PROBLEM SHEET ON DIFFERENTIAL GEOMETRY OF CURVES IN EUCLIDEAN SPACE

Problem 1. *Consider the parameterized curve*

$$\alpha(t) = (\frac{1}{3}(\sqrt{1+t})^3, \frac{1}{3}(\sqrt{1-t})^3, \frac{1}{\sqrt{2}}t)$$

which is defined for $t \in [-1, 1]$.

- *Check if* α *is parameterized by arclength;*
- *Find the normal vector and the curvature of* α *for every* $t \in \mathbb{R}$*;*
- *Find the binormal vector and the torsion of* α *for every* $t \in \mathbb{R}$*.*

Problem 2. *Consider the parameterized curve*

$$\beta(t) = (a\cosh(t), a\sinh(t), bt)\,.$$

(Recall that $\cosh(t) = (e^t + e^{-t})/2$ *is the* hyperbolic cosine *and* $\sinh(t) = (e^t - e^{-t})/2$ *is the* hyperbolic sine.*)*

- *Check if* β *is parameterized by arclength;*
- *Find the normal vector and the curvature of* β *for every* $t \in \mathbb{R}$*;*
- *Find the center of the osculating circle for every* $t \in \mathbb{R}$*;*
- *Find the binormal vector and the torsion of* β *for every* $t \in \mathbb{R}$*;*

Draw some pictures of the curve.

Problem 3. *Consider the parameterized curve*

$$\gamma(t) = (a\cos(t), a\sin(t), be^t)\,.$$

(You have already seen this curve in Problem 4, Problem sheet 1.)

- *Compute the tangent vector for every* $t \in \mathbb{R}$*.*
- *Compute the normal vector and the curvature of* γ *for every* $t \in \mathbb{R}$*. What is the limit of the curvature, when* $t \to \infty$ *and* $t \to -\infty$*?*

Problem 4. *Prove that, given a plane regular parameterized curve* $\alpha : \mathbb{R} \to \mathbb{R}^2$*, the curvature of* α *at* t *is the derivative of the angle formed by the tangent line to* α *at* t *with the horizontal line* $y = 0$*.* Hint: Write the unit tangent vector at $\alpha(t)$ as $(\cos\theta(t), \sin\theta(t))$, where $\theta(t)$ is the angle, and differentiate.

Is this still true if one takes any line in $\mathbb{R}^2$*, instead of the line* $y = 0$*?*

Problem 5 (Extra challenge)**.** *Find* all *planar curves in* $\mathbb{R}^2$ *parameterized by arclength, having constant curvature (i.e. for which the curvature is* $k(t) = k_0$*, where* k_0 *is a constant).*
Hint: Let $(\mathbf{t}_1(t), \mathbf{t}_2(t))$ be the tangent vector and $(\mathbf{n}_1(t), \mathbf{n}_2(t))$ the normal vector. Write the Frenet-Serret equations and remember from Problem Sheet 1 how to solve a linear system of ODEs, with the method of eigenvalues. You will need to impose a good initial condition: $(\mathbf{t}_1(0), \mathbf{t}_2(0)) = (x_0, y_0)$ with $x_0^2 + y_0^2 = 1$, and you need to distinguish two cases for $(\mathbf{n}_1(0), \mathbf{n}_2(0))$. Finally, if you know the tangent vector to a curve α parameterized by arclength, then you can find the curve α by integration. Interpret the result with some pictures.

Date: October 2, 2015.