The first non-umbilic examples of complete hyperbolic surfaces in $\mathbb{R}^{2,1}$ have been obtained by Hano and Nomizu in 1983 ([HN83]). By considering "surfaces of revolution" with spacelike axis, they reduced the problem of finding a function whose graph has constant intrinsic curvature -1 to an ordinary differential equation, whose maximal solutions give a one-parameter family of non-equivalent surfaces. It turns out that these surfaces are complete, and thus intrinsically isometric to $\mathbb{H}^2$.

Another large source of examples arose from the work [BBZ11] of Barbot-Béguin-Zeghib: they showed that for every $g \geq 2$ and every representation $\rho : \pi_1(S_g) \to$

$\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{O}(2,1) \ltimes \mathbb{R}^{2,1}$ (S_g being a closed orientable surface of genus g) whose linear part is Fuchsian, there exists a unique ρ-equivariant embedding of constant curvature -1. By cocompactness, the induced metric is complete and therefore isometric to $\mathbb{H}^2$; these surfaces are non-equivalent to the hyperboloid unless ρ has a global fixed point.

In [Li95], Li showed that for every smooth function ϕ on the circle, there exists a hyperbolic surface having null support function ϕ, which is moreover complete. The results of [Li95] actually hold in higher dimension, for hypersurfaces of constant Gauss-Kronecker curvature. In dimension three, the existence part of Li's results has been improved in [GJS06], for Lipschitz continuous ϕ, and in [BS17] for ϕ lower semicontinuous and bounded. Furthermore, in [BS17] we proved that this construction gives a bijection between the set of (convex, which we always assume here) entire hyperbolic surfaces in $\mathbb{R}^{2,1}$ with bounded second fundamental form and the (infinite-dimensional!) vector space of functions $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ having the Zygmund regularity. Since an entire hyperbolic surface with bounded second fundamental form is necessarily complete, the aforementioned results provide another large class of isometrically embedded copies of the hyperbolic plane, non-equivalent to one another.

6.2.2 Entire surfaces

The work [BSS19], improving the above results of the literature, characterized *all* entire hyperbolic surfaces in terms of their null support functions, by showing that the above construction gives a bijection between the set of *entire* hyperbolic surfaces in $\mathbb{R}^{2,1}$ and the set of lower semicontinuous functions $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ which are finite on at least three points.

Theorem 6.2.1. *For any lower semicontinuous functions $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ which is finite on at least three points, there exists a unique entire spacelike surface in $\mathbb{R}^{2,1}$ whose first fundamental form has constant curvature -1 and whose null support function is ϕ.*

Observe that the conditions of ϕ being lower semicontinuous and finite on at least three points are necessary conditions (for any *convex* entire spacelike surface), hence Theorem 6.2.1 is a full classification of entire hyperbolic surfaces in $\mathbb{R}^{2,1}$. Geometrically, this conditions means that the domain of dependence is a regular domain which is not the future of a spacelike line — which is the case when ϕ is finite on two points only.

Clearly, Theorem 6.2.1 could be restated by choosing any negative value of the (constant) curvature, up to applying a homothecy. We proved that, when the curvature varies in $(-\infty, 0)$ the corresponding surfaces provide a foliation of their domain of dependence:

Theorem 6.2.2. *Any regular domain in $\mathbb{R}^{2,1}$ which is not the future of a spacelike line is uniquely foliated by entire surfaces of constant intrinsic curvature K, for $K \in (-\infty, 0)$*

6.2.3 Completeness

Based on Theorem 6.2.1, the problem of characterizing all smooth isometric embeddings of the hyperbolic plane into $\mathbb{R}^{2,1}$ becomes equivalent to determining those lower semicontinuous functions ϕ which correspond to *complete* hyperbolic surfaces.

It is worth pausing to point out, once more, that an entire spacelike surface may be incomplete. Roughly speaking, this may happen if the surface approaches a null direction quickly enough. For instance, there exists a surface of revolution with respect to a lightlike axis, whose induced metric is hyperbolic and is incomplete, being intrinsically isometric to a half-plane in $\mathbb{H}^2$ ([BS17, Appendix A]). Its null support function is equal to minus the characteristic function of the point in $\mathbb{S}^1$ corresponding to the lightlike axis.

In [BSS22] we obtained several necessary and sufficient conditions for the completeness of an entire surface Σ in $\mathbb{R}^{2,1}$, under the assumption that the curvature of Σ is, more generally, bounded above and below by negative constants. Our first results are two conditions on the null support function which guarantee completeness of a convex entire spacelike surface.

Theorem 6.2.3 (Sequentially sublinear condition)**.** *Let $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and finite on at least three points. Suppose that for each $\theta_0 \in \mathbb{S}^1$ at which ϕ is finite, there exists $M > 0$ and a sequence $\theta_i \to \theta_0$ such that*

$$\phi(\theta_i) < \phi(\theta_0) + M|\theta_i - \theta_0| \ . \qquad \text{(Comp)}$$

If Σ is a convex entire spacelike surface in $\mathbb{R}^{2,1}$ with curvature bounded below and null support function ϕ, then Σ is complete.

Throughout, we identify $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{Z}$ in the standard way, and define $|\theta - \theta'|$ to be distance in $\mathbb{R}/2\pi\mathbb{Z}$. This is not an essential point, as the statement remains true for any reasonable notion of distance in the circle.

We remark the condition (Comp) has a simple geometric interpretation in terms of the domain of dependence $\mathcal{D}$ of the surface Σ which, as explained above, is entirely determined by the function ϕ on $\mathbb{S}^1$. Namely, (Comp) is equivalent to the condition that, whenever $\mathcal{D}$ has a null support plane P (which is of the form $\langle \vec{\theta}, \cdot \rangle = \phi(\theta)$ for some $\theta \in \mathbb{S}^1$), there exists a null line in P which does not intersect $\partial\mathcal{D}$.

If we restrict to constant curvature -1, Theorem 6.2.3 implies that for any function ϕ satisfying (Comp), the entire hyperbolic surface with null support function ϕ (which exists and is unique by Theorem 6.2.1) is complete, and hence gives an isometric embeddings of the hyperbolic plane in $\mathbb{R}^{2,1}$.

We stress that a function ϕ satisfying the condition (Comp) at every point can be highly discontinuous, and can take value $+\infty$ on large portions of the circle. For instance, by virtue of Theorem 6.2.3 the function taking value 0 on any Cantor set in $\mathbb{S}^1$, and $+\infty$ elsewhere, is the null support function of a complete hyperbolic surface.

Hence Theorem 6.2.3 is a remarkable improvement with respect to the state-of-the-art, since so far the most general result in this direction, which follows from [BS17], is that an entire hyperbolic surface with *Zygmund continuous* null support function is complete. Nevertheless, Theorem 6.2.3 is not sharp, as showed by the next statement, which gives another criterion for completeness.

Theorem 6.2.4 (Subloglogarithmic condition)**.** *Let $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous and finite on at least three points, and let $\lambda > 0$. Suppose that that for each $\theta_0 \in \mathbb{S}^1$ at which ϕ is finite, there is a one-sided neighbourhood U of θ_0 and such that*

$$\phi(\theta) \leq \phi(\theta_0) + \frac{\lambda}{4}|\theta - \theta_0| \log(-\log|\theta - \theta_0|) \tag{Comp'}$$

for every $\theta \in U$. If Σ is a convex entire spacelike surface in $\mathbb{R}^{2,1}$ with curvature bounded below by $-\lambda^2$ and null support function ϕ, then Σ is complete.

By one-sided neighbourhood of θ_0, we mean that U contains an interval either of the form $(\theta_0 - \epsilon, \theta_0]$ or $[\theta_0, \theta_0 + \epsilon)$ for $\epsilon > 0$.

In the other direction, we state now two conditions which are sufficient to guarantee incompleteness.

Theorem 6.2.5 (Power function condition)**.** *Let $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and finite on at least three points. Suppose that there exist $\theta_0 \in \mathbb{S}^1$ at which ϕ is finite, a neighborhood U of θ_0, and constants $\epsilon > 0$ and $0 < \alpha < 1$ such that*

$$\phi(\theta) - \phi(\theta_0) > \epsilon|\theta - \theta_0|^\alpha \tag{Inc}$$

for every $\theta \in U$. If Σ is a convex entire spacelike surface in $\mathbb{R}^{2,1}$ with null support function ϕ and curvature bounded above by a negative constant, then Σ is incomplete.

For example, taking any α, the conclusion holds if ϕ has a *two-sided jump* at θ_0, meaning that $\phi(\theta_0) < \liminf_{\theta \to \theta_0} \phi(\theta)$.

Theorem 6.2.6 (One-sided superlogarithmic condition)**.** *Let $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and finite on at least three points. Suppose that there exist $\theta_0 \in \mathbb{S}^1$ at which ϕ is finite, a neighbourhood U of θ_0, and $\epsilon > 0$ such that*

$$\begin{cases} \phi(\theta) = +\infty & \text{if } \theta \text{ is on one side of } \theta_0 \\ \phi(\theta) \geq \phi(\theta_0) + \epsilon|(\theta - \theta_0)\log|\theta - \theta_0|| & \text{if } \theta \text{ is on the other one side of } \theta_0. \end{cases} \tag{Inc'}$$

for every $\theta \in U$. If Σ is a convex entire spacelike surface in $\mathbb{R}^{2,1}$ with null support function ϕ and curvature bounded above by a negative constant, then Σ is incomplete.

Taking together our completeness and incompleteness theorems, we have a narrow window of local behaviors of ϕ for which we cannot yet determine completeness, such as

$\phi(\theta) = |\theta \log |\theta||$ near $\theta = 0$. From our perspective, the still-open problem of classifying isometric immersions of the hyperbolic plane by their asymptotic behavior amounts to closing this window.

6.3 Constant mean and scalar curvature

The study of spacelike hypersurfaces of constant mean curvature (CMC in short) in Minkowski space $\mathbb{R}^{n,1}$ has been widely developed since the 1980s, see for instance [Tre82, Mil83, BS83, CT88, CT90]. An important motivation is that among spacelike hypersurfaces in $\mathbb{R}^{n,1}$, CMC hypersurfaces are precisely those for which the Gauss map, with values in the hyperbolic space $\mathbb{H}^n$, is *harmonic*. Employing this idea for $n = 2$, many interesting results have been obtained on harmonic maps from $\mathbb{C}$ or $\mathbb{D}$ to $\mathbb{H}^2$ (see [CT88, AN90, Wan92, CT93, HTTW95, GMM03]). More recently several results appeared on CMC hypersurfaces in $\mathbb{R}^{n,1}$ admitting a co-compact action, thus giving rise to CMC compact Cauchy hypersurfaces in certain flat Lorentzian manifolds, in [And02a, ABBZ12], for $n = 2$ in [BBZ07, And05], and for manifolds with conical singularities in [KS07, CT19]. The generalization of this problem to general Lorentzian manifolds satisfying some additional conditions is also of importance to general relativity, for example [Ger83b]; see [Bar87] or Section 4.2 of [Ger06a] for a summary.

While the only entire hypersurfaces of *vanishing* mean curvature are spacelike planes ([CY76], also [Cal70] for $n \leq 4$), hypersurfaces of constant mean curvature $H \neq 0$ have a much greater flexibility, with many examples produced in [Tre82, CT90]. Still there is some rigidity: Cheng and Yau, in the same article [CY76], show that entire CMC hypersurfaces have complete induced metric and are convex (up to applying a time-reversing isometry).

6.3.1 Classification of entire CMC hypersurfaces

Perhaps surprisingly, although partial results were obtained in [Tre82, CT90], to our knowledge the literature lacked a complete classification of entire CMC hypersurfaces in Minkowski space.

In [BSS23], we proved a classification result.

Theorem 6.3.1. *For any lower semicontinuous functions $\phi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ which is finite on at least two points and any $H > 0$, there exists a unique entire hypersurface of constant mean curvature H in $\mathbb{R}^{2,1}$ whose null support function is ϕ.*

Unlike the case of constant Gaussian curvature (for $n = 2$), here ϕ is allowed to be finite on two points only, in which case the domain of dependence is the intersection of the future of two lightlike hyperplanes, and the corresponding CMC hypersurface is the product of a hyperbola and a $(n-1)$-dimensional spacelike affine subspace. Geometrically, Theorem 6.3.1 can be rephrased by saying that for every regular domain $\mathcal{D}$ in $\mathbb{R}^{n,1}$ and any $H > 0$, there exists a unique entire hypersurface $\Sigma \subset \mathbb{R}^{n,1}$ of constant mean curvature H such that the domain of dependence of Σ is $\mathcal{D}$. We also proved that, as H varies in $(0, +\infty)$, the entire hypersurfaces of constant mean curvature H analytically foliate $\mathcal{D}$:

Theorem 6.3.2. *Any regular domain in $\mathbb{R}^{n,1}$ is analytically foliated by entire hypersurfaces of constant mean curvature H, for $H \in (0, +\infty)$*

The result of [CT90] is an important predecessor to Theorem 6.3.1. To translate their result into the language of null support functions, say that a function valued in $\mathbb{R} \cup \{+\infty\}$ is nearly continuous if the set on which it is finite is closed and it continuous when restricted to that set. Then Choi and Treibergs prove that if φ is a lower semicontinuous function on $\mathbb{S}^{n-1}$ which is nearly continuous and finite on at least two points, then there exists an entire CMC hypersurface with null support function φ. Compared to [CT90], our contribution in [BSS23] was to extend the existence theorem to all lower semicontinuous functions finite on at least two points and crucially to prove uniqueness.

6.3.2 Constant scalar curvature

In a work in preparation with Pierre Bayard ([BS]), we are currently studying foliations of regular domains by hypersurfaces of constant scalar curvature. We expect to be able to prove that for any prescribed (negative) value c of the scalar curvature, every regular domain $\mathcal{D}$ which is the intersection of at least three non-parallel hyperplanes admits a hypersurface with constant scalar curvature c whose domain of dependence is $\mathcal{D}$. Uniqueness is more tricky to achieve in this context, and at the moment we expect to be able to obtain uniqueness under the assumption that the corresponding null support function ϕ is in $C^0(\mathbb{S}^{n-1}, \mathbb{R})$. Also the foliation result will probably hold under this assumption. Since this condition is satisfied in the cocompact case, this result will be an improvement of [Smi18], which proves the foliation result by constant scalar curvature hypersurfaces in the cocompact case for $\mathbb{R}^{3,1}$. Finally, it is still an open question whether the assumption that ϕ is finite on at least three points will be a necessary condition. Indeed, although this seems unlikely, a priori there might exist hypersurfaces of constant scalar curvature whose null support function is finite on precisely two points — equivalently, whose domain of dependence is the future of a spacelike $(n-1)$-dimensional affine subspace.

6.4 Induced metrics, and minimal Lagrangian maps again

As discussed in Section 6.2.3, entire surfaces of constant Gaussian curvature -1 are not necessarily complete. Surprisingly, Theorem 6.3.1 on constant *mean* curvature surfaces (in dimension $2+1$) leads to applications on hyperbolic surfaces (i.e. of constant *Gaussian* curvature), and to minimal Lagrangian maps, that we obtained in [BSS23].

For this purpose, let us first mention a relationship, in dimension $2+1$, between constant mean curvature and constant Gaussian curvature surfaces. This rests in the classical observation that if Σ has constant intrinsic curvature -1, then the surface which lies at Lorentzian distance one from Σ to the convex side has constant mean curvature $H = 1/2$.

Now, as mentioned before, among immersed spacelike surfaces Σ in $\mathbb{R}^{2,1}$ *hyperbolic* surfaces are exactly those whose Gauss map G is a minimal Lagrangian local diffeomorphism; that is, the graph of G is a minimal Lagrangian surface in $\Sigma \times \mathbb{H}^2$. If moreover Σ is embedded, then G is a diffeomorphism onto its image.

Our first result here is a description of the induced metric of entire surfaces of constant Gaussian curvature, when they are incomplete. Let us first introduce some definitions. First, a *straight convex domain* in $\mathbb{H}^2$ is the interior of the convex hull of a subset of $\partial_\infty \mathbb{H}^2$ consisting of at least 3 points. Second, let (S, h) and (S', h') be simply connected hyperbolic surfaces. We say that a smooth map $f : (S, h) \to (S', h')$ is *realizable* in $\mathbb{R}^{2,1}$ if there exists an isometric immersion $\sigma : (S, h) \to \mathbb{R}^{2,1}$ and a local isometry $d : (S', h') \to \mathbb{H}^2$ such that $d \circ f = G_\sigma$ where $G_\sigma : S \to \mathbb{H}^2$ is the Gauss map of σ. We say it is *properly realizable* if moreover the immersion is proper, which is equivalent to its image being entire.

It is known that realizability of f is equivalent to being a minimal Lagrangian local diffeomorphism. The following theorem from [BSS23] gives a characterization of properly realizable minimal Lagrangian maps, in terms of their graphs in the Riemannian product of (S, h) and (S', h').

Theorem 6.4.1. *Let $f : (S, h) \to (S', h')$ be a diffeomorphism between simply connected hyperbolic surface. Then f is properly realizable in $\mathbb{R}^{2,1}$ if and only if the graph of f is a complete minimal Lagrangian surface in $(S \times S', h \oplus h')$. In this case, both (S, h) and (S', h') are isometric to straight convex domains in $\mathbb{H}^2$.*

Observe that from the definition, it is easy to see that the inverse of a minimal Lagrangian diffeomorphism is again minimal Lagrangian. The following is then a straightforward corollary of Theorem 6.4.1:

Corollary 6.4.2. *Let $f : (S, h) \to (S', h')$ be a minimal Lagrangian diffeomorphism between simply connected hyperbolic surface. Then f is properly realizable in $\mathbb{R}^{2,1}$ if and only if f^{-1} is properly realizable in $\mathbb{R}^{2,1}$.*

Let us spend a few words here to outline the proof of Theorem 6.4.1. The basic observation is that for any entire hyperbolic surface Σ in $\mathbb{R}^{2,1}$, the surface Σ_+ at Lorentzian distance one with constant mean curvature $H = 1/2$ is still entire, and the two have the same domain of dependence. A consequence of the uniqueness in Theorems 6.3.1 and 6.2.1, is that the converse is almost always true: if Σ_+ is any entire CMC-(1/2) surface except for the trough, then the surface Σ at Lorentzian distance one to the past is still entire (with the same domain of dependence). To prove the first part of Theorem 6.4.1, it then suffices to observe that the first fundamental form of Σ_+ is bi-Lipschitz equivalent to the induced metric on the graph of the minimal Lagrangian map in the Riemannian product, and by the Cheng and Yau completeness theorem, entireness of the equidistant CMC-$(1/2)$ surface Σ_+ is equivalent to completeness of its first fundamental form. The second part of Theorem 6.4.1 follows applying the fact that the image of the Gauss map of any entire hyperbolic surface is a straight convex domain, which we have proved in [BSS19], and therefore the image of a minimal Lagrangian local diffeomorphism (injective but not surjective) from $\mathbb{H}^2$ to itself is a straight convex domain. Incidentally, in [FS21] we generalized this last results from minimal Lagrangian maps to a larger class of maps, called *one-harmonic*, when the source metric is allowed to be a metric of (non-constant) negative curvature; moreover, in [FS20] we further exploited the relations between Minkowski geometry and Teichmüller theory, leading to a new proof, and a generalization to a broader setting, to the classical Wolpert formula that expresses the Weil-Petersson symplectic form on two infinitesimal twists as the sum of the cosine of the angles of intersection.

To conclude, in [BSS23] we also showed that being a straight convex domain is not only a necessary condition, but also a sufficient condition for the induced metric.

Theorem 6.4.3. *A hyperbolic surface can be embedded isometrically and properly in $\mathbb{R}^{2,1}$ if and only if it is isometric to a straight convex domain.*

As a final comment, the hypothesis of entireness is clearly essential in Theorem 6.4.3, as any domain in $\mathbb{H}^2$ can be realized without the entireness assumption. But we remark here that the situation is even subtler, since also hyperbolic surfaces which are not isometric to a subset of $\mathbb{H}^2$ can be embedded as non-entire surfaces. In fact, in [BS17, Appendix A], an example of non-entire surface in $\mathbb{R}^{2,1}$ intrinsically isometric to the universal cover of the complement of a point in $\mathbb{H}^2$ is constructed.

Part III

Beyond pseudo-Riemannian: projective, affine and others

Chapter 7

Second topological interlude: geometric transition

In this chapter, we will discuss some phenomena of *geometric transition* in the context of the transition between hyperbolic and Anti-de Sitter structures introduced in Danciger's PhD thesis [Dan11]. This transition goes through the intermediate *half-pipe* geometry, which is also closely related with Minkowski geometry and, in dimension 3, with Teichmüller theory. In Section 7.1 we will discuss some novel examples of geometric transition on (singular) manifolds, that I constructed with Stefano Riolo. Section 7.2 will study the behaviour exhibited by these examples on the character variety. Finally, Section 7.3 will mention recent work in progress with Christian El Emam, Filippo Mazzoli and Andrea Tamburelli which studies a phenomenon induced by Danciger's geometric transition, but at the level of deformation spaces, related to the para-hyperKähler structure of [MST23a] discussed in Section 5.5.

7.1 Examples of geometric transition on manifolds

A major motivation for the study of geometric transition comes from a phenomenon called *degeneration* of hyperbolic structures, introduced in Thurston's famous notes [Thu79]. Several contribuitions have then been given on this topic [Hod86, Por98, HPS01, Por02, Ser05, PW07, Por13, Koz13, LMA15a, LMA15b, Koz16], which plays an important role in the proof of the celebrated Orbifold Theorem [BLP05, CHK00].

As an example, for some closed hyperbolic 3-orbifolds $\mathcal{X}$, singular along a knot $\Sigma \subset \mathcal{X}$ with cone angle $\frac{2\pi}{m}$, the following holds. There is a path $\theta \mapsto \mathcal{X}_\theta$ of hyperbolic cone-manifold structures on $\mathcal{X}$ with singular locus Σ and cone angle $\theta \in \left[\frac{2\pi}{m}, 2\pi\right)$, such that $\mathcal{X}_\theta$ collapses to a lower-dimensional orbifold as $\theta \to 2\pi$. This holds, for instance, when $\mathcal{X}$ is an exceptional Dehn filling of the figure-eight knot complement admitting a Seifert fibration $\mathcal{X} \to \mathcal{N}$ with base a hyperbolic 2-orbifold $\mathcal{N}$. As $\theta \to 2\pi$, the cone-manifold $\mathcal{X}_\theta$ collapses to $\mathcal{N}$, whose hyperbolic structure is said to *regenerate* to 3-dimensional hyperbolic structures.

The familiar idea of going from spherical to hyperbolic geometry, through Euclidean geometry, was known since Klein [AP15]. This is a continuous process inside projective geometry, seen as a common "ambient" geometry. This phenomenon, called *geometric transition*, has been recently studied in greater generality by Cooper Danciger and Wienhard [CDW18] (see also [Tre19]) through the notion of *limit geometry*. For example, among others, Euclidean geometry is a limit of both spherical and hyperbolic geometries inside projective geometry.

7.1.1 Some constructions in dimension three

Let us come back to the hyperbolic cone 3-manifolds $\mathcal{X}_\theta$ collapsing to the hyperbolic 2-orbifold $\mathcal{N}$. The work of Danciger [Dan11, Dan13, Dan14] shows that in many such cases the hyperbolic structure of $\mathcal{N}$ regenerates to anti-de Sitter (AdS for short, the Lorentzian analogue of hyperbolic geometry) structures on $\mathcal{X}$, where the singular locus Σ is a spacelike geodesic. Moreover, the two deformations are joined continuously via projective geometry so as to have geometric transition. To this purpose, Danciger introduced the so called *half-pipe* geometry, which is a limit geometry [CDW18] inside projective geometry of both hyperbolic and anti-de Sitter geometry. Half-pipe space naturally identifies with the space of spacelike hyperplanes in Minkowski space $\mathbb{R}^{1,n-1}$, and its group of transformations, which is a Chabauty limit of both $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, is isomorphic to $\mathrm{Isom}(\mathbb{R}^{1,n-1})$ by means of this duality. Suitable projective transformations are used to "rescale" the hyperbolic and AdS metric along the direction of collapse, thus obtaining geometric transition via half-pipe geometry.

In [Dan13, Theorem 1.1], Danciger provides an infinite class of Seifert 3-manifolds $\mathcal{X}$ (unit tangent bundles of some hyperbolic 2-orbifolds) supporting such a kind of geometric transition. Also, [Dan13, Theorem 1.2] is a regeneration result of half-pipe structures under a cohomological condition: the 1-dimensionality of the twisted cohomology group $H^1_{\mathrm{Ad}\,\rho}(\pi_1(\mathcal{X}\smallsetminus\Sigma), \mathfrak{so}(1,2))$, where $\rho\colon \pi_1(\mathcal{X}\smallsetminus\Sigma) \to \mathrm{Isom}(\mathbb{H}^2)$ is the representation associated to the degenerate structure and $\mathrm{Ad}\colon \mathrm{Isom}(\mathbb{H}^2) \to \mathrm{Aut}(\mathfrak{so}(1,2))$ is the adjoint representation. A few more examples, constructed *by hands*, appeared in my proceedings article [Sep19a].

My PhD student Farid Diaf, in his first article [Dia23b], has constructed a very large source of examples, obtained as doubles of hyperbolic/Anti-de Sitter/half-pipe convex cores such that the pleating locus is a rational lamination. His result shows that, for any closed oriented surface (possibly with a finite number of points removed) S, and any pair of filling rational laminations λ_1 and λ_2, $S\times\mathbb{S}^1\setminus(\lambda_1\times\{p_1\}\cup\lambda_2\times\{p_2\})$ admits a geometric transition from an Anti-de Sitter structure to a hyperbolic structure, going through a half-pipe structure, with conical singularities at λ_1 and λ_2 (and cusp structures in the product of $\mathbb{S}^1$ and a neighbourhood of the punctures). This results exhibits a recipe to construct a very large class of examples, although the singularities occur on a link instead of a knot. Moreover, in his second article [Dia23a], Farid exploited more deeply the relation of three-dimensional half-pipe geometry with Teichmüller theory. He achieved a number of *infinitesimal earthquake theorems* by means of applications of half-pipe geometry, relying on the idea that convex cores in half-pipe geometry lead to infinitesimal earthquakes, similarly to the manner by which convex cores in Anti-de Sitter geometry lead to earthquakes between hyperbolic surfaces.

7.1.2 A four-dimensional example

It seems natural to ask whether these transition phenomena are purely three-dimensional, or if they can occur also in higher dimension, where hyperbolic structures are typically more rigid. In the paper [RS22b] we answered affirmatively, by exhibiting novel examples in dimension four. We indeed built some examples of geometric transition from hyperbolic to AdS structures. The construction is explicitly obtained by gluing copies of a hyperbolic or AdS collapsing 4-polytope — which, however, is different from the doubling of a convex core.

The study of deformations of 4-dimensional hyperbolic cone-manifolds is quite recent, and in general very little is known on this topic. In [MR18, Theorem 1.2], Riolo and Martelli provided the first example of degeneration of hyperbolic cone structures on a 4-manifold to a 3-dimensional hyperbolic structure. We showed in [RS22b] that in this case there is geometric transition from hyperbolic to AdS structures, and provided an infinite class of such examples. Precisely, we showed the following:

Theorem 7.1.1. *Let $\mathcal{N}$ be a hyperbolic 3-manifold that finitely orbifold-covers the ideal right-angled cuboctahedron. There exists a C^1 family $\{\sigma_t\}_{t\in(-\epsilon,\epsilon]}$ of simple projective cone-manifold structures on the 4-manifold*

$$\mathcal{X} = \mathcal{N} \times S^1,$$

singular along a compact foam $\Sigma \subset \mathcal{X}$, such that σ_t is conjugated to a cusped, finite-volume,

- *hyperbolic orbifold structure with cone angles π as $t = \epsilon$,*
- *hyperbolic cone structure with decreasing cone angles $\alpha_t \in [\pi, 2\pi)$ as $t > 0$,*
- *half-pipe structure with spacelike singularity as $t = 0$,*
- *anti-de Sitter structure with spacelike singularity of increasing magnitude $\beta_t \in (-\infty, 0)$ as $t < 0$.*

As $t \to 0^+$ (resp. $t \to 0^-$), we have $\alpha_t \to 2\pi$ (resp. $\beta_t \to 0$) and the induced hyperbolic (resp. AdS) structures on $\mathcal{X} \setminus \Sigma$ degenerate to the complete hyperbolic structure of $\mathcal{N}$.

Similarly to Danciger's [Dan13, Theorem 1.1], but in higher dimension, there is a circle bundle over a hyperbolic orbifold (a 2-orbifold in his case, a 3-manifold in [RS22b]), and geometric transition from hyperbolic to AdS singular structures on the total space of the bundle with collapse to the base. Let us briefly explain some terminology used in the statement of Theorem 7.1.1.

The *cuboctahedron* is a well-known uniform polyhedron whose *ideal* hyperbolic counterpart $\mathcal{C} \subset \mathbb{H}^3$ is *right-angled*. As such, the polyhedron $\mathcal{C}$ can be seen as a cusped hyperbolic 3-orbifold. *Simple projective cone-manifolds* are singular real projective manifolds locally modelled on the double of a simple polytope in projective space. The singular locus $\Sigma \subset \mathcal{X}$ of an n-dimensional simple projective cone-manifold $\mathcal{X}$ is an $(n-2)$-complex with generic singularities: if $n = 1, 2, 3$ or 4, the set Σ is empty, discrete, a trivalent graph or a foam, respectively. A *foam* is a 2-complex locally modelled on the cone over the 1-skeleton of the tetrahedron. The singular locus in Theorem 7.1.1 is not a surface, as it has edges and vertices. However foams are quite natural objects in dimension four (like trivalent graphs in 3-manifolds). To the best of our knowledge, it is not known whether there can even exist deformations of 4-dimensional, finite-volume, hyperbolic cone-manifolds with singular locus an embedded surface.

The holonomy of a meridian $\gamma \in \pi_1(\mathcal{X} \smallsetminus \Sigma)$ of a 2-stratum of Σ has a totally geodesic 2-plane as fixed point set. We have a rotation in $\mathbb{H}^4$ of angle α_t when $t > 0$, and a Lorentz boost in $\mathbb{A}\mathrm{d}\mathbb{S}^4$ of magnitude β_t as $t < 0$. In the half-pipe case, we have a transformation that can be interpreted as an infinitesimal rotation (resp. boost) in $\mathbb{H}^4$ (resp. $\mathbb{A}\mathrm{d}\mathbb{S}^4$).

It is worth remarking that the cone-manifolds of Theorem 7.1.1 are non-compact, but of finite volume. Nevertheless, the singularity Σ is compact, or in other words, it does not enter into the ends of the cone-manifolds. These ends are (non-singular) *cusps* in a suitable sense. As a direct consequence of our methods, we achieve a nice description of the geometry of the cusps. A section of the cusps will indeed naturally support a geometric transition from Euclidean to Minkowski (non-singular) structures — going through an intermediate geometry which is a “flat version” of half-pipe geometry and is the so-called *Galilean geometry* [Yag79]. Finally, we remark that the statement of Theorem 7.1.1 can be made slightly more general by our methods, just assuming that $\mathcal{N}$ is a *cuboctahedral manifold*, namely a hyperbolic manifold tessellated by ideal right-angled cuboctahedra.

7.1.3 A key ingredient: the Kerckhoff-Storm polytope

The essential ingredient for the proof of Theorem 7.1.1 is a deforming 4-polytope $\mathcal{P}_t \subset \mathbb{H}^4$ parametrised by $t \in (0, 1]$, introduced by Kerckhoff and Storm [KS10]. For a particular choice of the 3-manifold $\mathcal{N}$, the hyperbolic cone structures σ_t that degenerate were shown to exist by Martelli and Riolo in [MR18, Theorem 1.2] by gluing eight copies of $\mathcal{P}_t$.

A fundamental property of $\mathcal{P}_t$ is that most of its dihedral angles are right for all values of t, while the remaining dihedral angles are all equal and tend to π as $t \to 0$, i.e. when $\mathcal{P}_t$ collapses to the aforementioned cuboctahedron. The presence of many right angles is essential in order to glue copies of $\mathcal{P}_t$ without creating a too complicated singular locus.

To prove Theorem 7.1.1, we first showed that the path of hyperbolic polytopes extends for negative times $t \in (-1, 0)$ to a path of AdS polytopes with the same combinatorics of $\mathcal{P}_t \subset \mathbb{H}^4$ with $t \in (0, \epsilon]$, and sharing similar properties on the dihedral angles and on the collapse. A remarkable difference is that, since the Anti-de Sitter metric is Lorentzian, some of the bounding hyperplanes are spacelike, and some others timelike.

The construction is however quite complicated and involves several computations. To prove that the combinatorics of the AdS polytopes remains constant, we needed to implement a Sage worksheet [RS]. The proof of the analogous property on the hyperbolic side [KS10, MR18] circumvented this amount of computations relying on Vinberg's theory of hyperbolic polytopes with non-obtuse dihedral angles.

By opportunely rescaling $\mathcal{P}_t$ inside projective space along the direction of collapse, as suggested by the work of Danciger, we showed that the resulting path of rescaled projective polytopes extends as $t = 0$ to a half-pipe 4-polytope. This whole deformation can be interpreted as a geometric transition of "cone-orbifold" structures. More precisely, the subset

$$\mathcal{P}_t^\times \subset \mathcal{P}_t$$

obtained by removing the ridges (the codimension-2 faces) with non-constant dihedral angles has a natural structure of hyperbolic (when $t > 0$) or AdS (when $t < 0$) orbifold. To show that these structures are linked by geometric transition, we construct an opportune half-pipe orbifold structure on the "rescaled limit" of $\mathcal{P}_t^\times$ as $t \to 0$.

Then, inspired by [MR18], we glue several copies of $\mathcal{P}_t$ in the following way. Any d-sheeted orbifold cover $\mathcal{N} \to \mathcal{C}$ of the the ideal right-angled cuboctahedron naturally induces a way to pair certain facets of d copies of $\mathcal{P}_t$. When $t < 0$, these facets are precisely the timelike facets of the AdS polytope. The resulting space is homeomorphic to $\mathcal{N} \times [0, 1]$, and its two boundary components contain all the ridges of the copies of $\mathcal{P}_t$ with non-constant dihedral angle. The final step is to double this manifold, thus obtaining $\mathcal{X} = \mathcal{N} \times S^1$ with a structure of hyperbolic, or AdS, cone-manifold. The singular locus Σ consists of the union of the copies of the ridges with non-constant dihedral angle.

We would like to stress here a particular caveat of this construction. The fact that the polytope $\mathcal{P}_t$, suitably rescaled, converges when $t \to 0$ to a half-pipe polytope is *not* sufficient to produce a half-pipe orbifold structure on the rescaled limit of $\mathcal{P}_t^\times$. Indeed, in contrast with the hyperbolic or AdS case, a hyperplane in half-pipe space does not uniquely determine a half-pipe reflection: there is a one-parameter family of reflections which fix a non-spacelike (i.e. *degenerate*) hyperplane. This counterintuitive phenomenon, which often occurs in the realm of real projective geometry, highlights the fact that half-pipe geometry is neither Riemannian, nor pseudo-Riemannian. Hence finding the "half-pipe glueings" is somehow subtler, and is achieved by analysing the behaviour of the holonomy representations of the hyperbolic and AdS structures infinitesimally, near the collapse.

7.2 Transition on character varieties

As discussed in Section 7.1 above, a key ingredient of the four-dimensional geometric transition in Theorem 7.1.1 is the Kerckhoff-Storm path of hyperbolic 4-polytopes which collapse as $t \to 0$ to a 3-dimensional ideal right-angled cuboctahedron. This induces a path of incomplete hyperbolic structures on a naturally associated 4-orbifold $\mathcal{O}$. The orbifold fundamental group of $\mathcal{O}$ is a rank-22 right-angled Coxeter group Γ_{22}, which embeds in $\mathrm{Isom}(\mathbb{H}^4)$ as a discrete reflection group when $t = 1$.

The holonomy representations of these hyperbolic structures on $\mathcal{O}$ then provide a smooth path $t \mapsto [\rho_t^G]$ in the character variety $X(\Gamma_{22}, G)$ for $G = \mathrm{Isom}(\mathbb{H}^4)$. (This path was originally defined in [KS10] when $G = \mathrm{Isom}(\mathbb{H}^4)$ only for $t \in (0, 1]$, and is easily continued analytically also for non-positive times.) Let, more generally, G be $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, or the group $G_{\mathbb{HP}^4}$ of transformations of half-pipe geometry, and let $G^+ < G$ be the subgroup of orientation-preserving transformations. Recall that $\mathrm{Hom}(\Gamma_{22}, G)$ is naturally a real algebraic affine set [Wei64]. We then call *character variety* of Γ_{22} the (topological) quotient

$$X(\Gamma_{22}, G) = \mathrm{Hom}(\Gamma_{22}, G)/G^+$$

by the action of G^+ by conjugation. When G is reductive, that is in the hyperbolic and AdS settings, it is also possible to define the GIT quotient, which has a structure of real semialgebraic set by general results [RS90].

The Anti-de Sitter path of polytopes introduced in [RS22b], is an Anti-de Sitter parallel of the one introduced by Kerckhoff and Storm in the following sense: it has the same combinatorics of the hyperbolic polytope $\mathcal{P}_t$ for $t \in (0, \varepsilon)$, such that the same orthogonality conditions between the bounding hyperplanes are satisfied, and again collapsing to an ideal right-angled cuboctahedron in a spacelike hyperplane $\mathbb{H}^3$ of $\mathbb{A}\mathrm{d}\mathbb{S}^4$. Some bounding hyperplanes are spacelike, and some others are timelike. This induces a path of Anti-de Sitter orbifold structures on $\mathcal{O}$, with holonomy representation $\rho_t^{\mathbb{A}\mathrm{d}\mathbb{S}^4} : \Gamma_{22} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ given by sending each generator to the corresponding AdS reflection. This path of representations is defined for $t \in (-1, 1)$ and diverges as $|t| \to 1^-$. Theorem 7.1.1 shows that the two paths, suitably rescaled, can be joined so as to give geometric transition on the orbifold $\mathcal{O}$, and in particular there is a transitional half-pipe orbifold structure on $\mathcal{O}$ joining the two paths. For $G = G_{\mathbb{HP}^4}$ there is a "trivial" path of non-equivalent half-pipe representations (defined for $t \in \mathbb{R}$, and diverging as $|t| \to +\infty$) differing from one another by "stretching" in the ambient real projective space. Indeed, a half-pipe structure is never rigid, because one can always conjugate with a transformation which "stretches" the degenerate direction, and obtain a new structure equivalent to the initial one *as a real projective structure*, but inequivalent *as a half-pipe structure*.

The representations obtained at $t = 0$ correspond geometrically to a "collapse" and play a special role in two ways. First, they correspond to a "symmetry" in the character varieties, since the representations ρ_t^G and ρ_{-t}^G are conjugated in G but not in G^+. Second, interpreting $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ and $G_{\mathbb{HP}^4}$ as subgroups of $\mathrm{PGL}(5,\mathbb{R})$, the three representations ρ_0^G coincide. They correspond to a representation (we omit the superscript G here)

$$\rho_0 \colon \Gamma_{22} \to \mathrm{Stab}(\mathbb{H}^3) < G \ ,$$

for a fixed copy of $\mathbb{H}^3$ in $\mathbb{H}^4$, $\mathbb{A}\mathrm{d}\mathbb{S}^4$, or $\mathbb{HP}^4$, respectively. Projecting the image of ρ_0 in $\mathrm{Stab}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3)\times\mathbb{Z}/2\mathbb{Z}$ to $\mathrm{Isom}(\mathbb{H}^3)$ gives the reflection group of an ideal right-angled cuboctahedron.

The goal of the paper [RS22a] was to describe the hyperbolic, Anti-de Sitter, and half-pipe character varieties of the right-angled Coxeter group Γ_{22}, including a study of the behaviour at the collapse. Let us now summarize these results.

Theorem 7.2.1. *Let G be $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, or $G_{\mathbb{HP}^4}$. A neighbourhood $\mathcal{U}$ of $[\rho_0]$ in $X(\Gamma_{22}, G)$ consists of two smooth, transverse, components $\mathcal{V}$ and $\mathcal{H}$ satisfying $\mathcal{V}\cap\mathcal{H} = \{[\rho_0]\}$:*

- *the curve $\mathcal{V}$ of the conjugacy classes of all the holonomy representations ρ_t^G;*
- *a 12-dimensional ball $\mathcal{H}$, identified to a neighbourhood of the complete hyperbolic orbifold structure of the ideal right-angled cuboctahedron in its deformation space.*

The group $G/G^+ \cong \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{U}$ fixing $\mathcal{H}$ point-wise and sending $[\rho_t^G]$ to $[\rho_{-t}^G]$.

Let us include some comments to elucidate the content of Theorem 7.2.1. First, our proofs actually show that the representation ρ_0 has a neighbourhood in $\mathrm{Hom}(\Gamma_{22}, G)$ that is homeomorphic to $(\mathcal{H}\cup\mathcal{V})\times G^+$, in such a way that the action of G^+ corresponds to obvious left multiplication by G^+ on the second factor.

Let $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ be the preimages in $\mathrm{Hom}(\Gamma_{22}, G)$ of $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{H}$, respectively. By "smoothness" of the "components" $\mathcal{V}$ and $\mathcal{H}$ of $\mathcal{U}$ we actually refer to $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{U}}$, respectively. In particular, $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ are smooth manifolds (of dimension 11 and 22, respectively). The smoothness of $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ together with the local product structure in a neighbourhood of ρ_0 induce a smooth structure on the components $\mathcal{H}$ and $\mathcal{V}$ in the quotient.

The "transversality" of $\mathcal{V}$ and $\mathcal{H}$ is defined as follows: $\widetilde{\mathcal{V}}\cap\widetilde{\mathcal{H}}$ is the G-orbit of ρ_0, and the Zariski tangent spaces of $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ intersect transversely in the Zariski tangent space of $\mathrm{Hom}(\Gamma_{22}, G)$ at ρ_0 (and hence at any other point of its orbit). In particular, every infinitesimal deformation tangent to both $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{H}}$ is tangent to the G^+-orbit of ρ_0.

Our analysis showed that, when G is $\mathrm{Isom}(\mathbb{H}^4)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, the character variety $X(\Gamma_{22}, G)$ is homeomorphic to the GIT quotient $\mathrm{Hom}(\Gamma_{22}, G)/\!\!/G^+$ near each $[\rho_t]$. In other words, $X(\Gamma_{22}, G)$ is Hausdorff near $[\rho_t]$. Moreover, the natural smooth structure of each component is coherent with the real semialgebraic structure of the GIT quotient.

While the smoothness of the $\mathrm{Isom}(\mathbb{H}^4)$-character variety for $t > 0$ was proved in [KS10], the smoothness on the Anti-de Sitter and half-pipe sides is completely new. In the half-pipe picture, we also discover, *a posteriori*, that the "stretching" deformations mentioned above are the only possible deformations of the half-pipe orbifold structure found in [RS22b], which is therefore essentially unique. The presence of many commutation relations forces the rigidity of the half-pipe structures.

Together with the hyperbolic and Anti-de Sitter picture, this shows that "nearby" there is no collapsing path of hyperbolic or Anti-de Sitter orbifold structures other than the ones we found (up to reparameterisation). This should be compared with some 3-dimensional examples found by Danciger [Dan13, Section 6], where the transitional HP structure deforms non-trivially to nearby half-pipe structures that regenerate to non-equivalent AdS structures, despite not regenerating to hyperbolic structures.

All in all, Theorem 7.2.1 exhibits a strong lack of flexibility around this example and suggests that this could be more generally due to dimension issues, confirming the usual feeling that "the rigidity increases with the dimension".

7.3 Transition on deformation spaces

In the work in preparation [EMST] with Christian El Emam, Filippo Mazzoli and Andrea Tamburelli, we study the effect of Danciger's geometric transition *on the deformation spaces*. Roughly speaking, we show that geometric transition from hyperbolic to Anti-de Sitter geometry induces a continuous deformation from Donaldson's hyperKähler structure to the para-hyperKähler structure of [MST23a] (described in Section 5.5), going through a *degenerate* hyperKähler structure on the deformation space of half-pipe manifolds.

More concretely, we produce a path of symmetric tensors g_t and paths I_t, J_t, K_t of smooth sections of the endomorphism bundle, each defined on a space of "solutions of an t-Gauss-Codazzi equation", which correspond to the Gauss-Codazzi equations in hyperbolic space when $t = 1$, in Anti-de Sitter space when $t = -1$, and in half-pipe space when $t = 0$. The symmetric tensors g_t are Riemannian metrics for $t > 0$, of neutral signature for $t < 0$, and degenerate for $t = 0$, and the paths I_t, J_t, K_t satisfy $I^2 = -1$ and $J^2 = K^2 = -t$, in such a way that (g_1, I_1, J_1, K_1) coincides with Donaldson's hyperKähler structure, and $(g_{-1}, I_{-1}, J_{-1}, K_{-1})$ coincides with the para-hyperKäher structure of [MST23a]. This construction is obtained by identifying deformation spaces to subsets of the cotangent bundle of Teichmüller space via minimal or maximal surfaces, and their analogues in half-pipe geometry, which have been introduced in [FS19, BF20]. At $t = 0$, (g_0, I_0, J_0, K_0) has a transparent interpretation in terms of half-pipe manifolds: for instance I_0 will again coincide with the complex structure of the cotangent bundle of Teichmüller space, while J_0 and K_0, namely the endomorphisms whose square vanishes, arise from an algebraic construction on the representation variety in the spirit of [Dan13].

Chapter 8

Equiaffine manifolds

The results of this chapter place in the context of *affine differential geometry* [NS94, LSZH15], which is the study of convex (hyper)surfaces in affine space $\mathbb{R}^n$ by means of geometric notions that are invariant by the group of affine transformations of $\mathbb{R}^n$ preserving the volume (called also *equiaffine transformations*). We will mostly focus on a class of surfaces in $\mathbb{R}^3$ called of *constant affine Gaussian curvature*. The results presented, which have been obtained in collaboration with Xin Nie, develop two different directions: the first is the study of such surfaces invariant under the action of discrete groups of equiaffine transformations (Section 8.2, [NS23a]), while the second aims to a classification result in a universal setting, without any group action (Section 8.3, [NS22]).

8.1 Affine spheres and affine Gaussian curvature

The crucial fact in order to develop the theory of affine differential geometry is the existence of a canonical transverse vector field to any convex (hyper)surface, called *affine normal*, which can be used to develop differential geometric invariants, such as the *(affine) shape operator*.

The main object of study of this chapter will be surfaces of *constant affine Gaussian curvature*, namely such that the determinant of their shape operator is a (positive) constant, whose study has been started in [LSC97, LSZ00, WZ11]. Although they are well-defined objects in any dimensions, we will restrict here to surfaces of constant affine Gaussian curvature in affine three-space $\mathbb{R}^3$. These can be considered at the same time as a generalization of *affine spheres* and of surfaces of *constant Gaussian curvature* in Minkowski space $\mathbb{R}^{2,1}$.

On the one hand, surfaces of constant Gaussian curvature in $\mathbb{R}^{2,1}$ were largely discussed in Chapter 6, and Theorem 6.2.1 shows that they are in 1-to-1 correspondence with regular domains in $\mathbb{R}^{2,1}$ which are not the future of a spacelike line. On the other hand, let us briefly recall the theory of affine spheres. These are convex surfaces (or more in general, convex hypersurfaces in $\mathbb{R}^n$) such that their affine shape operator is a multiple of the identity. Equivalently, their affine normals all meet at a single point, and if this point lies on the concave side of the hypersurface, then the affine sphere is called *hyperbolic*.

Now, a convex domain is said to be *proper* if it does not contain any entire straight line. Cheng and Yau [CY77] provided a classification theorem for hyperbolic affine spheres, by showing that in every proper convex cone $C \subset \mathbb{R}^{n+1}$ there exists a unique complete hyperbolic affine sphere Σ_C asymptotic to the boundary ∂C with affine shape operator the identity. Conversely, every properly embedded affine sphere is asymptotic to such a proper convex cone. See also [Lof10].

From the analytic viewpoint, the result of Cheng and Yau amounts to solving the Dirichlet problem of Monge-Ampère equation

$$\begin{cases} \det \mathrm{D}^2 w = (-w)^{-n-2} \quad \text{in } \Omega, \\ w|_{\partial\Omega} = 0, \end{cases} \tag{8.1}$$

on any bounded convex domain $\Omega \subset \mathbb{R}^n$, which is the dual of $\mathbb{P}(C)$. Now, given a proper convex cone C, we define a *C-regular domain* in the same way as regular domains in $\mathbb{R}^{n,1}$ mentioned earlier, namely as the intersection of at least two half-spaces bounded by affine hyperplanes which are C-lightlike, meaning that they are parallel to linear hyperplanes that contain a half-line in ∂C. The role of the future light cone $C_0 \subset \mathbb{R}^{n,1}$ is then replaced by C.

In this context, the problem of finding a surface of constant affine Gaussian curvature asymptotic to the boundary of the C-regular domain is governed by a lower semicontinuous function $\phi : \partial\Omega \to \mathbb{R} \cup \{+\infty\}$ is equivalent to the following Monge-Ampère equation:

$$\begin{cases} \det \mathrm{D}^2 u = c_k(-w_\Omega)^{-n-2} \text{ in } \Omega, \\ u|_{\partial\Omega} = \phi, \end{cases} \tag{8.2}$$

where w_Ω is the Cheng-Yau solution of the affine sphere problem 8.1, with the additional important condition that $|\nabla u(x)| \to +\infty$ as $x \in U$ tends to ∂U, which ensures that the corresponding surface is properly embedded (or *Euclidean complete*, in the classical terminology of affine differential geometry). For this reason, this problem is sometimes called a *two-step Monge-Ampère problem* ([LSC97]).

Observe that, if $\phi = 0$, then $u = w_\Omega$ is itself the solution to (8.2), which geometrically corresponds to the fact that C itself is trivially a C-regular domain, and an affine sphere is itself a surface of constant affine Gaussian curvature asymptotic to C. More remarkably, a spacelike surface in Minkowski space $\mathbb{R}^{2,1}$ has constant Gaussian curvature in the sense of Minkowski geometry if and only if its Minkowski normal coincides with the affine normal, and therefore this particular class also has constant *affine* Gaussian curvature. For this reason, constant affine Gaussian curvature is the right condition to generalize both K-surfaces in Minkowski space and affine spheres.

8.2 Affine deformations of quasi-divisible cones

Let us now present the results of [NS23a], in the context of group actions. From the viewpoint of higher Teichmüller theory, given a Fuchsian representation $\rho : \pi_1(S) \to \mathrm{SO}_0(2,1) = \mathrm{Isom}^+(\mathbb{H}^2)$ where S is a closed hyperbolic surface, one can consider the inclusion of ρ into a larger Lie group G, for example:

- the isometry group $\mathrm{SO}(2,1) \ltimes \mathbb{R}^{2,1}$ of the Minkowski space $\mathbb{R}^{2,1}$, where the deformations give rise to *maximal globally hyperbolic flat spacetimes*, introduced by [Mes07] (see also [BG01, Bar05, Bon05, KS07, BB09]);

- the special linear group $\mathrm{SL}(3,\mathbb{R})$, where the deformations yield *convex real projective structures* (see for example [Gol90, Ben08, KP14, CLM18]).

It is in the former case that surfaces of constant Gaussian curvatures occur to produce foliations of regular domains in $\mathbb{R}^{2,1}$, while the latter case has been largely studied via the theory of affine spheres. It is therefore natural to study representations in $\mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^{2,1}$ via surfaces of constant affine Gaussian curvature, as a generalization of both the above situations. Moreover, the results of [NS23a] also extend to the setting of punctured surfaces S.

Let us introduce the necessary definitions. Given a proper convex cone C in $\mathbb{R}^3$, we let $\mathrm{Aut}(C) < \mathrm{SL}(3,\mathbb{R})$ denote the group of special linear transformations preserving C, which is also the group of orientation-preserving projective automorphisms of the convex domain $\mathbb{P}(C) \subset \mathbb{RP}^2$. The projectivised cone $\mathbb{P}(C)$ is said to be *divisible* (resp. *quasi-divisible*) by a group $\Gamma < \mathrm{SL}(3,\mathbb{R})$ if Γ is discrete, contained in $\mathrm{Aut}(C)$, and the quotient $\mathbb{P}(C)/\Gamma$ is compact (resp. has finite volume with respect to the Hilbert metric). Furthermore, we always assume Γ is torsion-free, so that the quotient is a closed (resp. finite-volume) *convex projective surface*. Abusing the terminology, we also say that C is (quasi-)divisible by Γ if $\mathbb{P}(C)$ is.

Given a map $\tau : \Gamma \to \mathbb{R}^3$, a subgroup in $\mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^3$ of the form

$$\Gamma_\tau := \big\{(A, \tau(A)) \in \mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^3 \,\big|\, A \in \Gamma\big\}$$

is called an *affine deformation* of Γ. The group relation forces τ to be an element in the space $Z^1(\Gamma, \mathbb{R}^3)$ of cocycles. We call τ *admissible* if for every $A \in \Gamma$ given by a peripheral loop of the surface (such an A is parabolic), $\tau(A)$ is contained in the 2-dimensional subspace of $\mathbb{R}^3$ preserved by A.

Probably the most important result of [NS23a] is the following.

Theorem 8.2.1. *Let $C \subset \mathbb{R}^3$ be a proper convex cone quasi-divisible by a torsion-free group $\Gamma < \mathrm{SL}(3,\mathbb{R})$, $\tau \in Z^1(\Gamma, \mathbb{R}^3)$ be an admissible cocycle and D be a C-regular domain preserved by Γ_τ. Then for any $k > 0$, D contains a unique Euclidean complete affine*

(C,k)-surface Σ_k asymptotic to ∂D, which is preserved by Γ_τ. Moreover, $(\Sigma_k)_{k>0}$ is a foliation of D, and the function $K : D \to \mathbb{R}$ given by $K|_{\Sigma_k} = \log k$ is convex.

We moreover provided a more precise description of the action of Γ_τ. Indeed, we showed that:

1. There exists a C-regular domain in $\mathbb{R}^3$ preserved by Γ_τ if and only if τ is admissible. In this case, there is a unique continuous map f from $\partial\mathbb{P}(C)$ to the space of C-null planes in $\mathbb{R}^3$ which is equivariant in the sense that $f(A.x) = (A, \tau(A)).f(x)$ for all $x \in \partial\mathbb{P}(C)$ and $A \in \Gamma$. The complement of the union of planes $\bigcup_{x\in\partial\mathbb{P}(C)} f(x)$ in $\mathbb{R}^3$ has two connected components D^+ and D^-, which are C-regular and $(-C)$-regular domains preserved by Γ_τ, respectively.

2. If C is divisible by Γ, then D^+ is the unique C-regular domain preserved by Γ_τ. Otherwise, assume the surface $S := \mathbb{P}(C)/\Gamma$ has $n \geq 1$ punctures and τ is admissible, then all the C-regular domains preserved by Γ_τ form a family (D_μ) parameterized by $\mu \in \mathbb{R}^n_{\geq 0}$, such that $D_{(0,\cdots,0)} = D^+$ and we have $D_\mu \subset D_{\mu'}$ if and only if μ is coordinate-wise larger than or equal to μ'.

3. Γ_τ acts freely and properly discontinuously on every C-regular domain preserved by it, with quotient homeomorphic to $S \times \mathbb{R}$.

When C is the future light cone $C_0 \subset \mathbb{R}^{2,1}$, the divisible case of this theorem is part of the seminal work of Mess [Mes07]. Brunswic [Bru16, Bru21] has obtained results in the quasi-divisible case for C_0 as well. For general C, in the divisible case, the equivariant continuous map given in the first item is related to the Anosov property of Γ_τ, studied by Barbot [Bar10] and Danciger-Guéritaud-Kassel [DGK18] in different but related settings.

When S is closed (i.e. C is divisible by Γ), some of the above statements are contained in the works of Barbot-Béguin-Zeghib [BBZ11] for C_0 and Labourie [Lab07, §8] for general C. We remark moreover that the class of representations $\rho : \pi_1(S) \to \mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^3$ considered here is extremely rich. Indeed, by Marquis [Mar12], when the orientable surface $S = \mathbb{P}(C)/\Gamma$ is homeomorphic to either a surface $S_{g,n}$ of negative Euler characteristic with genus g and n punctures (which excludes the case of the torus, which is very simple), the deformation space $\mathcal{P}_{g,n}$ of finite-volume convex projective structures on $S_{g,n}$ is homeomorphic to a ball of dimension $16g - 16 + 6n$ (see [Mar10] and [BH13]). Moreover, if $\widehat{\mathcal{P}}_{g,n}$ is the deformation space of representations $\rho : \pi_1(S_{g,n}) \to \mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^3$ such that the $\mathrm{SL}(3,\mathbb{R})$-component of ρ defines a finite-volume convex projective structure and the $\mathbb{R}^3$-component is given by an admissible cocycle, then (assuming $2 - 2g - n < 0$), $\widehat{\mathcal{P}}_{g,n}$ is a topological vector bundle over $\mathcal{P}_{g,n}$ of rank $6g - 6 + 2n$.

8.3 Regular domains in the universal setting

In the article [NS22], we studied the problem of existence of surfaces of constant affine gaussian curvature in C-regular domains, without any assumption of group invariance. The result, however, requires a technical assumption, as follows. Given a convex planar domain Ω, we say that Ω satisfies the *exterior circle condition* if for every $x_0 \in \partial\Omega$ there is a round disk $B \subset \mathbb{R}^2$ containing Ω such that $x_0 \in \partial B$.

Theorem 8.3.1. *Let $C \subset \mathbb{R}^3$ be a proper convex cone such that the projectivized dual cone $\mathbb{P}(C^*) \subset \mathbb{RP}^{*2}$ satisfies the exterior circle condition. Let $D \subset \mathbb{R}^3$ be a proper C-regular domain. Then for every $k > 0$ there exists a unique complete affine (C, k)-surface $\Sigma_k \subset D$ which is asymptotic to ∂D.*

Moreover, similarly to the cocompact case, we showed that the family of surfaces $(\Sigma_k)_{k>0}$ from Theorem 8.3.1 is a foliation of D and the function $K : D \to \mathbb{R}$ defined by $K|_{\Sigma_k} = \log k$ is convex.

From the analytic viewpoint, given a bounded convex domain $\Omega \subset \mathbb{R}^2$ satisfying the exterior circle condition and $\phi : \partial\Omega \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function that is finite on at least three points, Theorem 8.3.1 asserts the existence of a unique lower semicontinuous convex function $u : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\}$ which is smooth in the interior of its domain $\{u < +\infty\}$, satisfies (8.2), and moreover has the gradient blowup property, namely the condition that its gradient diverges as one approaches $\partial\Omega$. Moreover, $\{u < +\infty\}$ coincides with the convex hull of $\{\phi < +\infty\}$ in $\mathbb{R}^2$. Theorem 8.3.1 largely improves in terms of regularity assumptions, but in dimension 3, the results of Li, Simon and Chen in [LSC97], where they proved unique solvability in any dimension when $\partial\Omega$ and ϕ are both *smooth*.

In a later article [NS23b] we achieved a partial generalization of these results, partly also in higher dimensions, for a more general class of two-step Monge-Ampère equations, which correspond geometrically to the class of surfaces of *constant Gaussian curvature* (or, in higher dimensions, *constant Gaussian curvature*), but with respect to the so-called *Li-normalization* (introduced and studied in [Xu08, XLL09, WZ11, XY11]) instead of the classical affine normal vector field.

Let us conclude by discussing the role of the exterior circle condition in Theorem 8.3.1 This is a technical condition, that ensures that the right-hand side of the first equation in (8.2) goes to $+\infty$ fast enough near $\partial\Omega$, which in turn ensures of u, namely the last condition in (8.2). Although this might seem a technical assumption, we produced a counterexample to Theorem 8.3.1 without the exterior circle condition assumption. Given a bounded convex domain $\Omega \subset \mathbb{R}^2$, and $\Delta \subset \Omega$ an open triangle with vertices on $\partial\Omega$, let ϕ be the function on $\partial\Omega$ vanishing at the vertices of Δ with $\phi = +\infty$ everywhere else. We showed that:

1. If Ω satisfies the exterior circle condition at every vertex of Δ, then there exists a unique u satisfying (8.2) and the gradient blowup property.

2. If $\partial\Omega$ contains an open line segment meeting $\partial\Delta$ exactly at a vertex, then any u satisfying (8.2) does not fulfill the gradient blowup property.

In other words, in the situation of the second item, the corresponding C-regular domain does not admit any Euclidean complete surface of constant affine Gaussian curvature. This is achieved by showing that the gradient of u does not blowup at the vertex of Δ on the line segment because the right-hand side of the Equation (8.2), which in turn is obtained as a solution to Equation 8.1, does not go to $+\infty$ fast enough near the segment.

We emphasize that Theorems 8.2.1 and 8.3.1 are transverse to each other, in the sense that none of the two results is covered by the other. To elaborate this, clearly Theorem 8.3.1 does not involve any group action, hence it might appear as more general than Theorems 8.2.1, However, when Ω is a *quasi-divisible projectivised cone*, then $\partial\mathbb{P}(C)$ is known to have at most $\mathbb{C}^{1,\alpha}$-regularity [Ben04, Gui05], and this implies that the dual domain $\partial\mathbb{P}(C^*)$ does not satisfy the exterior circle condition. However, in that setting the exterior circle condition is replaced by the fact that the quotient of $\partial\mathbb{P}(C)$ by Γ_τ is compact when one takes away a neighbourhood of every puncture, and that the domain Ω still has enough regularity at the fixed points of the action of peripheral loops in $\pi_1(S)$ on $\partial\Omega$ corresponding to any puncture of S.

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