A NOTE ON PLANAR CURVES WITH CONSTANT CURVATURE

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The aim of this note is to provide an elementary proof of the following:

Theorem 1. *All planar differentiable curves with constant curvature $c > 0$ are circles.*

1. The setting of the problem

We will reduce the problem to finding the solutions of a linear system of ODEs. Let $\gamma : I \to \mathbb{R}^2$ a differentiable curve. We can assume γ is parameterized by arclength. Let $t(s) = \gamma'(s)$ be the tangent vector of γ and let $n(s) = t'(s)/|t'(s)|$ be the normal vector.

The Frenet-Serret equations of γ are

$$\begin{cases} t'(s) = cn(s) \\ n'(s) = -ct(s) \end{cases}$$

where c is the curvature, which is constant by hypothesis.

Writing $t = (t_1, t_2)$ and $n = (n_1, n_2)$, the Frenet-Serret equations provide the following linear systems of ODEs:

$$\begin{cases} t_1'(s) = cn_1(s) \\ t_2'(s) = cn_2(s) \\ n_1'(s) = -ct_1(s) \\ n_2'(s) = -ct_2(s) \end{cases} . \tag{1}$$

We further need to impose suitable initial conditions. Since γ is parameterized by arclength, the initial tangent vector $t(0)$ is a unit vector, which we denote (α, β) with $\alpha^2 + \beta^2 = 1$. Moreover, the normal vector $n(0)$ is a unit vector orthogonal to $t(0)$, hence there are two possibilities: $n(0) = (-\beta, \alpha)$ or $n(0) = (\beta, -\alpha)$.

Hence we will study the system (1) with the following two sets of initial data:

$$\begin{cases} t_1(0) = \alpha \\ t_2(0) = \beta \\ n_1(0) = -\beta \\ n_2(0) = \alpha \end{cases} \tag{2}$$

$$\begin{cases} t_1(0) = \alpha \\ t_2(0) = \beta \\ n_1(0) = \beta \\ n_2(0) = -\alpha \end{cases} . \tag{3}$$

2. The resolution of the system of ODEs

Let us observe that the matrix associated to the linear system (1) is:

$$\begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \\ -c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{pmatrix} .$$

The characteristic polynomial is:

$$p(\lambda) = \lambda^4 + 2c^2\lambda^2 + c^4 = (\lambda^2 + c^2)^2 .$$

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Therefore the complex eigenvalues are ci and $-ci$, each with algebraic multiplicity 2. It can be checked that the geometric multiplicity of the eigenvalues of ci and $-ci$ is again 2, and two linearly independent eigenvectors for ci are:

$$\begin{pmatrix}1\\0\\i\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\i\end{pmatrix}.$$

On the other hand, two linearly independent eigenvectors for $-ci$ are:

$$\begin{pmatrix}1\\0\\-i\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\-i\end{pmatrix}.$$

By taking the real and imaginary part of each eigenpair, one obtains the four linearly independent solutions:

$$\begin{pmatrix}\cos(cs)\\0\\-\sin(cs)\\0\end{pmatrix}, \quad \begin{pmatrix}\sin(cs)\\0\\\cos(cs)\\0\end{pmatrix}, \quad \begin{pmatrix}0\\\cos(cs)\\0\\-\sin(cs)\end{pmatrix}, \quad \begin{pmatrix}0\\\sin(cs)\\0\\\cos(cs)\end{pmatrix}.$$

Therefore the general solution of (1) is

$$\begin{pmatrix}t_1(s)\\t_2(s)\\n_1(s)\\n_2(s)\end{pmatrix} = A\begin{pmatrix}\cos(cs)\\0\\-\sin(cs)\\0\end{pmatrix} + B\begin{pmatrix}\sin(cs)\\0\\\cos(cs)\\0\end{pmatrix} + C\begin{pmatrix}0\\\cos(cs)\\0\\-\sin(cs)\end{pmatrix} + D\begin{pmatrix}0\\\sin(cs)\\0\\\cos(cs)\end{pmatrix}.$$

Observe that

$$\begin{pmatrix}t_1(0)\\t_2(0)\\n_1(0)\\n_2(0)\end{pmatrix} = \begin{pmatrix}A\\C\\B\\D\end{pmatrix}.$$

Imposing the initial condion (2), one obtains $A = \alpha, C = \beta, B = -\beta, D = \alpha$. Therefore the solution has the form

$$\begin{pmatrix}t_1(s)\\t_2(s)\\n_1(s)\\n_2(s)\end{pmatrix} = \begin{pmatrix}\alpha\cos(cs) - \beta\sin(cs)\\\beta\cos(cs) + \alpha\sin(cs)\\-\alpha\sin(cs) - \beta\cos(cs)\\-\beta\sin(cs) + \alpha\cos(cs)\end{pmatrix}.$$

By an analogous computation, the initial condition (3) provides $A = \alpha, C = \beta, B = \beta, D = -\alpha$ and therefore the solution

$$\begin{pmatrix}t_1(s)\\t_2(s)\\n_1(s)\\n_2(s)\end{pmatrix} = \begin{pmatrix}\alpha\cos(cs) + \beta\sin(cs)\\\beta\cos(cs) - \alpha\sin(cs)\\-\alpha\sin(cs) + \beta\cos(cs)\\-\beta\sin(cs) - \alpha\cos(cs)\end{pmatrix}.$$

3. The explicit form of planar curves of constant curvature

The explicit expression of the curve γ is now easily recovered by integrating the tangent vector t, and of course imposing the initial point $\gamma(0) = (x_0, y_0)$. In the first case we get

$$\gamma(s) = \begin{pmatrix}x_0 + \alpha\sin(cs) + \beta\cos(cs)\\y_0 + \beta\sin(cs) - \alpha\cos(cs)\end{pmatrix},$$

while in the second case

$$\gamma(s) = \begin{pmatrix}x_0 + \alpha\sin(cs) - \beta\cos(cs)\\y_0 + \beta\sin(cs) + \alpha\cos(cs)\end{pmatrix}.$$

4. Interpretation of the results

We have shown that, given any initial point (x_0, y_0), any initial tangent vector (α, β) and any choice of unit normal vector (among the two possibilities), there is a unique curve γ of constant curvature, parameterized by arclength, such that the initial point is

$$\gamma(0) = (x_0, y_0),$$

the initial tangent vector is

$$\gamma'(0) = (\alpha, \beta),$$

and the normal vector is the chosen one. A direct inspection shows that the resulting curves are circles, as in the following pictures:

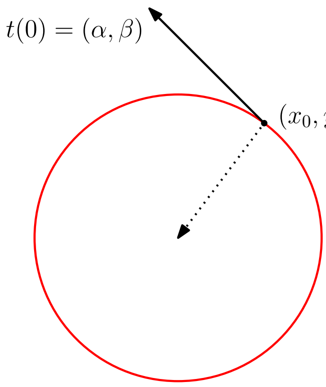

Figure 1. The solution with the choice $n(0) = (-\beta, \alpha)$. The radius of curvature is $1/c$.

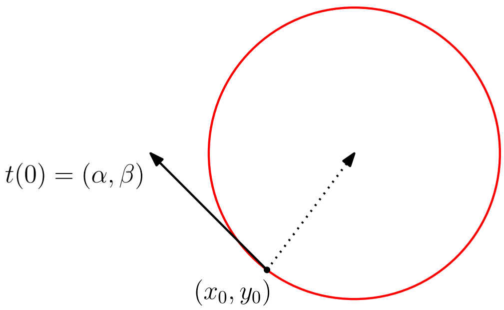

Figure 2. The solution with the choice $n(0) = (\beta, -\alpha)$.

This coincides with the intuitive idea (which you have probably experimented by using compasses) that there are only two circles passing through a given point, with a given tangent line at that point, and with a fixed radius.