HOMEWORK ON TENSOR ALGEBRA

In the following problems $\mathcal{B} := (\boldsymbol{e}_1, \boldsymbol{e}_1, \boldsymbol{e}_3)$ will always denote a basis of $\mathcal{V}$.

Problem 1. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{V}$. What are $\det(\boldsymbol{a} \otimes \boldsymbol{b})$ and $(\boldsymbol{a} \otimes \boldsymbol{b})^\top$?

Problem 2. Let $\mathbf{A}$ and $\mathbf{B}$ two tensors. Prove that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$.

Problem 3. Let $\mathbf{C}$ be an invertible tensor. Prove that $\det(\mathbf{C}^{-1}) = (\det \mathbf{C})^{-1}$.

Problem 4. Let $\mathbf{A} = \sum_{i,j=1}^{3} A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\mathbf{A}^\top = \sum_{i,j=1}^{3} A_{ij}^\top \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Prove that

(1) $A_{ij}^\top = A_{ji}$ for each $i, j = 1, 2, 3$;
(2) $\det \mathbf{A}^\top = \det \mathbf{A}$.

Problem 5. Let $\mathbf{L}$ be a tensor such that

$$\begin{cases} \mathbf{L}\boldsymbol{e}_1 = 2\boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \mathbf{L}\boldsymbol{e}_2 = \boldsymbol{e}_2 + 3\boldsymbol{e}_3 \\ \mathbf{L}\boldsymbol{e}_3 = \boldsymbol{e}_1 + 3\boldsymbol{e}_2 + 3\boldsymbol{e}_3 \end{cases}$$

and let $\boldsymbol{v} = -\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3$.

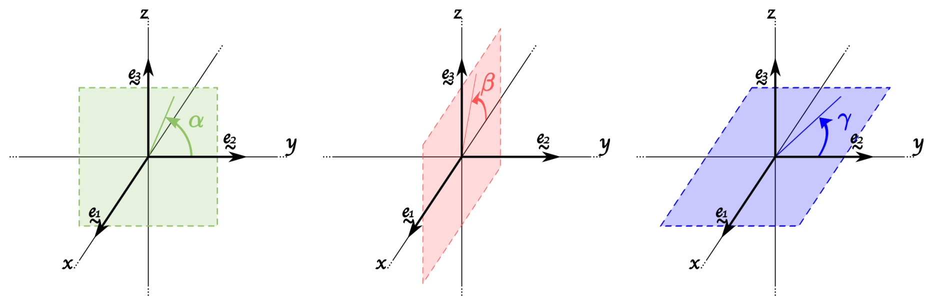
(1) Write the matrices L, $L^\top$ and L^{-1} that represent, respectively, $\mathbf{L}$, $\mathbf{L}^\top$ and $\mathbf{L}^{-1}$ in the basis $\mathcal{B}$.
(2) Write (in the basis $\mathcal{B}$) $\mathbf{L}^\top \boldsymbol{v}$, $\mathbf{L}^{-1}\boldsymbol{v}$ and $\mathbf{L}^\star \boldsymbol{v}$.

Let $\mathcal{B}' = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$ with

$$\boldsymbol{b}_1 = \frac{\sqrt{3}}{3}\boldsymbol{e}_1 + \frac{\sqrt{3}}{3}\boldsymbol{e}_2 + \frac{\sqrt{3}}{3}\boldsymbol{e}_3,$$
$$\boldsymbol{b}_2 = -\frac{\sqrt{6}}{3}\boldsymbol{e}_1 + \frac{\sqrt{6}}{6}\boldsymbol{e}_2 + \frac{\sqrt{6}}{6}\boldsymbol{e}_3,$$
$$\boldsymbol{b}_3 = -\frac{\sqrt{2}}{2}\boldsymbol{e}_2 + \frac{\sqrt{2}}{2}\boldsymbol{e}_3.$$

(3) Check that $\mathcal{B}'$ is a basis of $\mathcal{V}$ [i.e.: $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ are linearly independent, pairwise orthogonal and of length 1].
(4) Write the matrix L' that represents the tensor $\mathbf{L}$ in the basis $\mathcal{B}'$.
(5) Check $\det \mathbf{L} = \det L = \det L'$. [Hint: you can compute $\det \mathbf{L}$ by using $\boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_3$ and the very definition of determinant.]

Problem 6. Let the angles α, β, γ and the lines x, y, z be as in the following pictures:



Let the tensor $\mathbf{R}_{x\alpha}$ be the rotation around the line x by the angle α.
Let the tensor $\mathbf{R}_{y\beta}$ be the rotation around the line y by the angle β.
Let the tensor $\mathbf{R}_{z\gamma}$ be the rotation around the line z by the angle γ.

(1) What are the images (expressed in the basis $\mathcal{B}$) of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ under $\mathbf{R}_{x\alpha}, \mathbf{R}_{y\beta}$ and $\mathbf{R}_{z\gamma}$ respectively?
(2) Find the eigenvalues and the associated eigenspaces of $\mathbf{R}_{x\alpha}, \mathbf{R}_{y\beta}$ and $\mathbf{R}_{z\gamma}$.
(3) What happens if $\alpha = 0$ or $\beta = 0$ or $\gamma = 0$?
(4) Write the matrices that represent $\mathbf{R}_{x\alpha}, \mathbf{R}_{y\beta}$ and $\mathbf{R}_{z\gamma}$ in $\mathcal{B}$.

Note: *a generic* rotation *in $L(\mathcal{V})$ is always a tensor of the form $\mathbf{R}_{x\alpha}\mathbf{R}_{y\beta}\mathbf{R}_{z\gamma}$, i.e. it is always a composition of a rotation around x, a rotation around y and a rotation around z.*

(5) What are $\mathbf{R}_{x\frac{\pi}{2}}\mathbf{R}_{y\frac{\pi}{2}}\mathbf{R}_{z\frac{\pi}{2}}\boldsymbol{e}_1$, $\mathbf{R}_{x\frac{\pi}{2}}\mathbf{R}_{y\frac{\pi}{2}}\mathbf{R}_{z\frac{\pi}{2}}\boldsymbol{e}_2$ and $\mathbf{R}_{x\frac{\pi}{2}}\mathbf{R}_{y\frac{\pi}{2}}\mathbf{R}_{z\frac{\pi}{2}}\boldsymbol{e}_3$?

Problem 7. Let $\boldsymbol{u} = \boldsymbol{e}_2 - \boldsymbol{e}_3$ and $\boldsymbol{v} = \boldsymbol{e}_1 - \boldsymbol{e}_3$, and let $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ be the subspace of $\mathcal{V}$ generated by $\boldsymbol{u}$ and $\boldsymbol{v}$ (i.e. the plane where both $\boldsymbol{u}$ and $\boldsymbol{v}$ lie). Let the tensor $\mathbf{R}_{\boldsymbol{uv}}$ be the reflection through $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$.

(1) What are $\mathbf{R}_{\boldsymbol{uv}}\boldsymbol{u}$ and $\mathbf{R}_{\boldsymbol{uv}}\boldsymbol{v}$?
(2) What is the image of $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ under $\mathbf{R}_{\boldsymbol{uv}}$? What happens to any vector $\boldsymbol{w} \in \operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ when $\mathbf{R}_{\boldsymbol{uv}}$ is applied to it?
(3) What is $\mathbf{R}_{\boldsymbol{uv}}(\boldsymbol{u} \times \boldsymbol{v})$?
(4) Write the matrix that represents $\mathbf{R}_{\boldsymbol{uv}}$ in the basis $\mathcal{B}$.

Problem 8. Consider the shear tensor $\mathbf{F} = \boldsymbol{I} + 3\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}_3$.

(1) Write the matrix that represents $\mathbf{F}^{\star}$ in the basis $\mathcal{B}$.
(2) Which is the area dilation factor of $\mathbf{F}$ for surfaces parallel to $\operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_3\}$?
(3) Which is the area dilation factor of $\mathbf{F}$ for surfaces parallel to $\operatorname{span}\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$?